Zero forcing and its applications

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System of linear equations

$$\begin{array}{rrrl} 2x & +3y & -z & = 4 \\ x & -y & +2z & = 3 \\ -3x & +2y & +z & = 2 \end{array}$$

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Easy to see $y = 2$, then $z = 1$, and then $x = 0$.
Easy to know the solution exists and is unique.

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Main philosophy

$$2x + y + 3z = 7$$

$$x = 1, y = 2 \implies z = 1$$

In a linear equation, if all but one variable are known, then this remaining variable is also known.

$$2x + y + 3z = 0$$

$$x = 0, y = 0 \implies z = 0$$

In a homogeneous linear equation, if all but one variable are zero, then this remaining variable is also zero.

Hidden triangle in a system

$$\begin{array}{lllllll} 1. & x & & +z & & +u & = 0 \\ 2. & & y & +z & & & = 0 \\ 3. & x & +y & +z & +w & +u & = 0 \\ 4. & & & z & +w & & = 0 \\ 5. & x & & +z & & +u & = 0 \end{array}$$

Given information: $x = y = 0$. Then

$$\begin{aligned} 2. &\implies z = 0, \\ 4. &\implies w = 0, \\ 3. &\implies u = 0. \end{aligned}$$

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As long as the red terms has nonzero coefficients and the orange terms are zero, the same argument always works.

Application to algebra

Find the inverse of a formal power series.

$$\begin{array}{rcccccc} & 1 & +2x & +3x^2 & +4x^3 & +5x^4 & +\cdots \\ \times) & b_0 & +b_1x & +b_2x^2 & +b_3x^3 & +b_4x^4 & +\cdots \\ \hline & 1 & & & & & \end{array}$$

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\hline
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1b_0 & & & & = 1 \\
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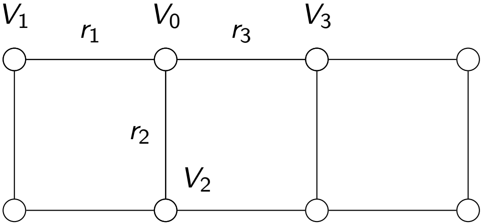
A formal power series has an inverse if and only if the constant term is nonzero.

Electronic circuit

$$\frac{1}{r_1}(V_0 - V_1) + \frac{1}{r_2}(V_0 - V_2) = \frac{1}{r_3}(V_3 - V_0) + \epsilon V_0$$



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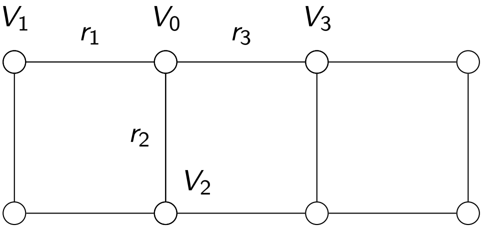

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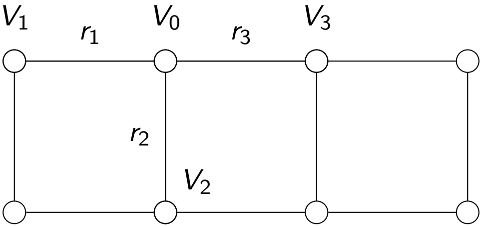
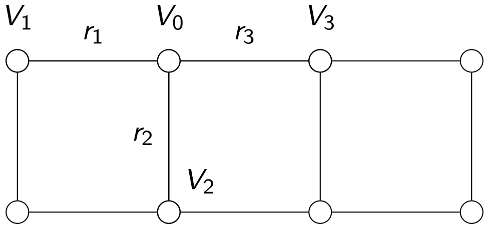
$$a_1 V_1 + \underbrace{a_2}_{\text{nonzero}} V_2 + a_3 V_3 + \underbrace{a_0}_{\text{zero or nonzero}} V_0 = 0$$



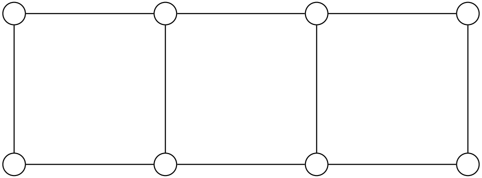
Electronic circuit

$$\underbrace{a_1 V_1 + a_2 V_2 + a_3 V_3}_{\text{nonzero}} + \underbrace{a_0 V_0}_{\text{zero or nonzero}} = 0$$

The conservation law leads to a linear equation on each node; itself and its neighbours represent the variables.



How many sensors required to monitor the voltages?



- Each vertex represents a linear equation; variables are itself and its neighbors (closed neighbourhood).
- If in a closed neighbourhood, all but one voltages are known, then this remaining one are also known.

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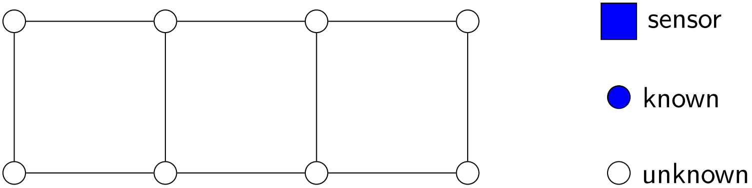

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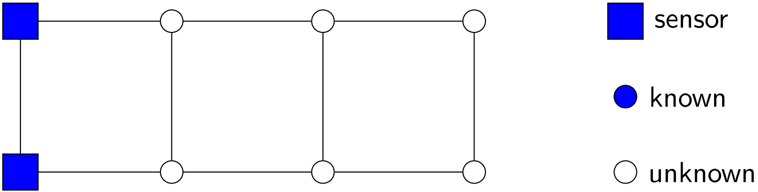

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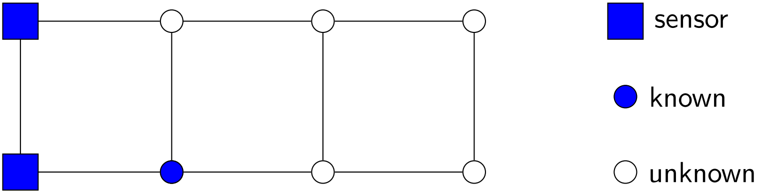

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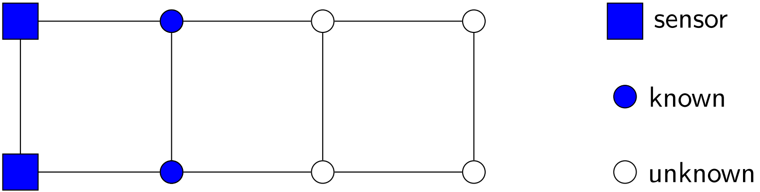

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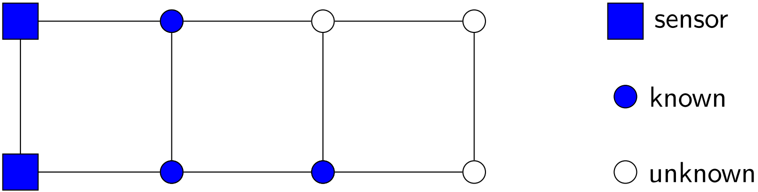

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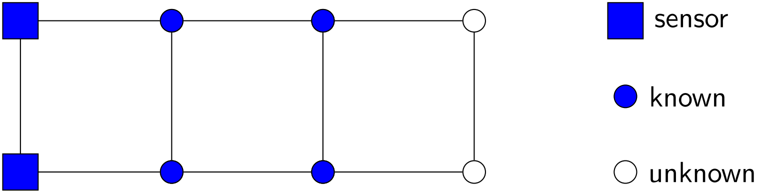

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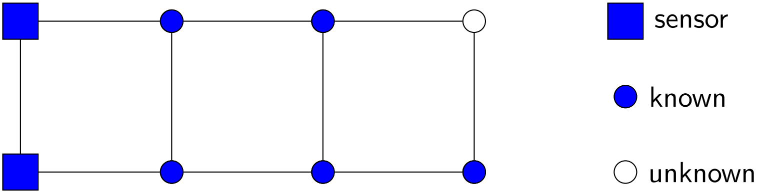

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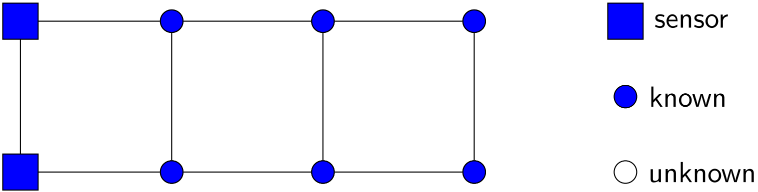

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Model by graphs and matrices

- A electronic circuit can be represented by a graph; each vertex represents a node, and each edge represents a connection.
- The linear equations can be recorded into a matrix; each row represents a equation, and each column represents an unknown voltage.
- This is a symmetric matrix where rows and columns are both indexed by the vertices.

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Let G be a simple graph on n vertices. The family $\mathcal{S}(G)$ consists of all $n \times n$ real symmetric matrix $M = \left[M_{i,j}\right]$ with

$$\begin{cases} M_{i,j} = 0 & \text{if } i \neq j \text{ and } \{i,j\} \text{ is not an edge,} \\ M_{i,j} \neq 0 & \text{if } i \neq j \text{ and } \{i,j\} \text{ is an edge,} \\ M_{i,j} \in \mathbb{R} & \text{if } i = j. \end{cases}$$

$$\mathcal{S}(\circ\!-\!\circ\!-\!\circ) \ni \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0.1 & 0 \\ 0.1 & 1 & \pi \\ 0 & \pi & 0 \end{bmatrix}, \cdots$$

Zero forcing

Zero forcing process:

- Start with a given set of blue vertices (sensors).
- If for some x, the closed neighbourhood $N_G[x]$ are all blue except for one vertex y and $y \neq x$, then y turns blue.

An initial blue set that can make the whole graph blue is called a zero forcing set. The zero forcing number $Z(G)$ of a graph G is the minimum size of a zero forcing set.

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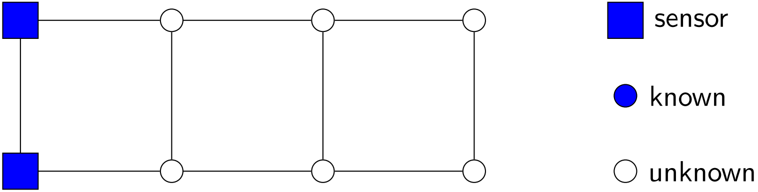

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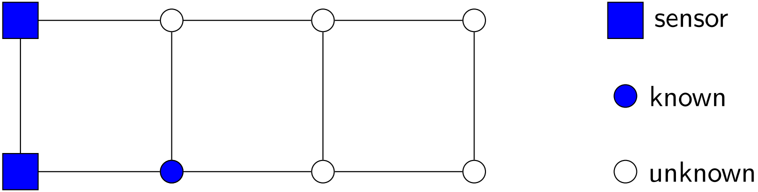

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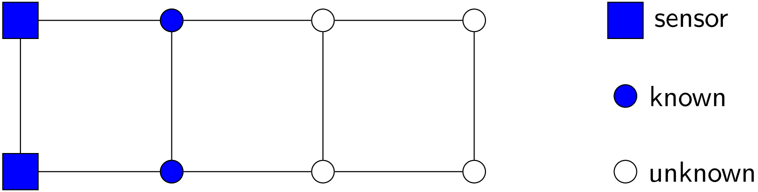

How to deploy the sensors?

Any zero forcing set is a good deployment of sensors that can monitor the whole graph.

The zero forcing number is the minimum number of sensors required.

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Proposition (Kenter and L 2018)

Let G be a graph on the vertex set V. The following are equivalent:

1. *B is a zero forcing set.*
2. *For any $A \in \mathcal{S}(G)$, the columns corresponding to $V \setminus B$ hides a lower triangular matrix.*
3. *For any $A \in \mathcal{S}(G)$, the columns corresponding to $V \setminus B$ are linearly independent.*

Theorem (AIM Work Group 2008)

Let G be a graph on n vertices. For any matrix $A \in \mathcal{S}(G)$, $n - Z(G) \leq \operatorname{rank}(A)$.

More zero forcing

- Same argument works for non-symmetric matrices.
- When more information are known on the matrices, the design of the zero forcing process can be improved.
- For example, nonnegative matrices, zero diagonal entries, or nonzero diagonal entries.
- They all follow the same philosophy.
- Zero forcing is related to the minimum rank problem (Math), quantum control (Physics), building logic circuit (Physics), the graph searching problem (ComS).

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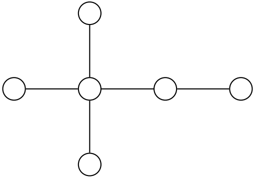
Domination number and zero forcing

Zero forcing with nonzero diagonal

Color change rule:

> *If for some x, the closed neighbourhood $N_G[x]$ are all blue except for one vertex y, then y turns blue.*

$Z_{\dot{\ell}}(G)$ =minimum size of a zero forcing set.



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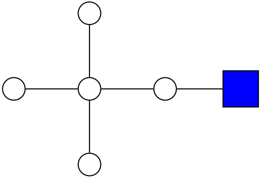

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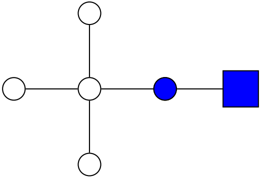

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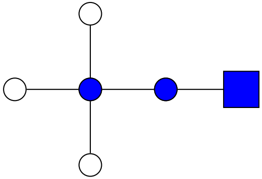

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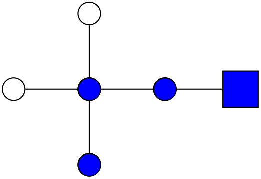

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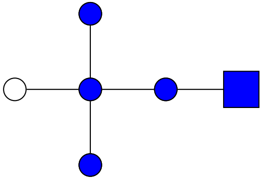

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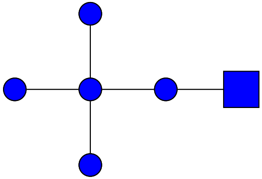

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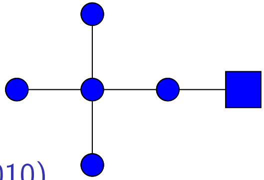

Theorem (Hogben 2010)

Let G be a graph on n vertices. Then $n - Z_{\dot{\ell}}(G) \leq \operatorname{rank}(A)$ for any $A \in \mathcal{S}(G)$ with nonzero diagonal entries.

Zero forcing with zero diagonal

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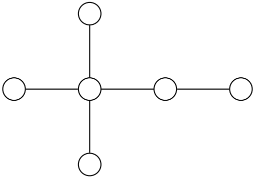
> *If for some x, the open neighbourhood $N_G(x)$ are all blue except for one vertex y, then y turns blue.*

$Z_-(G)$ =minimum size of a zero forcing set.



Zero forcing with zero diagonal

Color change rule:

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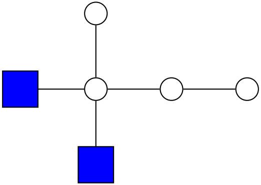

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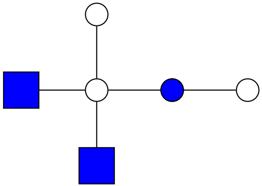

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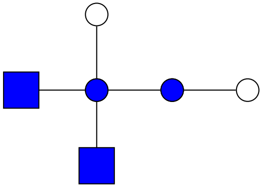

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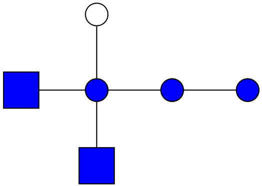

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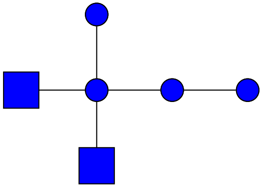

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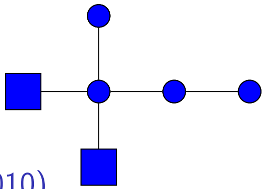


Theorem (Hogben 2010)

Let G be a graph on n vertices. Then $n - Z_-(G) \leq \text{rank}(A)$ for any $A \in \mathcal{S}(G)$ with zero diagonal entries.

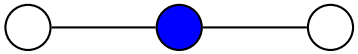
Domination number

Let G be a graph. The domination number $\gamma(G)$ is the minimum cardinality of a set X such that

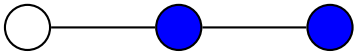
$$\bigcup_{x\in X} N_G[x] = V(G).$$

The total domination number $\gamma^t(G)$ is the minimum cardinality of a set X such that

$$\bigcup_{x\in X} N_G(x) = V(G).$$



$\gamma(P_3) = 1$



$\gamma^t(P_3) = 2$

Greedy algorithm

- Greedy algorithm follows the problem solving heuristic of making the locally optimal choice at each stage with the hope of finding a global optimum.
- Greedy algorithm for domination number: When X are chosen and not yet dominate the whole graph, pick a vertex v such that

$$N_G[v] \setminus \bigcup_{x \in X} N_G[x] \neq \emptyset.$$

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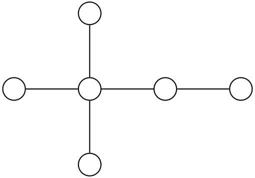
Grundy domination number

The Grundy domination number $\gamma_{\mathrm{gr}}(G)$ is the length of the longest sequence $(v_1, v_2, \ldots, v_k)$ such that

$$N_G[v_i] \setminus \bigcup_{j=1}^{i-1} N_G[v_j] \neq \emptyset.$$



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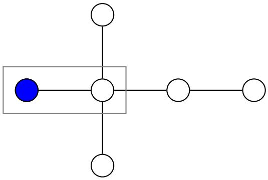

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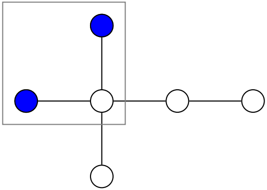

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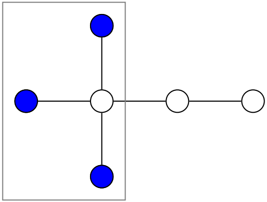

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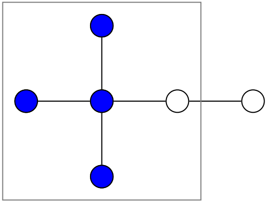

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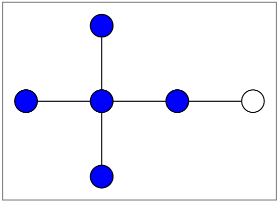

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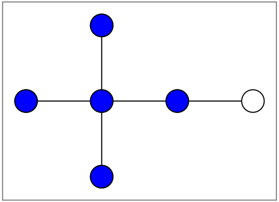

So $\gamma_{\mathrm{gr}}(G) = 5$.

Grundy total domination number

The Grundy total domination number $\gamma_{\mathrm{gr}}^t(G)$ is the length of the longest sequence $(v_1, v_2, \ldots, v_k)$ such that

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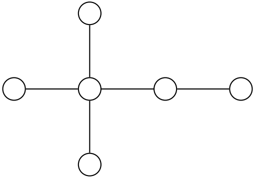

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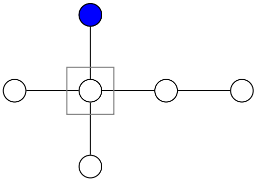

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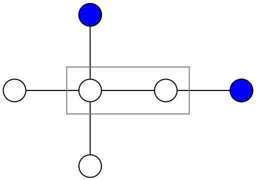

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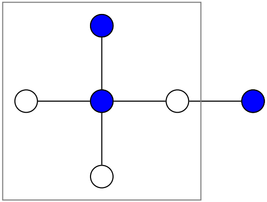

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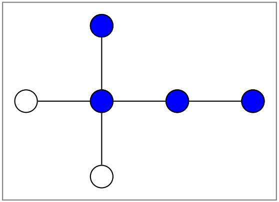

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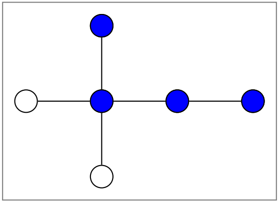


So $\gamma_{\mathrm{gr}}^{t}(G) = 4$.

Grundy domination number, zero forcing number, and the rank bound

Theorem (L 2017)

Let G be a graph on n vertices. Then

$$\gamma_{\mathrm{gr}}(G) = n - Z_{\dot{\ell}}(G) \text{ and } \gamma_{\mathrm{gr}}^{t}(G) = n - Z_{-}(G).$$

Therefore,

$$\gamma_{\mathrm{gr}}(G) \leq \mathrm{rank}(A)$$

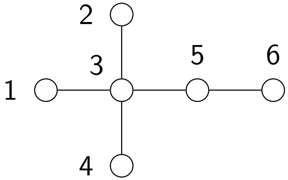
for any $A \in \mathcal{S}(G)$ with diagonal entries all nonzero; and

$$\gamma_{\mathrm{gr}}^{t}(G) \leq \mathrm{rank}(A)$$

for any $A \in \mathcal{S}(G)$ with zero diagonal.

Proof of the theorem

Key: Reverse the forcing process!



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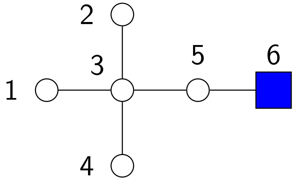

$6 \to 5$
$5 \to 3$
$4 \to 4$
$2 \to 2$
$3 \to 1$

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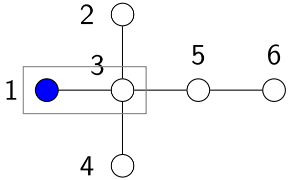

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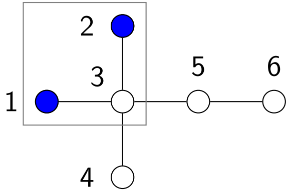

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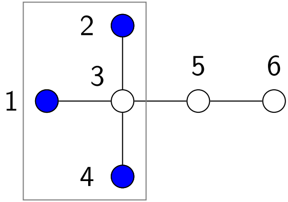

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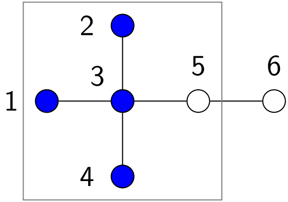

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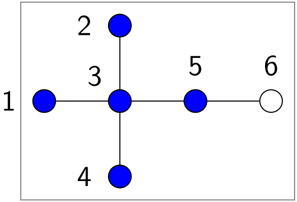

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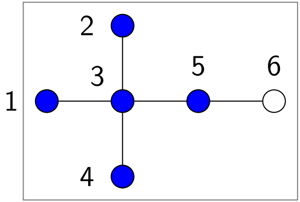

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Thank you!

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