

General spectral graph theory: The inverse eigenvalue problem of a graph

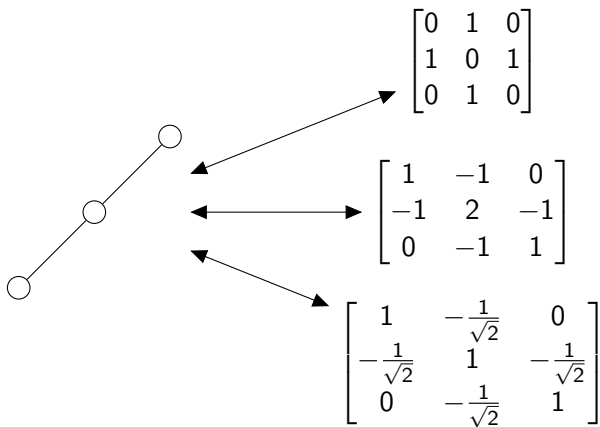
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Dec 10, 2017

2017 年中華民國數學年會, Chiayi City, Taiwan

Spectral graph theory



Cvetković's inertia bound

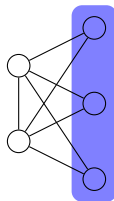
The **inertia** of a matrix A is $(n_+(A), n_-(A), n_0(A))$, which are the number of positive, negative, and zero eigenvalues of A , respectively.

Theorem (Cvetković 1971)

Let G be a graph and A its adjacency matrix. Then

$$\alpha(G) \leq \min\{n - n_+(A), n - n_-(A)\},$$

where $\alpha(G)$ is the independence number.



$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} n = 5 \\ n_+ = 1 \\ n_- = 2 \end{array}$$

Godsil's Lemma

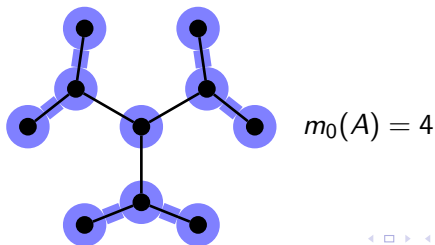
Let G be a graph. The **path cover number** $P(G)$ is the minimum number of disjoint induced paths that can cover G .

Theorem (Godsil 1984)

Let G be a tree with adjacency matrix A . Then

$$m_\lambda(A) \leq P(G)$$

for any eigenvalue λ of A .

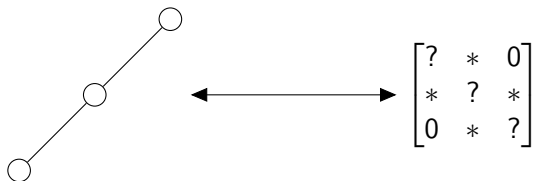


General spectral graph theory

Given a graph G on n vertices, consider the family $\mathcal{S}(G)$ of $n \times n$ real symmetric matrices M with

$$\begin{cases} M_{i,j} = 0 & \text{if } i \neq j \text{ and } \{i,j\} \text{ is not an edge,} \\ M_{i,j} \neq 0 & \text{if } i \neq j \text{ and } \{i,j\} \text{ is an edge,} \\ M_{i,j} \in \mathbb{R} & \text{if } i = j. \end{cases}$$

Thus, $\mathcal{S}(G)$ includes the adjacency matrix, the Laplacian matrix, and so on.



The general version of Cvetković's inertia bound

Theorem

Let G be a graph and $A \in \mathcal{S}(G)$ with zero diagonal entries. Then

$$\alpha(G) \leq \min\{n - n_+(A), n - n_-(A)\},$$

where $\alpha(G)$ is the independence number.

- ▶ Sinkovic (2017) proved Paley 17 is an example where the inertia bound is not tight. (So far, all known constructions are related to Paley 17.)
- ▶ He is going to talk about it at the Joint Meeting 2018 in San Diego!

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Theorem (Johnson and Leal Duarte 1999)

Let G be a tree and $A \in \mathcal{S}(G)$. Then

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- ▶ Indeed, for any tree, there is a matrix A with an eigenvalue λ such that $m_\lambda(A) = P(G)$.

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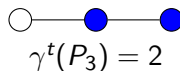
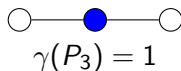
Domination number

Let G be a graph. The **domination number** $\gamma(G)$ is the minimum cardinality of a set X such that

$$\bigcup_{x \in X} N_G[x] = V(G).$$

The **total domination number** $\gamma^t(G)$ is the minimum cardinality of a set X such that

$$\bigcup_{x \in X} N_G(x) = V(G).$$



Greedy algorithm

- ▶ Greedy algorithm follows the problem solving heuristic of **making the locally optimal choice** at each stage with the hope of finding a global optimum.
- ▶ For solving a maze, you may keep going straight at fork. But it might lead you to a dead end.
- ▶ For a coloring problem, you may keep using the smallest free number to color the next vertex, showing $\chi(G) \leq \Delta(G) + 1$.
- ▶ Greedy algorithm for domination number: When X are chosen and not yet dominate the whole graph, pick a vertex v such that

$$N_G[v] \setminus \bigcup_{x \in X} N_G[x] \neq \emptyset.$$

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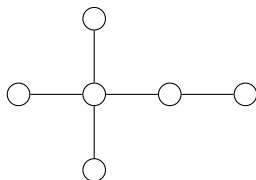
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Grundy domination number

The **Grundy domination number** $\gamma_{\text{gr}}(G)$ is the length of the longest sequence (v_1, v_2, \dots, v_k) such that

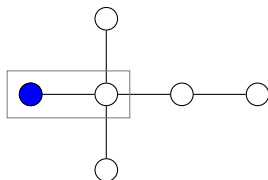
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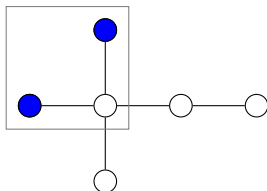
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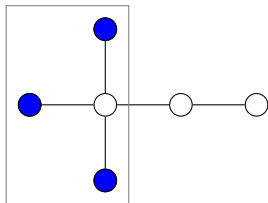
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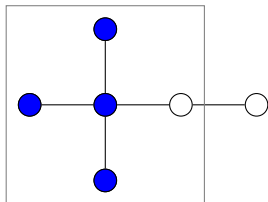
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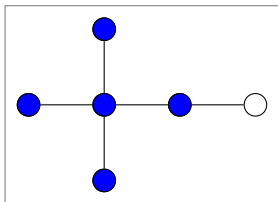
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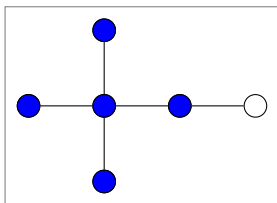
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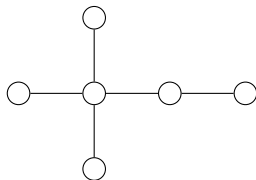


So $\gamma_{\text{gr}}(G) = 5$.

Grundy total domination number

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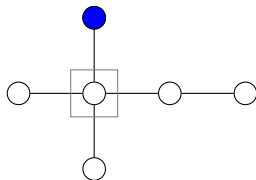
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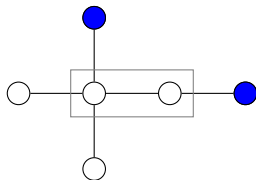
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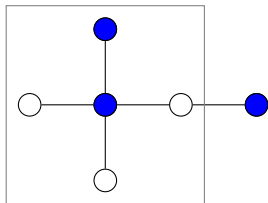
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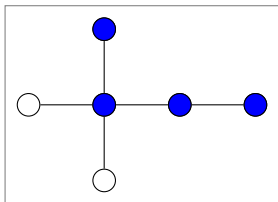
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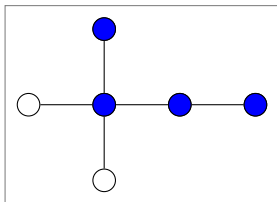
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So $\gamma_{\text{gr}}^t(G) = 4$.

Rank bound

Theorem (L 2017)

Let G be a graph. Then

$$\gamma_{\text{gr}}(G) \leq \text{rank}(A)$$

for any $A \in S(G)$ with diagonal entries all nonzero; and

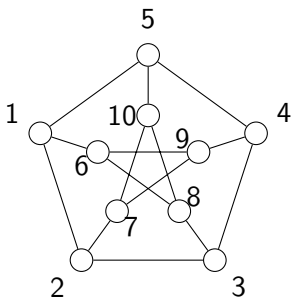
$$\gamma_{\text{gr}}^t(G) \leq \text{rank}(A)$$

for any $A \in S(G)$ with zero diagonal.

Let P be the Petersen graph. Consider

$$A = \begin{bmatrix} C - I & I_5 \\ I_5 & C' - I \end{bmatrix} \text{ and } B = \begin{bmatrix} C & I_5 \\ I_5 & -C' \end{bmatrix},$$

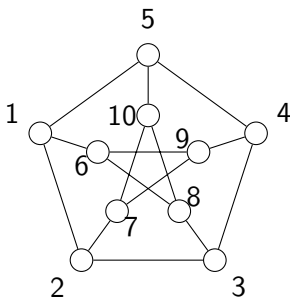
where C and C' are the adjacency matrix of C_5 and $\overline{C_5}$, respectively. Then $\gamma_{\text{gr}}(P) \leq \text{rank}(A) = 5$ and the sequence $(1, 2, 3, 4, 5)$ is optimal.



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where C and C' are the adjacency matrix of C_5 and $\overline{C_5}$, respectively. Then $\gamma_{\text{gr}}^t(G) \leq \text{rank}(B) = 6$ and the sequence $(9, 1, 2, 3, 4, 5)$ is optimal.



Proof of the theorem

- ▶ Goal: Show $\gamma_{\text{gr}}(G) \leq \text{rank}(A)$ for all $A \in \mathcal{S}(G)$ with nonzero diagonal entries.
- ▶ Key: **Permutation** does not change the rank, and the dominating sequence gives an **echelon form**.

Pick an optimal sequence (v_1, \dots, v_k) and a matrix A . Let N_j be the vertices dominated by v_j but not any vertex before v_j .

$$\begin{array}{l} v_1 \\ v_2 \\ \vdots \\ v_k \\ \text{other vertices} \end{array} \begin{bmatrix} & N_1 & N_2 & \dots & N_k \\ * \cdots * & 0 & \dots & 0 \\ ? & * \cdots * & 0 & \vdots \\ ? & ? & \ddots & 0 \\ ? & \dots & ? & * \cdots * \\ ? & ? & ? & ? \end{bmatrix}$$

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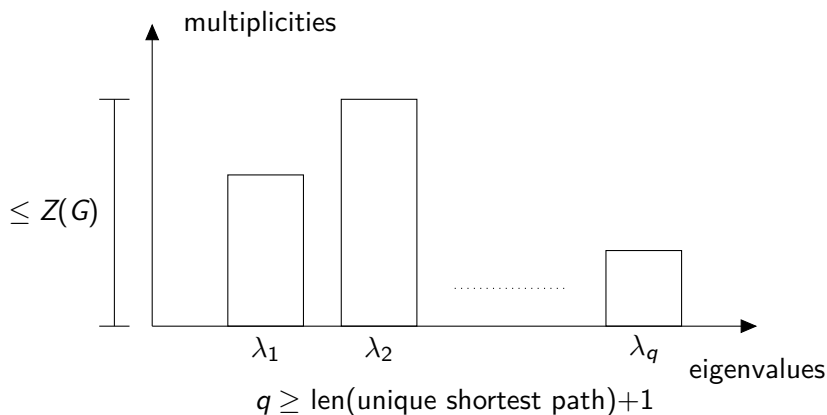
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Inverse eigenvalue problem of a graph

The **inverse eigenvalue problem of a graph** (IEPG) aims to find all spectra in $\mathcal{S}(G)$ for a given graph.





(Idea from Totem Poles in Canada.)

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