General spectral graph theory: The inverse eigenvalue problem of a graph

Jephian C.-H. Lin

Department of Mathematics and Statistics, University of Victoria

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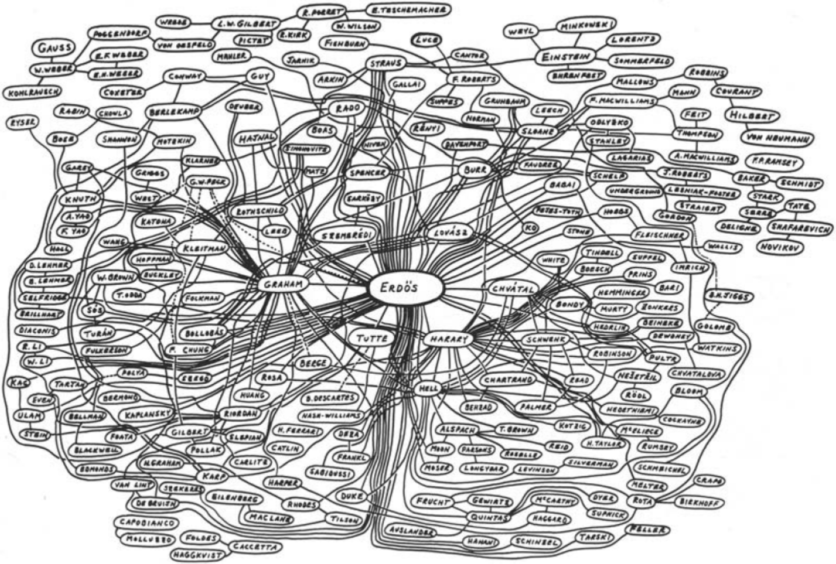
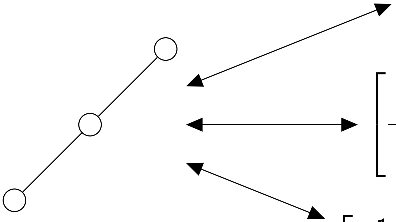


Sources: Easley and Kleinberg (2010)

Spectral graph theory



$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 1 & -\frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & 1 \end{bmatrix}$$

Dragoš Cvetković

Serbian Academy
of Sciences and Arts

Sources: Cvetković's website

Cvetković's inertia bound

The inertia of a matrix A is $(n_+(A), n_-(A), n_0(A))$, which are the number of positive, negative, and zero eigenvalues of A, respectively.

Theorem (Cvetković 1971)

Let G be a graph and A its adjacency matrix. Then

$$\alpha(G) \leq \min\{n - n_+(A), n - n_-(A)\},$$

where $\alpha(G)$ is the independence number.

Chris Godsil

University of Waterloo

Sources: FreeTechBooks

Godsil's Lemma

Let G be a graph. The path cover number $P(G)$ is the minimum number of disjoint induced paths that can cover G.

Theorem (Godsil 1984)

Let G be a tree with adjacency matrix A. Then

$$m_\lambda(A) \leq P(G)$$

for any eigenvalue λ of A.

General spectral graph theory

Given a grpah G on n vertices, consider the family $\mathcal{S}(G)$ of $n \times n$ real symmetric matrices M with

$$\begin{cases} M_{i,j} = 0 & \text{if } i \neq j \text{ and } \{i,j\} \text{ is not an edge,} \\ M_{i,j} \neq 0 & \text{if } i \neq j \text{ and } \{i,j\} \text{ is an edge,} \\ M_{i,j} \in \mathbb{R} & \text{if } i = j. \end{cases}$$

Thus, $\mathcal{S}(G)$ includes the adjacency matrix, the Laplacian matrix, and so on.

The general version of Cvetković's inertia bound

Theorem

Let G be a graph and $A \in \mathcal{S}(G)$ with zero diagonal entries. Then

$$\alpha(G) \leq \min\{n - n_+(A), n - n_-(A)\},$$

where $\alpha(G)$ is the independence number.

- Sinkovic (2017) proved Paley 17 is an example where the inertia bound is not tight. (So far, all known constructions are related to Paley 17.)
- He is going to talk about it at the Joint Meeting 2018 in San Diego!

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The general version of Godsil's lemma

Theorem (Johnson and Leal Duarte 1999)
Let G be a tree and $A \in \mathcal{S}(G)$. Then

$$m_\lambda(A) \leq P(G)$$

for any eigenvalue λ of A.

- Indeed, for any tree, there is a matrix A with an eigenvalue λ such that $m_\lambda(A) = P(G)$.

The general version of Godsil's lemma

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Domination number

Let G be a graph. The domination number $\gamma(G)$ is the minimum cardinality of a set X such that

$$\bigcup_{x \in X} N_G[x] = V(G).$$

The total domination number $\gamma^t(G)$ is the minimum cardinality of a set X such that

$$\bigcup_{x \in X} N_G(x) = V(G).$$

For example, $\gamma(P_3) = 1$ but $\gamma^t(P_3) = 2$.

Greedy algorithm

- Greedy algorithm follows the problem solving heuristic of making the locally optimal choice at each stage with the hope of finding a global optimum.
- For solving a maze, you may keep going straight at fork. But it might lead you to a dead end.
- For a coloring problem, you may keep using the smallest free number to color the next vertex, showing $\chi(G) \leq \Delta(G) + 1$.
- Greedy algorithm for domination number: When X are chosen and not yet dominate the whole graph, pick a vertex v such that

$$N_G[v] \setminus \bigcup_{x \in X} N_G[x] \neq \emptyset.$$

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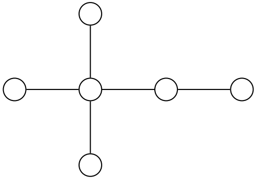
Grundy domination number

The Grundy domination number $\gamma_{\mathrm{gr}}(G)$ is the length of the longest sequence $(v_1, v_2, \ldots, v_k)$ such that

$$N_G[v_i] \setminus \bigcup_{j=1}^{i-1} N_G[v_j] \neq \emptyset.$$



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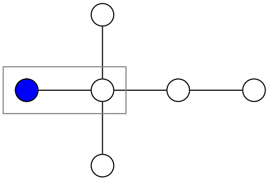

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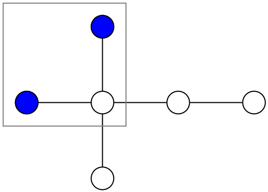

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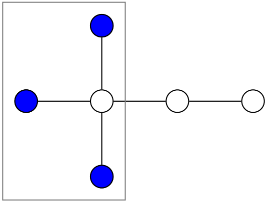

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The Grundy domination number $\gamma_{\rm gr}(G)$ is the length of the longest sequence $(v_1, v_2, \ldots, v_k)$ such that

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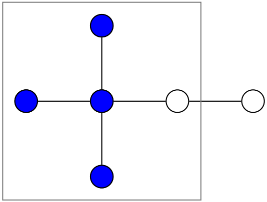

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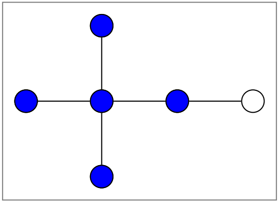

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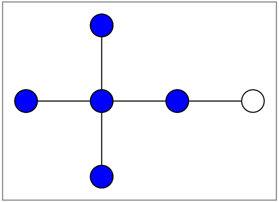

So $\gamma_{\mathrm{gr}}(G) = 5$.

Grundy total domination number

The Grundy total domination number $\gamma_{\mathrm{gr}}^{t}(G)$ is the length of the longest sequence $(v_1, v_2, \ldots, v_k)$ such that

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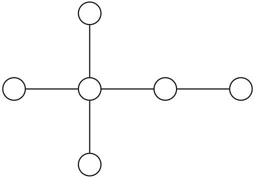

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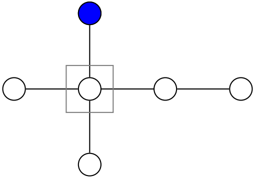

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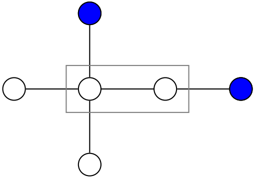

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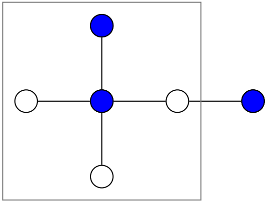

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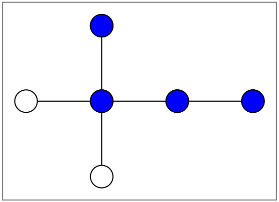

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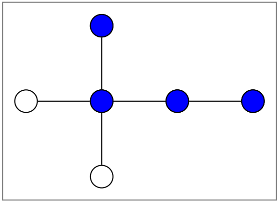


So $\gamma_{\mathrm{gr}}^t(G) = 4$.

Rank bound

Theorem (L 2017)

Let G be a graph. Then

$$\gamma_{\mathrm{gr}}(G) \leq \mathrm{rank}(A)$$

for any $A \in \mathcal{S}(G)$ with diagonal entries all nonzero; and

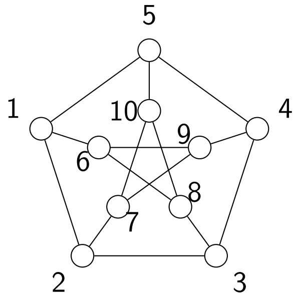
$$\gamma_{\mathrm{gr}}^{t}(G) \leq \mathrm{rank}(A)$$

for any $A \in \mathcal{S}(G)$ with zero diagonal.

Let P be the Petersen graph. Consider

$$A = \begin{bmatrix} C - I & I_5 \\ I_5 & C' - I \end{bmatrix} \text{ and } B = \begin{bmatrix} C & I_5 \\ I_5 & -C' \end{bmatrix},$$

where C and C' are the adjacency matrix of C_5 and $\overline{C_5}$, respectively. Then $\gamma_{\mathrm{gr}}(P) \leq \mathrm{rank}(A) = 5$ and the sequence $(1, 2, 3, 4, 5)$ is optimal.



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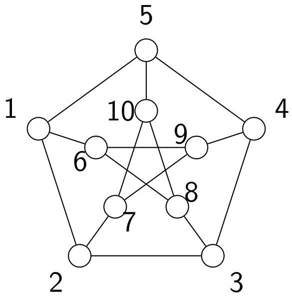
where C and C' are the adjacency matrix of C_5 and $\overline{C_5}$, respectively. Then $\gamma_{\mathrm{gr}}^t(G) \leq \mathrm{rank}(B) = 6$ and the sequence $(9, 1, 2, 3, 4, 5)$ is optimal.



Proof of the theorem

- Goal: Show $\gamma_{\mathrm{gr}}(G) \leq \mathrm{rank}(A)$ for all $A \in \mathcal{S}(G)$ with nonzero diagonal entries.
- Key: Permutation does not change the rank, and the dominating sequence gives an echelon form.

Pick an optimal sequence $(v_1, \ldots, v_k)$ and a matrix A. Let N_i be the vertices dominated by v_i but not any vertex before v_i.

$$
\begin{array}{c}
\\ v_1 \\ v_2 \\ \vdots \\ v_k \\ \text{other vertices}
\end{array}
\begin{array}{c}
\begin{array}{cccc} N_1 & N_2 & \cdots & N_k \end{array} \\
\begin{bmatrix}
* \cdots * & 0 & \cdots & 0 \\
? & * \cdots * & 0 & \vdots \\
? & ? & \ddots & 0 \\
? & \cdots & ? & * \cdots * \\
? & ? & ? & ?
\end{bmatrix}
\end{array}
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\end{array}
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Z-Grundy domination number and zero forcing number

The Z-Grundy domination number $\gamma_{\rm gr}^Z(G)$ is the length of the longest sequence $(v_1, v_2, \ldots, v_k)$ such that

$$N_G(v_i) \setminus \bigcup_{j=1}^{i-1} N_G[v_j] \neq \emptyset.$$

Theorem (Brešar et al. 2017; L 2017)

For any graph, $\gamma_{\rm gr}^Z(G) \leq \operatorname{rank}(A)$ *for any matrix* $A \in \mathcal{S}(G)$.

Theorem (AIM 2008)

For any graph, $\operatorname{null}(A) \leq Z(G)$ *for any matrix* $A \in \mathcal{S}(G)$.

- Here $Z(G)$ is the zero forcing number defined through a graph searching process.
- Brešar et al. proved that $Z(G) = |V(G)| - \gamma_{\rm gr}^Z(G)$.

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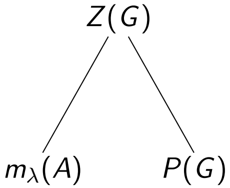
Upper bound for the multiplicity

- Recall that $\text{null}(A - \lambda I) = m_\lambda(A)$, while $A \in \mathcal{S}(G)$ if and only if $A - \lambda I \in \mathcal{S}(G)$.

Theorem (AIM 2008)

For any graph G and a matrix $A \in \mathcal{S}(G)$,

$$m_\lambda(A) \le Z(G) \text{ for all eigenvalue } \lambda \text{ of } A.$$



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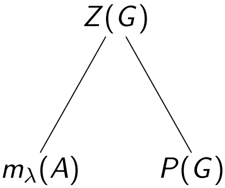
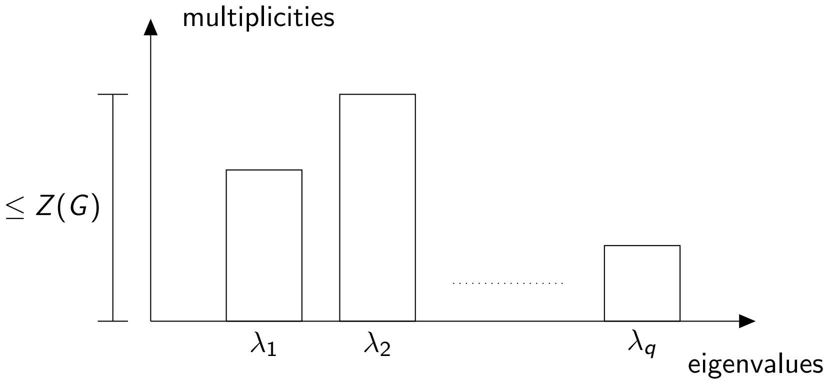


Inverse eigenvalue problem of a graph

The inverse eigenvalue problem of a graph (IEPG) aims to find all spectra in $\mathcal{S}(G)$ for a given graph.



$q \geq$ len(unique shortest path)$+1$

Thanks!

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