The sieving process and lower bounds of the minimum rank problem

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$$\begin{pmatrix} ? & * & 0 & 0 & 0 \\ * & ? & * & * & 0 \\ 0 & * & ? & * & 0 \\ 0 & * & * & ? & * \\ 0 & 0 & 0 & * & ? \end{pmatrix}$$

- For a real symmetric matrix of the pattern above, what is the smallest possible rank?

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

- For a real symmetric matrix of the pattern above, what is the smallest possible rank?
- 3 is possible.

$$\begin{pmatrix} ? & * & 0 & 0 & 0 \\ * & ? & * & * & 0 \\ 0 & * & ? & * & 0 \\ 0 & * & * & ? & * \\ 0 & 0 & 0 & * & ? \end{pmatrix}$$

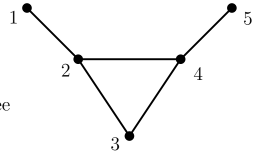
- For a real symmetric matrix of the pattern above, what is the smallest possible rank?
- 3 is possible.
- rank ≥ 3.

Minimum Rank (for simple graphs)

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} ? & * & 0 & 0 & 0 \\ * & ? & * & * & 0 \\ 0 & * & ? & * & 0 \\ 0 & * & * & ? & * \\ 0 & 0 & 0 & * & ? \end{pmatrix} \end{array}$$

$\longleftrightarrow$

no edge = zero
edge = nonzero
diagonal terms = free
(1)



$$\mathcal{S}(G) = \{A \in M_{n\times n}(\mathbb{R}) : A = A^t,\ A \text{ satisfies } (1)\}.$$

- The minimum rank of a simple graph G is

$$\text{mr}(G) = \min\{\text{rank}(A) : A \in \mathcal{S}(G)\}.$$

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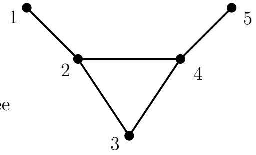
$$\begin{aligned} \text{no edge} &= \text{zero} \\ \text{edge} &= \text{nonzero} \\ \text{diagonal terms} &= \text{free} \end{aligned} \quad (1)$$



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- The minimum rank of a simple graph G is

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- The minimum rank problem of a graph G is to determine the value $\text{mr}(G)$.

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- Finding $\text{mr}(G) \cong$ Finding $M(G)$.
- Finding lower bounds of $\text{mr}(G) \cong$ Finding upper bounds of $M(G)$.

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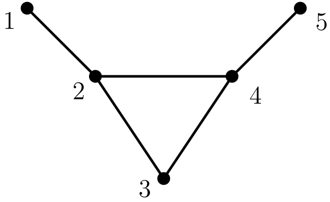
Theorem (AIM, 08)

For all graph G, $M(G) \leq Z(G)$.

Example for $M(G)$ and $Z(G)$

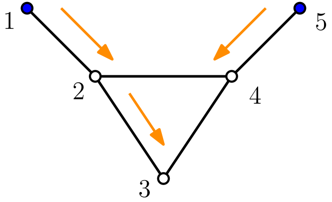
$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$



- $\mathrm{mr}(G) = 3$, and $M(G) = 5 - 3 = 2$.

Example for $M(G)$ and $Z(G)$

	1	2	3	4	5
1	1	1	0	0	0
2	1	2	1	1	0
3	0	1	1	1	0
4	0	1	1	2	1
5	0	0	0	1	1



- $\mathrm{mr}(G) = 3$, and $M(G) = 5 - 3 = 2$.
- $B = \{1, 5\}$ is a zero forcing set.

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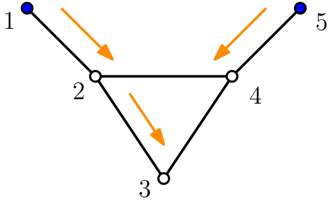

- $\text{mr}(G) = 3$, and $M(G) = 5 - 3 = 2$.
- $B = \{1, 5\}$ is a zero forcing set.
- $2 = M(G) \leq Z(G) = 2$.

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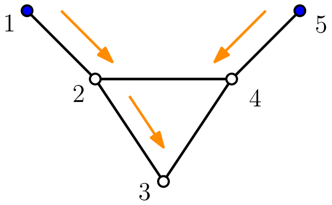
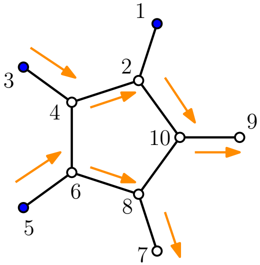


- $\mathrm{mr}(G) = 3$, and $M(G) = 5 - 3 = 2$.
- $B = \{1, 5\}$ is a zero forcing set.
- $2 = M(G) \leq Z(G) = 2$.
- $M(G) = Z(G)$ when G is a tree [AIM, 08], or $|V(G)| \leq 7$ [DeLoss et. al. 10].

Sketch of the proof of $M(G) \leq Z(G)$

- Find a minimum zero forcing set.

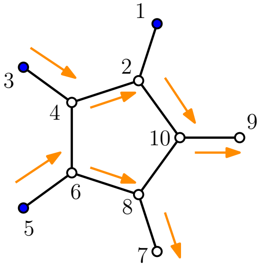
	1	2	3	4	5	6	7	8	9	10
1	?	*	0	0	0	0	0	0	0	0
2	*	?	0	*	0	0	0	0	0	*
3	0	0	?	*	0	0	0	0	0	0
4	0	*	*	?	0	*	0	0	0	0
5	0	0	0	0	?	*	0	0	0	0
6	0	0	0	*	*	?	0	*	0	0
7	0	0	0	0	0	0	?	*	0	0
8	0	0	0	0	0	*	*	?	0	*
9	0	0	0	0	0	0	0	0	?	*
10	0	*	0	0	0	0	0	*	*	?



Sketch of the proof of $M(G) \leq Z(G)$

- Find a minimum zero forcing set.
- Write down all forces $x_i \to y_i$ in order.

	1	2	3	4	5	6	7	8	9	10
1	?	*	0	0	0	0	0	0	0	0
2	*	?	0	*	0	0	0	0	0	*
3	0	0	?	*	0	0	0	0	0	0
4	0	*	*	?	0	*	0	0	0	0
5	0	0	0	0	?	*	0	0	0	0
6	0	0	0	*	*	?	0	*	0	0
7	0	0	0	0	0	0	?	*	0	0
8	0	0	0	0	0	*	*	?	0	*
9	0	0	0	0	0	0	0	0	?	*
10	0	*	0	0	0	0	0	*	*	?

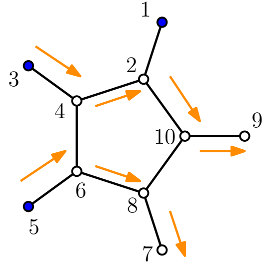


$3 \to 4$
$5 \to 6$
$4 \to 2$
$2 \to 10$
$6 \to 8$
$10 \to 9$
$8 \to 7$

Sketch of the proof of $M(G) \leq Z(G)$

- Find a minimum zero forcing set.
- Write down all forces $x_i \to y_i$ in order.
- Make x_i be the i-th column; y_i be the i-th row.

	3	5	4	2	6	10	8	1	7	9
4	*	0	?	*	*	0	0	0	0	0
6	0	*	*	0	?	0	*	0	0	0
2	0	0	*	?	0	*	0	*	0	0
10	0	0	0	*	0	?	*	0	0	*
8	0	0	0	0	*	*	?	0	*	0
9	0	0	0	0	0	*	0	0	0	?
7	0	0	0	0	0	0	*	0	?	0
1	0	0	0	*	0	0	0	?	0	0
3	?	0	*	0	0	0	0	0	0	0
5	0	?	0	0	*	0	0	0	0	0



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Sketch of the proof of $M(G) \leq Z(G)$

- Number of forces ≅ size of a triangle.

	3	5	4	2	6	10	8	1	7	9
4	$*$	0	?	$*$	$*$	0	0	0	0	0
6	0	$*$	$*$	0	?	0	$*$	0	0	0
2	0	0	$*$	?	0	$*$	0	$*$	0	0
10	0	0	0	$*$	0	?	$*$	0	0	$*$
8	0	0	0	0	$*$	$*$	?	0	$*$	0
9	0	0	0	0	0	$*$	0	0	0	?
7	0	0	0	0	0	0	$*$	0	?	0
1	0	0	0	$*$	0	0	0	?	0	0
3	?	0	$*$	0	0	0	0	0	0	0
5	0	?	0	0	$*$	0	0	0	0	0

Sketch of the proof of $M(G) \leq Z(G)$

- Number of forces $\cong$ size of a triangle.
- Finding minimum zero forcing set $\cong$ Finding largest triangle.

	3	5	4	2	6	10	8	1	7	9
4	$*$	0	?	$*$	$*$	0	0	0	0	0
6	0	$*$	$*$	0	?	0	$*$	0	0	0
2	0	0	$*$	?	0	$*$	0	$*$	0	0
10	0	0	0	$*$	0	?	$*$	0	0	$*$
8	0	0	0	0	$*$	$*$	?	0	$*$	0
9	0	0	0	0	0	$*$	0	0	0	?
7	0	0	0	0	0	0	$*$	0	?	0
1	0	0	0	$*$	0	0	0	?	0	0
3	?	0	$*$	0	0	0	0	0	0	0
5	0	?	0	0	$*$	0	0	0	0	0

$M(G) \neq Z(G)$

- The previous graph H_5 is called a 5-sun. It is an example of $M(G) \lneqq Z(G)$.

$$2 = M(H_5) \lneqq Z(H_5) = 3$$

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- 8 is the minimum rank; but maximum size of triangle is 7!
- How do we know?
- Proven by examining the number of zeros on the diagonal [Barioli, Fallat, and Hogben, 04].

Control diagonal pattern by loops

- For a simple graph G, pick $I \subseteq V(G)$ to add one loop, and call it $\widehat{G}_I$.

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- $Z(\widehat{G}_I)$: If x ~~is blue and~~ has exactly one white neighbor y, change the color of y to blue at next step.
- Triangle argument still can prove $M(\widehat{G}_I) \leq Z(\widehat{G}_I)$

Interpretation on matrices

Pattern for H_5

$$
\begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8\\9\\10 \end{array}
\begin{array}{c}
\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\
\begin{pmatrix}
0/* & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & ? & 0 & * & 0 & 0 & 0 & 0 & 0 & * \\
0 & 0 & ? & * & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & * & * & ? & 0 & * & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & ? & * & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & * & * & ? & 0 & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & ? & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & * & * & ? & 0 & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ? & * \\
0 & * & 0 & 0 & 0 & 0 & 0 & * & * & ?
\end{pmatrix}
\end{array}
$$

Pattern for H_5

	1	2	3	4	5	6	7	8	9	10
1	0	$*$	0	0	0	0	0	0	0	0
2	$*$	?	0	$*$	0	0	0	0	0	$*$
3	0	0	?	$*$	0	0	0	0	0	0
4	0	$*$	$*$	?	0	$*$	0	0	0	0
5	0	0	0	0	?	$*$	0	0	0	0
6	0	0	0	$*$	$*$	?	0	$*$	0	0
7	0	0	0	0	0	0	?	$*$	0	0
8	0	0	0	0	0	$*$	$*$	?	0	$*$
9	0	0	0	0	0	0	0	0	?	$*$
10	0	$*$	0	0	0	0	0	$*$	$*$	?

Interpretation on matrices

Pattern for H_5

	1	3	9	2	4	10	6	8	5	7
2	*	0	0	?	*	*	0	0	0	0
4	0	*	0	*	?	0	*	0	0	0
10	0	0	*	*	0	?	0	*	0	0
1	0	0	0	*	0	0	0	0	0	0
6	0	0	0	0	*	0	?	*	*	0
8	0	0	0	0	0	*	*	?	0	*
5	0	0	0	0	0	0	*	0	?	0
7	0	0	0	0	0	0	0	*	0	?
3	0	?	0	0	*	0	0	0	0	0
9	0	0	?	0	0	*	0	0	0	0

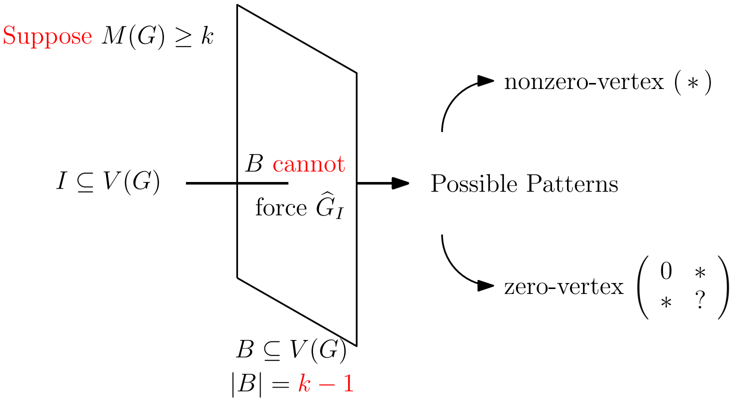
Pattern for H_5

	1	2	3	4	5	6	7	8	9	10
1	$*$	$*$	0	0	0	0	0	0	0	0
2	$*$	?	0	$*$	0	0	0	0	0	$*$
3	0	0	?	$*$	0	0	0	0	0	0
4	0	$*$	$*$	?	0	$*$	0	0	0	0
5	0	0	0	0	?	$*$	0	0	0	0
6	0	0	0	$*$	$*$	?	0	$*$	0	0
7	0	0	0	0	0	0	?	$*$	0	0
8	0	0	0	0	0	$*$	$*$	?	0	$*$
9	0	0	0	0	0	0	0	0	?	$*$
10	0	$*$	0	0	0	0	0	$*$	$*$	?

Pattern for H_5

	3	5	4	6	1	2	8	10	7	9
4	*	0	?	*	0	*	0	0	0	0
6	0	*	*	?	0	0	*	0	0	0
2	0	0	*	0	*	?	0	*	0	0
8	0	0	0	*	0	0	?	*	*	0
1	0	0	0	0	*	*	0	0	0	0
10	0	0	0	0	0	*	*	?	0	*
7	0	0	0	0	0	0	*	0	?	0
9	0	0	0	0	0	0	0	*	0	?
3	?	0	*	0	0	0	0	0	0	0
5	0	?	0	*	0	0	0	0	0	0

Sieving Process



- Many known graphs G with $M(G) \lneqq Z(G)$ can be explained by this process.

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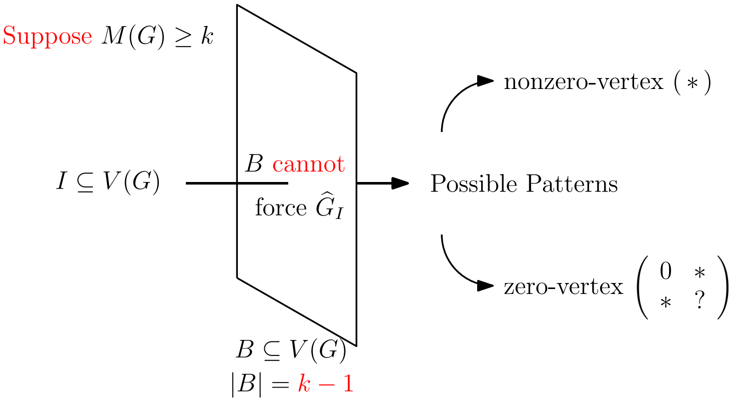

- Many known graphs G with $M(G) \lneqq Z(G)$ can be explained by this process.
- Thanks for your attention!

- AIM Minimum Rank – Special Graphs Work Group (F. Barioli, W. Barrett, S. Butler, S. M. Cioaba, D. Cvetković, S. M. Fallat, C. Godsil, W. Haemers, L. Hogben, R. Mikkelson, S. Narayan, O. Pryporova, I. Sciriha, W. So, D. Stevanović, H. van der Holst, K. Vander Meulen, and A. Wangsness). Zero forcing sets and the minimum rank of graphs. Lin. Alg. Appl., 428: 1628–1648, 2008.
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Looped Multigraph

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- In 2008, Holst gave a reduction formula on cut sets of size two, by considering multigraphs.

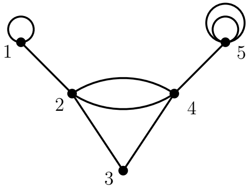
Looped Multigraph

- In 2013, Barioli et. al. proposed enhanced zero forcing number $\widehat{Z}(G)$,
$$M(G) \leq \widehat{Z}(G) \leq Z(G),$$
by considering looped graphs.
- In 2008, Holst gave a reduction formula on cut sets of size two, by considering multigraphs.
- Considering looped multigraphs $\widehat{G}$ is a natural extension.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} * & * & 0 & 0 & 0 \\ * & 0 & * & ? & 0 \\ 0 & * & 0 & * & 0 \\ 0 & ? & * & 0 & * \\ 0 & 0 & 0 & * & ? \end{pmatrix} \end{array}$$

$\longleftrightarrow$

no edge = zero
edge = nonzero
≥ 2 edges = free
(2)



$$\mathcal{S}(\widehat{G}) = \{A \in M_{n\times n}(\mathbb{R}) : A = A^t, \ A \text{ satisfies } (2)\}.$$

- The minimum rank of a looped multigraph $\widehat{G}$ is

$$\text{mr}(\widehat{G}) = \min\{\text{rank}(A) : A \in \mathcal{S}(\widehat{G})\}.$$

Minimum Rank for Looped Multigraphs

	1	2	3	4	5
1	*	*	0	0	0
2	*	0	*	?	0
3	0	*	0	*	0
4	0	?	*	0	*
5	0	0	0	*	?

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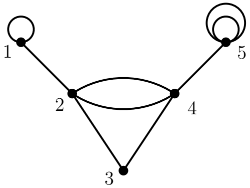

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$$\mathrm{mr}(\widehat{G}) = \min\{\mathrm{rank}(A) : A \in \mathcal{S}(\widehat{G})\}.$$
- Similarly,
$$M(\widehat{G}) = \max\{\mathrm{null}(A) : A \in \mathcal{S}(\widehat{G})\}.$$
$$\mathrm{mr}(\widehat{G}) + M(\widehat{G}) = |V(\widehat{G})|$$

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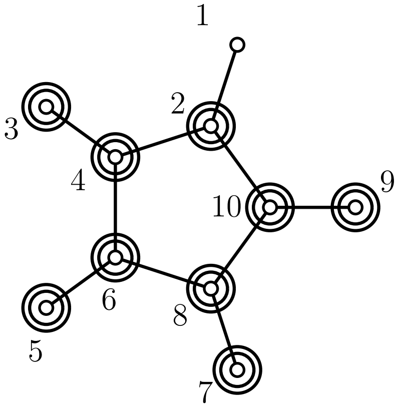
- The zero forcing number $Z(\widehat{G})$ of a looped multigraph $\widehat{G}$ is the minimum cardinality of a zero forcing set.
- For all looped multigraph $\widehat{G}$, $M(\widehat{G}) \leq Z(\widehat{G})$.

What happens on H_5? (if zero)

- Suppose $A = (a_{ij}) \in \mathcal{S}(H_5)$ with $a_{11} = 0$, then $A \in \mathcal{S}(\widehat{G})$, $\widehat{G}$ is the looped multigraph below.



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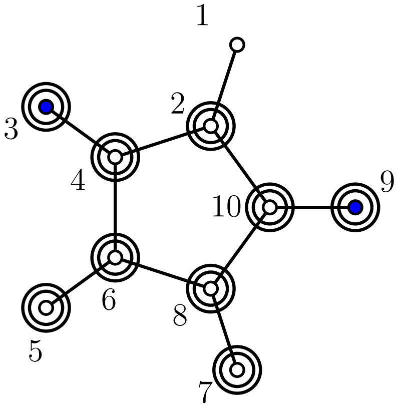

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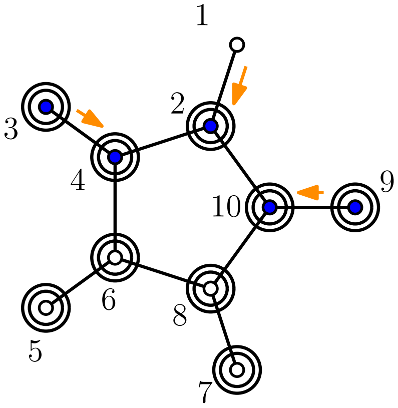

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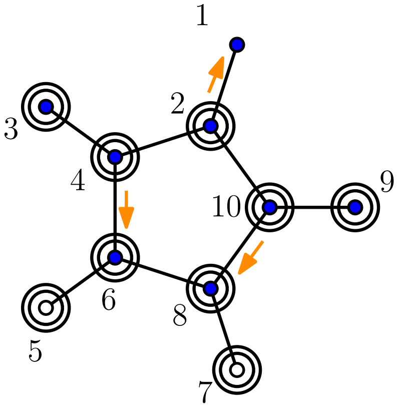

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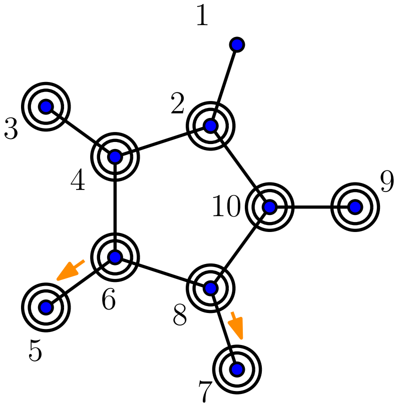

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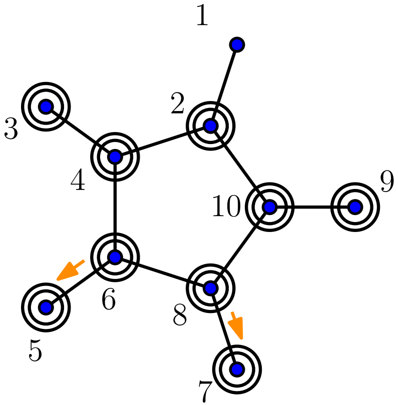
- That is, if $M(G) = 3$, then the corresponding a_{11} is nonzero!



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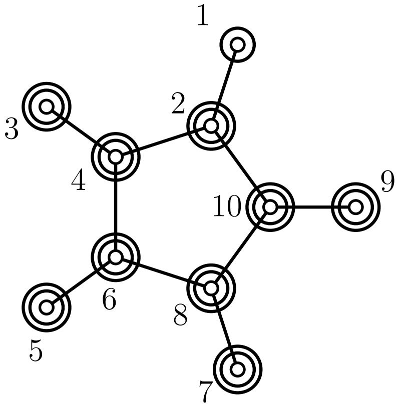

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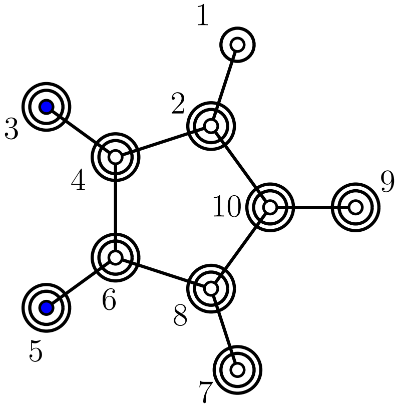

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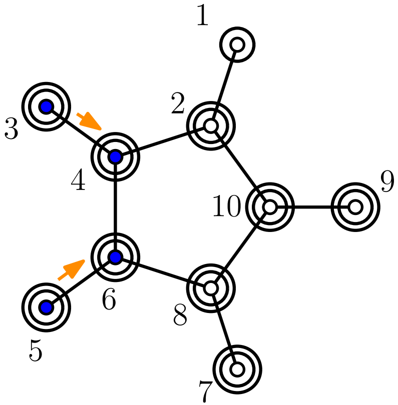

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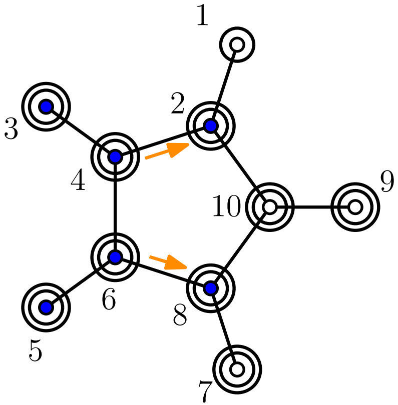

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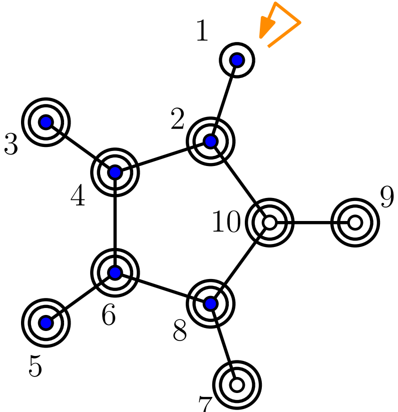

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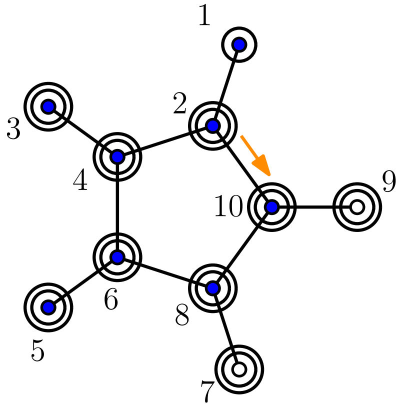

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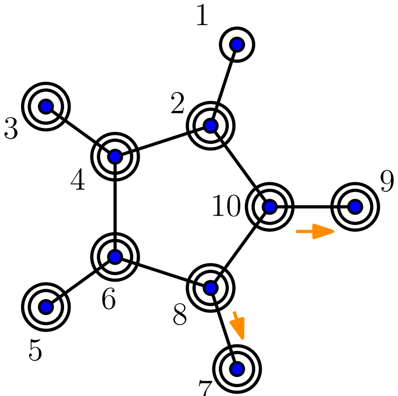

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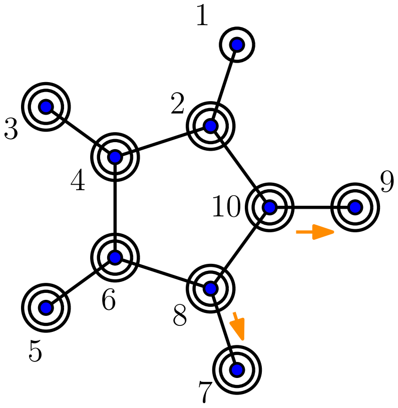

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- Thanks for your attention!