On the inverse eigenvalue problem for block graphs

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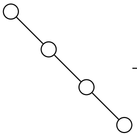
Inverse eigenvalue problem of a graph (IEP-G)

Let G be a graph. Define $\mathcal{S}(G)$ as the family of all real symmetric matrices $A = \left[a_{ij}\right]$ such that

$$a_{ij} \begin{cases} \neq 0 & \text{if } ij \in E(G), i \neq j; \\ = 0 & \text{if } ij \notin E(G), i \neq j; \\ \in \mathbb{R} & \text{if } i = j. \end{cases}$$



$$\longrightarrow \begin{bmatrix} ? & * & 0 & 0 \\ * & ? & * & 0 \\ 0 & * & ? & * \\ 0 & 0 & * & ? \end{bmatrix} \underset{?}{\overset{\text{spec}}{\rightleftarrows}} \{1, 2, 3, 4\}$$

IEP-G: What are the possible spectra of a matrix in $\mathcal{S}(G)$?

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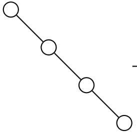

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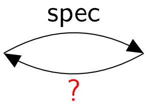
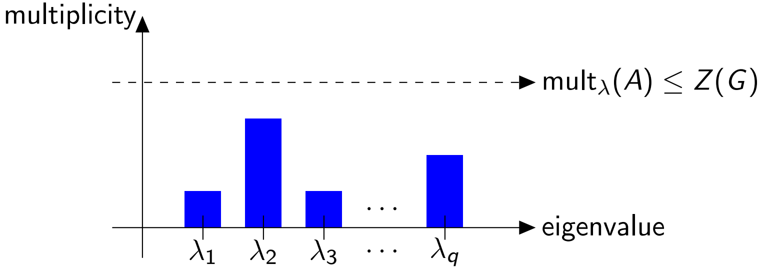


$$\{1, 2, 3, 4\}$$

IEP-G: What are the possible spectra of a matrix in $\mathcal{S}(G)$?

Ordered multiplicity list



$$\text{spec}(A) = \{\lambda_1^{(m_1)}, \ldots, \lambda_q^{(m_q)}\} \implies \begin{array}{l} \text{m}(A) = (m_1, \ldots, m_q), \\ q(A) = q \end{array}$$

Supergraph Lemma

Lemma (BFHHLS 2017)

Let G and H' be two graphs with $V(G) = V(H')$ and $E(G) \subseteq E(H')$. If $A \in \mathcal{S}(G)$ has the SSP, then there is a matrix $A' \in \mathcal{S}(H')$ such that

- $\text{spec}(A') = \text{spec}(A)$,
- *A' has the SSP, and*
- *$\|A' - A\|$ can be chosen arbitrarily small.*

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} \sim 1 & \epsilon & 0 & 0 \\ \epsilon & \sim 2 & \epsilon & 0 \\ 0 & \epsilon & \sim 3 & \epsilon \\ 0 & 0 & \epsilon & \sim 4 \end{bmatrix}$$

SSP will be defined later

Matrix derivative

Definition

Let U and W be open subsets in vector spaces over $\mathbb{R}$ and $F : U \to W$ a function.

The derivative of F at a point $\mathrm{u}_0 \in U$ is

$$\dot{F} \cdot \mathrm{d} = \lim_{t \to 0} \frac{F(\mathrm{u}_0 + t\mathrm{d}) - F(\mathrm{u}_0)}{t},$$

which is a linear operator sending a direction to the directional derivative.

Example: $F(K) = e^K$

Define $F : \mathrm{Skew}_n(\mathbb{R}) \to \mathrm{Mat}_n(\mathbb{R})$ by $F(K) = e^K$.

Then $\dot{F}$ at O is $\dot{F} \cdot K = K$ since

$$\begin{aligned}\dot{F} \cdot K &= \lim_{t\to 0} \frac{e^{O+Kt} - e^{O}}{t} \\ &= \lim_{t\to 0} \frac{1}{t}\left[\frac{(Kt)^0}{0!} + \frac{(Kt)^1}{1!} + \frac{(Kt)^2}{2!} + \frac{(Kt)^3}{3!} + \cdots - I\right] \\ &= \lim_{t\to 0}\left[\frac{K^1}{1!} + \frac{K^2t^1}{2!} + \frac{K^3t^2}{3!} + \cdots\right] = K.\end{aligned}$$

Inverse function theorem

Theorem (Inverse function theorem)

Let $F : U \to W$ be a smooth function. If $\dot{F}$ at a point $\mathrm{u}_0 \in U$ is invertible, then F is locally invertible around u_0.

Theorem (FHLS 2021+)

Let $F : U \to W$ be a smooth function. If $\dot{F}$ at a point $\mathrm{u}_0 \in U$ is surjective, then F is locally surjective around u_0.

Sketch of the proof

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}}_{A} \longrightarrow \underbrace{\begin{bmatrix} \sim 1 & \epsilon & 0 & 0 \\ \epsilon & \sim 2 & \epsilon & 0 \\ 0 & \epsilon & \sim 3 & \epsilon \\ 0 & 0 & \epsilon & \sim 4 \end{bmatrix}}_{A' = M - B}$$

- $\mathcal{S}$: symmetric matrices that is nonzero only on the blue entries
- Define $F : \mathcal{S} \times \mathrm{Skew}_n(\mathbb{R}) \to \mathrm{Sym}_n(\mathbb{R})$ by $F(B, K) = e^{-K}Ae^{K} + B$.
- SSP $\iff$ $\dot{F}$ is surjective!
- For any M nearby A, there is B' and K' such that

$$e^{-K'}Ae^{K'} + B' = M.$$

- Choose proper M and let $A' = M - B'$.

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The derivative of $F(B, K) = e^{-K} A e^{K} + B$

At (O, O),

$$\dot{F} = K^{\top} A + AK + B$$

- $K \in \mathrm{Skew}_n(\mathbb{R})$
- $B \in \mathcal{S}^{\mathrm{cl}}(G)$, where $\mathcal{S}^{\mathrm{cl}}(G)$ is the topological closure of $\mathcal{S}(G)$. That is,

$$\mathcal{S}^{\mathrm{cl}}(G) = \{A = \left[a_{i,j}\right] \in \mathrm{Sym}_n(\mathbb{R}) : a_{i,j} = 0 \iff \{i, j\} \in E(\overline{G})\}.$$

$\dot{F}$ is surjective at (O, O) $\iff$
$\{K^{\top} A + AK : K \in \mathrm{Skew}_n(\mathbb{R})\} + \mathcal{S}^{\mathrm{cl}}(G) = \mathrm{Sym}_n(\mathbb{R})$.

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$$\dot{F} \text{ is surjective at } (O,O) \iff$$
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Strong spectral property (SSP)

Definition

A symmetric matrix A has the strong spectral property (SSP) if $X = O$ is the only real symmetric matrix that satisfies the following matrix equations:

- $A \circ X = O, I \circ X = O,$
- $AX - XA = O.$

Proposition (FHLS 2021+)

A symmetric matrix $A \in \mathcal{S}(G)$ has the SSP if and only if

$$\{K^\top A + AK : K \in \mathrm{Skew}_n(\mathbb{R})\} + \mathcal{S}^{\mathrm{cl}}(G) = \mathrm{Sym}_n(\mathbb{R}).$$

Extended SSP

Definition

Let G and H be two graphs such that $V(G) = V(H)$ and $E(G) \subseteq E(H)$. A matrix $A \in \mathcal{S}(G)$ has the SSP with respect to H if $X = O$ is the only real symmetric matrix that satisfies the following matrix equations:

- $X \in \mathcal{S}^{\mathrm{cl}}(\overline{H})$, $I \circ X = O$,
- $AX - XA = O$.

Proposition (FHLS 2021+)

A symmetric matrix $A \in \mathcal{S}(G)$ has the SSP with respect to H if and only if

$$\{K^\top A + AK : K \in \mathrm{Skew}_n(\mathbb{R})\} + \mathcal{S}^{\mathrm{cl}}(H) = \mathrm{Sym}_n(\mathbb{R}).$$

Extended supergraph lemma

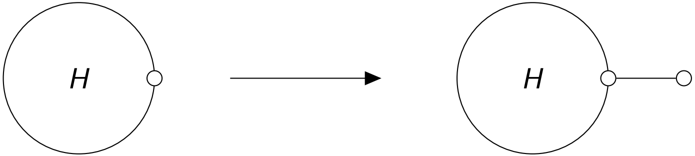
Lemma (L, Oblak, and Šmigoc 2021)

Let G, H, and H' be three graphs such that $V(G) = V(H) = V(H')$ and $E(G) \subseteq E(H) \subseteq E(H')$. If $A \in \mathcal{S}(G)$ has the SSP with respect to H, then there is a matrix $B \in \mathcal{S}^{\mathrm{cl}}(H')$ such that

- $\mathrm{spec}(A) = \mathrm{spec}(A')$,
- *A' has the SSP, and*
- *$\|A' - A\|$ can be chosen arbitrarily small.*

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} \sim 1 & \sim 0 & \epsilon & 0 \\ \sim 0 & \sim 1 & 0 & \epsilon \\ \epsilon & 0 & \sim 2 & \sim 0 \\ 0 & \epsilon & \sim 0 & \sim 2 \end{bmatrix}$$

Appending a leaf



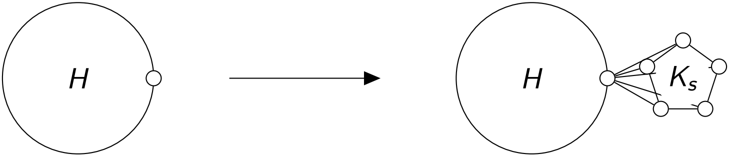
$A \in \mathcal{S}(H)$
SSP

$A' \in \mathcal{S}(H')$
SSP
$\text{spec}(A') = \text{spec}(A) \cup \{\lambda\}$

Theorem (BFHHLS 2017)

Let H be a graph and H' be obtained from H by appending a leaf. If $A \in \mathcal{S}(H)$ has the SSP and $\lambda \notin \text{spec}(A)$, then there is a matrix $A' \in \mathcal{S}(H')$ such that $\text{spec}(A') = \text{spec}(A) \cup \{\lambda\}$.

Appending a clique

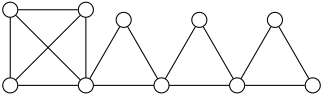


$A \in \mathcal{S}(H)$
SSP

$A' \in \mathcal{S}(H')$
SSP
$\mathrm{spec}(A') = \mathrm{spec}(A) \cup \{\lambda^{(s)}\}$

Theorem (L, Oblak, and Šmigoc 2021)

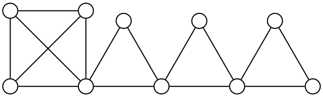
Let H be a graph and H' be obtained from H by appending a clique K_s. If $A \in \mathcal{S}(H)$ and $\lambda \notin \mathrm{spec}(A) \cup \mathrm{spec}(A(v))$ for all v, then there is a matrix $A' \in \mathcal{S}(H')$ such that $\mathrm{spec}(A') = \mathrm{spec}(A) \cup \{\lambda^{(s)}\}$.



allows ordered multiplicity list $(2, 2, 2, 2, 2)$

$$\begin{array}{c|ccc} A & & O & \\ \hline & \lambda & & 0 \\ O & & \ddots & \\ & 0 & & \lambda \end{array} \longrightarrow \begin{array}{cc|ccc} \sim A & & & O & \\ & & \epsilon & \cdots & \epsilon \\ \hline & \epsilon & \sim \lambda & & \sim 0 \\ O & \vdots & & \ddots & \\ & \epsilon & \sim 0 & & \sim \lambda \end{array}$$

Thanks!



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$$
\left[\begin{array}{c|ccc}
A & & O & \\
\hline
 & \lambda & & 0 \\
O & & \ddots & \\
 & 0 & & \lambda
\end{array}\right]
\longrightarrow
\left[\begin{array}{cc|ccc}
\sim A & & & O & \\
 & & \epsilon & \cdots & \epsilon \\
\hline
 & \epsilon & \sim \lambda & & \sim 0 \\
O & \vdots & & \ddots & \\
 & \epsilon & \sim 0 & & \sim \lambda
\end{array}\right]
$$

Thanks!

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