Distance Spectra of Graphs

Jephian C.-H. Lin
Joint work supported by GRWC

Department of Mathematics, Iowa State University

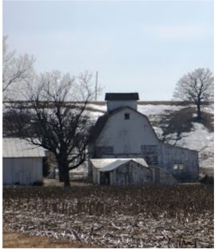
Oct 30, 2016
2016 AMS Fall Central Sectional Meeting, Minneapolis, MN

Rocky Mountain-Great Plains Graduate Research Workshop in Combinatorics

- **GRWC 2016 - Laramie, WY**
- **Application Instructions**
- **Funding**
- **Organizing Committee**
- **Participants**
- **Previous Workshops**
  - **GRWC 2014 - Denver, CO**
  - GRWC 2015 - Ames, IA
- **Problem Gardens**
- **Research Papers**
- **Workshop Format**
- **Sitemap**

Previous Workshops >

GRWC 2014 - Denver, CO



July 27 – August 9, 2014 --- Denver, CO

Co - Hosted by

The University of Colorado Denver and

The University of Denver

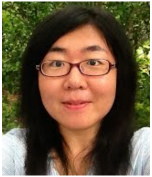
Ghodratollah Aalipour

Aida Abiad

Zhanar Berikkyzy

Jay Cummings

Jessica De Silva



Wei Gao

Kristin Heysse

Leslie Hogben

Franklin Kenter

Michael Tait

Distance matrix

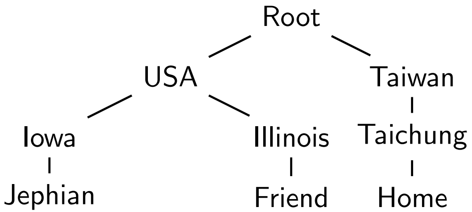
- Let G be a connected simple graph on vertex set $V = \{1, \ldots, n\}$.
- The distance $d_G(i,j)$ between two vertices i, j on G is the length of the shortest path.
- The distance matrix of G is an $n \times n$ matrix

$$\mathcal{D} = \left[d_G(i,j) \right].$$

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 2 & 3 \\ 2 & 1 & 0 & 1 & 2 \\ 3 & 2 & 1 & 0 & 1 \\ 4 & 3 & 2 & 1 & 0 \end{bmatrix}$$

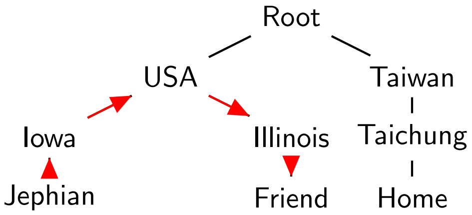
Motivation: Pierce's loop switching scheme

- How two build a phone call between two persons?
  - Root-USA-Iowa-Jephian
  - Root-USA-Illinois-Friend
  - Root-Taiwan-Taichung-Home



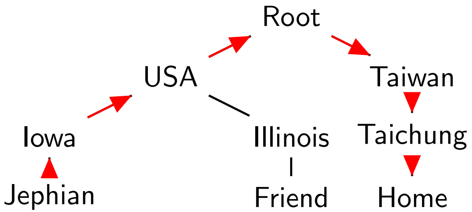
Motivation: Pierce's loop switching scheme

- How two build a phone call between two persons?
  - Root-USA-Iowa-Jephian
  - Root-USA-Illinois-Friend
  - Root-Taiwan-Taichung-Home



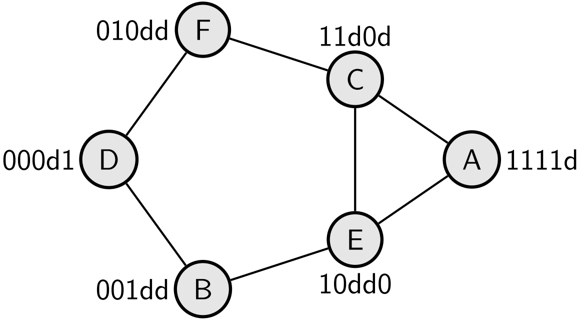
Motivation: Pierce's loop switching scheme

- How two build a phone call between two persons?
  - Root-USA-Iowa-Jephian
  - Root-USA-Illinois-Friend
  - Root-Taiwan-Taichung-Home



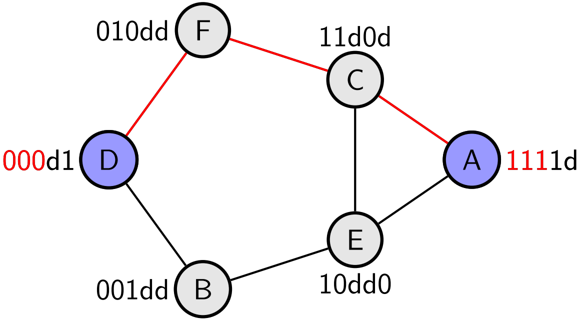
Graham and Pollak's model

- A model works for all graphs, not limited to trees.
- Each vertex is assigned with an address, and the distance between two vertices is the Hamming distance of the address.
- Find the neighbor that decrease the Hamming distance.



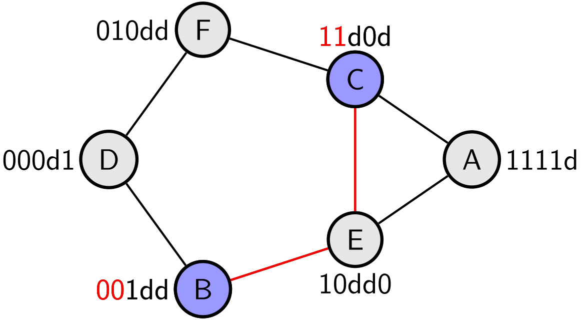
Graham and Pollak's model

- A model works for all graphs, not limited to trees.
- Each vertex is assigned with an address, and the distance between two vertices is the Hamming distance of the address.
- Find the neighbor that decrease the Hamming distance.



Graham and Pollak's model

- A model works for all graphs, not limited to trees.
- Each vertex is assigned with an address, and the distance between two vertices is the Hamming distance of the address.
- Find the neighbor that decrease the Hamming distance.



Length of the address

Theorem (Graham and Pollak 1971)

Let G be a graph and $\mathcal{D}$ its distance matrix. Then such an address always exist and its length is at least

$$\max\{n_-, n_+\},$$

where n_-, n_+ are the negative and positive inertia.

Corollary (Graham and Pollak 1971)

When G is a complete graph or a tree, then the minimum length of the address is $|V(G)| - 1$.

Length of the address

Conjecture (Graham and Pollak 1971)

For any graph on n vertices, the address can be chosen with length at most $n - 1$.

Theorem (Winkler 1983)

The squashed cube conjecture is true.

Number of distinct eigenvalues

- Suppose A is a matrix. Let $q(A)$ be the number of distinct eigenvalues.
- If A is the adjacency matrix of graph G, then

$$q(A) \geq \operatorname{diam}(G) + 1.$$

- Key: When a matrix M is diagonalizable, then

$$q(M) = \text{degree of min polynomial.}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, A^2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, A^3 = \begin{bmatrix} 0 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 0 & 2 \\ 1 & 0 & 2 & 0 \end{bmatrix}.$$

How about distance matrices?

- Distance matrices are dense (all off-diagonal entries are non-zero).
- Let Q_d be the d-dimensional hypercube. Then $q(\mathcal{D}(Q_d)) = 3$ and $\mathrm{diam}(Q_d) = d$ for $d \geq 2$.
- What is the relation between $q(\mathcal{D}(G))$ and $\mathrm{diam}(G)$?

How about distance matrices?

- Distance matrices are dense (all off-diagonal entries are non-zero).
- Let Q_d be the d-dimensional hypercube. Then $q(\mathcal{D}(Q_d)) = 3$ and $\operatorname{diam}(Q_d) = d$ for $d \geq 2$.
- What is the relation between $q(\mathcal{D}(G))$ and $\operatorname{diam}(G)$?

Theorem (Aalipour et al 2016)

Let T be a tree and $\mathcal{D}$ its distance matrix. Then

$$q(\mathcal{D}) \geq \left\lceil \frac{\operatorname{diam}(T)}{2} \right\rceil.$$

Proof.

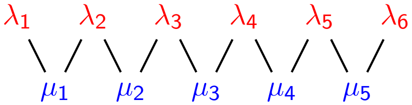
- Let $L(T)$ be the line graph of T and A the adjacency matrix of $L(T)$.
- Note that A is $(n-1)\times(n-1)$ and $\mathcal{D}$ is $n \times n$.
- $\operatorname{spec}(-2(2I+A)^{-1})$ interlaces $\operatorname{spec}(\mathcal{D})$. [Merris 1990]
- $q(-2(2I+A)^{-1}) = q(A) \geq \operatorname{diam}(L(T)) + 1 = \operatorname{diam}(T)$.
- $q(\mathcal{D}) \geq \left\lceil \frac{q(A)}{2} \right\rceil$.

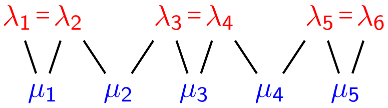
Interlacing

$\mu_1 \le \mu_2 \le \cdots \le \mu_{n-1}$ interlaces $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ if

$$\lambda_1 \le \mu_1 \le \lambda_2 \le \mu_2 \le \cdots \le \mu_{n-1} \le \lambda_n$$

Examples of interlacing:





Distinct eignvalues of trees

It is true that $q(\mathcal{D}) \geq \operatorname{diam}(T) + 1$?
Possible approaches:

- Show the interlacing does not collapse.
- Or consider the inverse:

$$\mathcal{D}^{-1} = -\frac{1}{2}L + \frac{1}{2(n-1)}\delta\delta^{\top},$$

where $\delta_i = 2 - d_i$. [Graham and Lovász 1978]

Checked by *Sage* up to 20 vertices; graphs with the inequality tight are extremely rare; e.g. when $n = 15$, only 7 graphs has $q(\mathcal{D}) = \operatorname{diam}(T) + 1$.

Thank you!

Distinct eignvalues of trees

It is true that $q(\mathcal{D}) \geq \mathrm{diam}(T) + 1$?

Possible approaches:

- Show the interlacing does not collapse.
- Or consider the inverse:

$$\mathcal{D}^{-1} = -\frac{1}{2}L + \frac{1}{2(n-1)}\delta\delta^{\top},$$

where $\delta_i = 2 - d_i$. [Graham and Lovász 1978]

Checked by *Sage* up to 20 vertices; graphs with the inequality tight are extremely rare; e.g. when $n = 15$, only 7 graphs has $q(\mathcal{D}) = \mathrm{diam}(T) + 1$.

Thank you!

References I

G. Aalipour, A. Abiad, Z. Berikkyzy, J. Cummings, J. De Silva, W. Gao, K. Heysse, L. Hogben, F. H. J. Kenter, J. C.-H. Lin, and M. Tait.
On the distance spectra of graphs.
Linear Algebra Appl., 497:66–87, 2016.

R. L. Graham and L. Lovász.
Distance matrix polynomials of trees.
Adv. Math., 29:60–88, 1978.

R. L. Graham and H. O. Pollak.
On the addressing problem for loop switching.
The Bell System Technical Journal, 50:2495–2519, 1971.

References II

R. Merris.
The distance spectrum of a tree.
J. Graph Theory, 14:365–369, 1990.

P. M. Winkler.
Proof of the squashed cube conjecture.
Combinatorica, 3:135–139, 1983.