Odd cycle zero forcing parameters and the minimum rank problem

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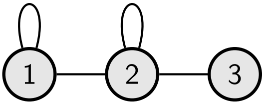
Connections in Discrete Mathematics, Vancouver

Minimum rank problem (simple and loop)



$$\begin{bmatrix} ? & * & 0 \\ * & ? & * \\ 0 & * & ? \end{bmatrix} \qquad \begin{bmatrix} * & * & 0 \\ * & * & * \\ 0 & * & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

smallest possible rank
largest possible nullity

$\mathrm{mr}(P_3) = 2$

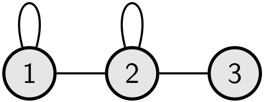
$M(P_3) = 1$

$\mathrm{mr}(\mathfrak{P}_3) = 3$

$M(\mathfrak{P}_3) = 0$

Zero forcing number (simple and loop)

$M(P_3) = 1$
$Z(P_3) = 1$



$M(\mathfrak{P}_3) = 0$
$Z(\mathfrak{P}_3) = 0$

minimum number of blue vertices
to force all vertices blue

simple: If y is the only white neighbor of x and x is blue, then $x \to y$.

loop: If y is the only white neighbor of x ~~and x is blue~~, then $x \to y$. (x, y are possibly the same.)

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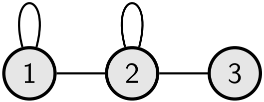

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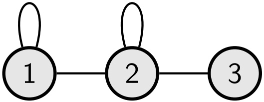

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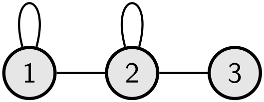

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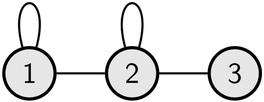

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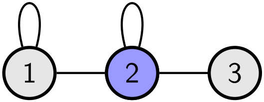

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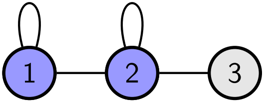

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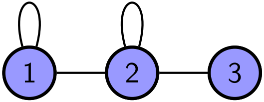

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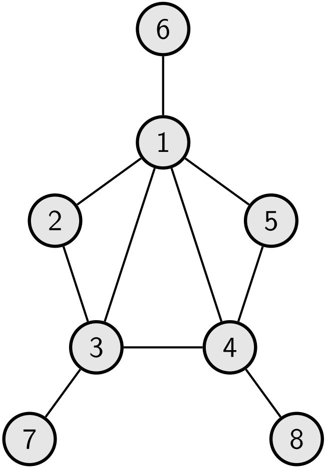
Max Nullity vs Zero Forcing

- $M(G) \leq Z(G)$ for all simple graph [AIM 2008]; $M(\mathfrak{G}) \leq Z(\mathfrak{G})$ for all loop graph [Hogben 2010].
- $M(G) = Z(G)$ whenever $|V(G)| \leq 7$ or G is a tree, a cycle; not always true for outerplanar graphs.
- $M(\mathfrak{G}) = Z(\mathfrak{G})$ whenever $|V(\mathfrak{G})| \leq 2$ or $\mathfrak{G}$ is a loop configuration of a tree; not always true for cycles or outerplanar graphs.

An example with $M(G) \neq Z(G)$



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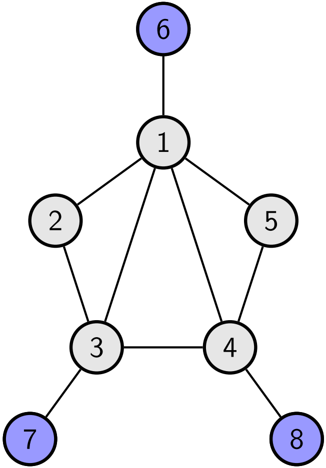

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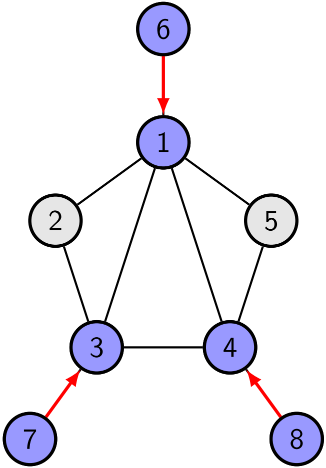

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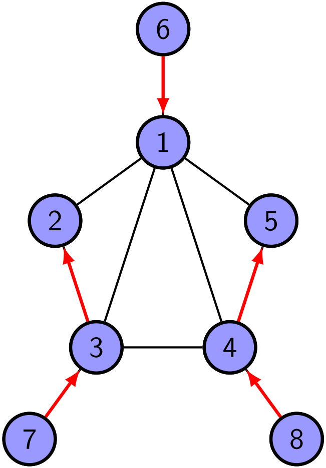

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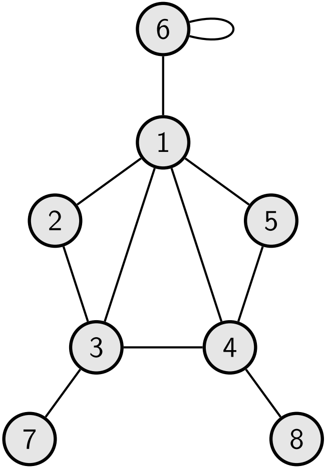

6 has a loop
others unknown

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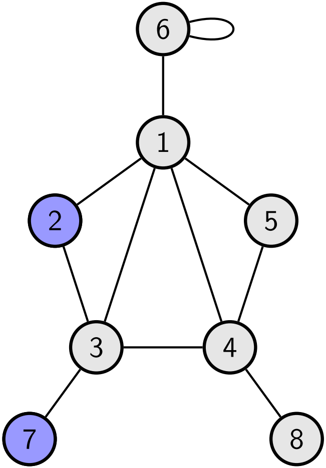

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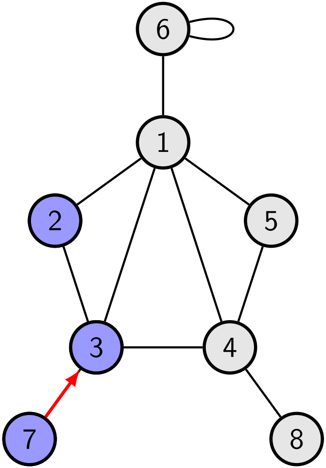

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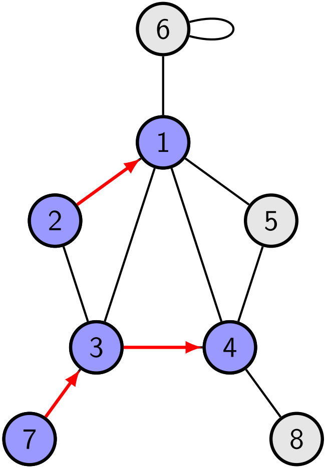

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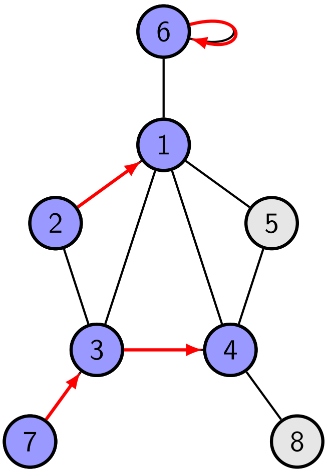

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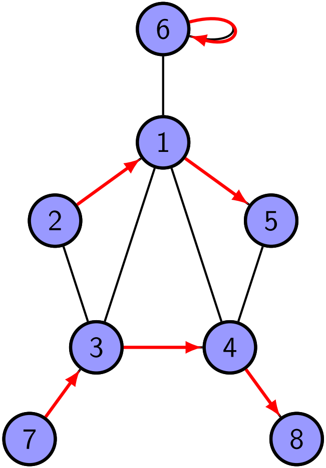

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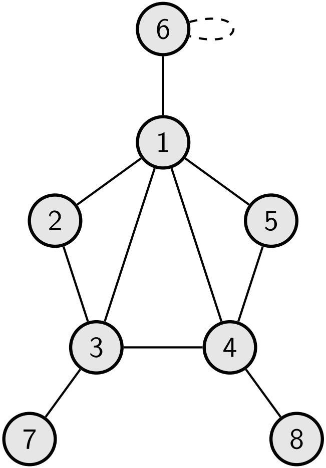

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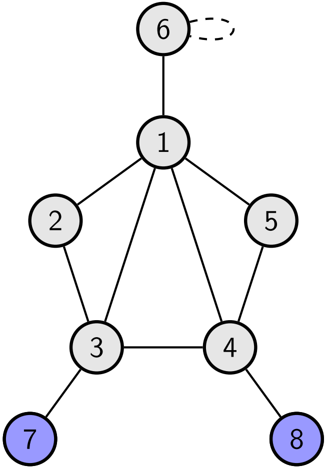

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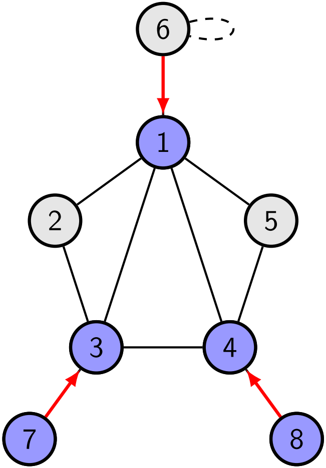

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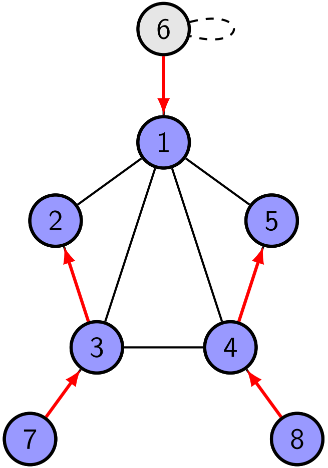

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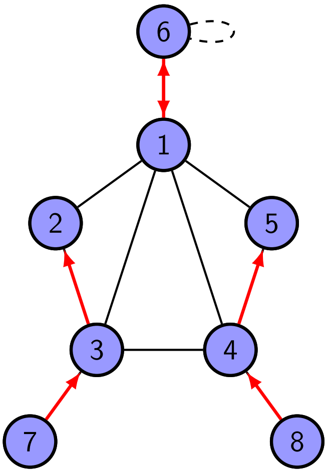

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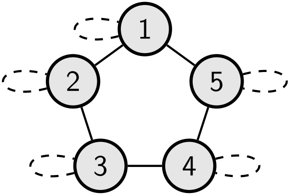
An example with $M(\mathfrak{G}) \neq Z(\mathfrak{G})$

- $M(\mathfrak{C}_n) = Z(\mathfrak{C}_n)$ if $\mathfrak{C}_n$ is not a loopless odd cycle; $M(\mathfrak{C}^0_{2k+1}) = 0$ but $Z(\mathfrak{C}^0_{2k+1}) = 1$.
- $M^{\mathbb{R}}(\mathfrak{C}^0_{2k+1}) = 0$ but $M^{\mathbb{F}_2}(\mathfrak{C}^0_{2k+1}) = 1$.



$$\det \begin{bmatrix} 0 & e_1 & & & e_{2k+1} \\ e_1 & 0 & e_2 & & \\ & e_2 & \ddots & \ddots & \\ & & \ddots & & e_{2k} \\ e_{2k+1} & & & e_{2k} & 0 \end{bmatrix} = 2 \prod_{i=1}^{2k+1} e_i$$

Max Nullity vs Zero Forcing Revisit

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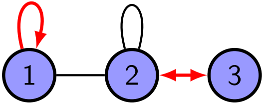
- For simple graphs with $|V(G)| \leq 7$, $M(G) = Z(G)$.
- For the simple graph $K_{3,3,3}$, $M(K_{3,3,3}) = 6$ and $\widehat{Z}(K_{3,3,3}) = 7$.
- For the loop graph $\mathfrak{C}_3^0$, $M(\mathfrak{C}_3^0) = 0$ and $Z(\mathfrak{C}_3^0) = 1$.
- The fact $M^F(\mathfrak{C}_{2k+1}^0) = 0$ is true whenever the considered matrix is symmetric and char $\neq 2$.

Proof of $M \leq Z$



$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} * & * & 0 \\ * & * & * \\ 0 & * & 0 \end{bmatrix} \end{array}$$

$$\begin{array}{l} 3 \to 2 \\ 1 \to 1 \\ 2 \to 3 \end{array}$$

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Try to generalize the "triangle"

$$\operatorname{rank}\begin{bmatrix} a_{1,1} & ? & ? & ? & ? \\ 0 & a_{2,2} & ? & ? & ? \\ 0 & 0 & a_{3,3} & ? & ? \\ ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \end{bmatrix} \geq 3$$

Try to generalize the "triangle"

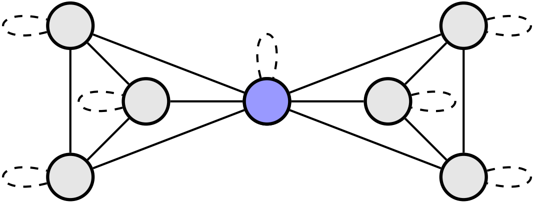
$$\operatorname{rank}\begin{bmatrix} A_{1,1} & ? & ? & ? & ? \\ O & A_{2,2} & ? & ? & ? \\ O & O & A_{3,3} & ? & ? \\ ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \end{bmatrix} \geq \sum_{i=1}^{3} \operatorname{rank}(A_{i,i})$$

Try to generalize the "triangle"

$$\operatorname{rank}\begin{bmatrix} A(\mathfrak{C}_5^0) & ? & ? & ? & ? \\ O & A(\mathfrak{C}_7^0) & ? & ? & ? \\ O & O & A(\mathfrak{C}_3^0) & ? & ? \\ ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \end{bmatrix} \geq 5 + 7 + 3 = 15$$

Odd cycle zero forcing number

- The color-change rule for loop graphs is:
  - if y is the only white neighbor of x ~~and x is blue~~, then $x \to y$. (x, y are possibly the same.)
  - if W is the set of white vertices, and $\mathfrak{G}[W]$ has a connected component $\mathfrak{C}$ such that $\mathfrak{C} \cong \mathfrak{C}^0_{2k+1}$, then all vertices in $V(\mathfrak{C})$ turn blue.



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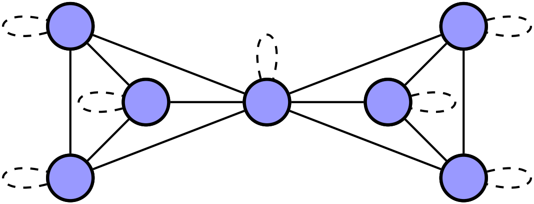

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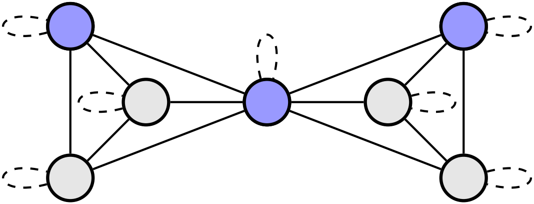

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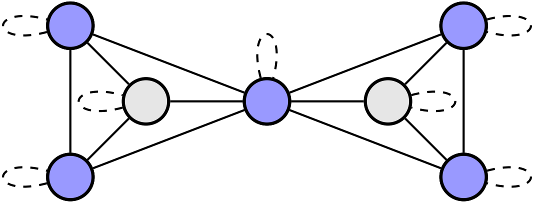

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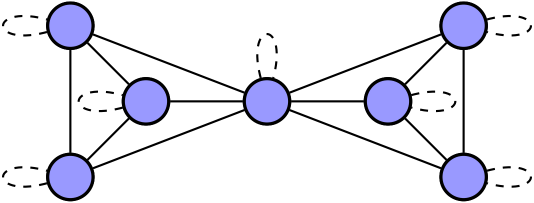

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  - if y is the only white neighbor of x ~~and x is blue~~, then $x \to y$. (x, y are possibly the same.)
  - if W is the set of white vertices, and $\mathfrak{G}[W]$ has a connected component $\mathfrak{C}$ such that $\mathfrak{C} \cong \mathfrak{C}^0_{2k+1}$, then all vertices in $V(\mathfrak{C})$ turn blue.
- $Z_{oc}(\mathfrak{G})$ is the minimum number of blue vertices required to force all graph blue.
- $M^F(\mathfrak{G}) \leq Z_{oc}(\mathfrak{G})$ whenever char $F \neq 2$ and matrices are symmetric.
- The enhanced odd cycle zero forcing number is defined as $\widehat{Z}_{oc}(G) = \max_{\mathfrak{G}} Z_{oc}(\mathfrak{G})$, where $\mathfrak{G}$ runs over all loop configurations of G.

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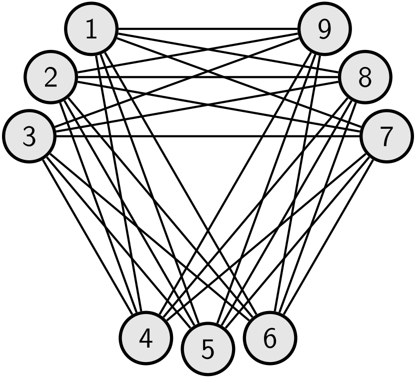
- For the loop graph $\mathfrak{C}_3^0$, $M(\mathfrak{C}_3^0) = 0$ and $Z(\mathfrak{C}_3^0) = 1$.
- $M(K_{3,3,3}) = 6 = \widehat{Z}_{oc}(K_{3,3,3})$.
- $M(\mathfrak{C}_{2k+1}^0) = 0 = Z_{oc}(\mathfrak{C}_{2k+1}^0)$.

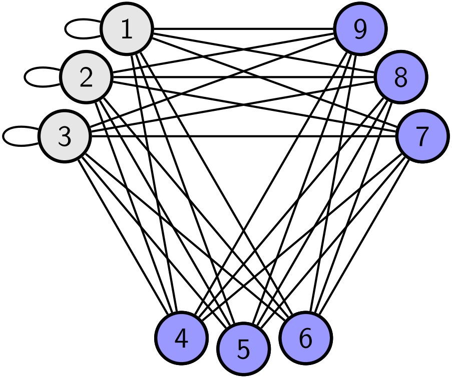
Example: $K_{3,3,3}$



$$\widehat{Z}_{oc}(K_{3,3,3}) = 6 \text{ and } \widehat{Z}(K_{3,3,3}) = Z(K_{3,3,3}) = 7.$$

Example: $K_{3,3,3}$

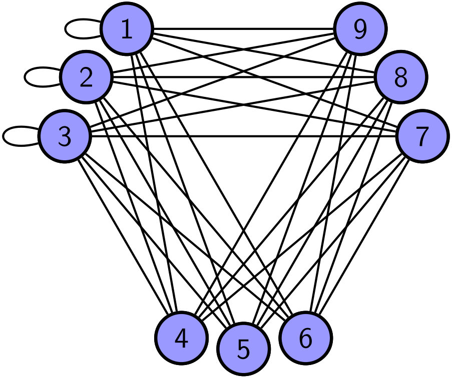
1,2,3 have loops
others are unknown



$$\widehat{Z}_{oc}(K_{3,3,3}) = 6 \text{ and } \widehat{Z}(K_{3,3,3}) = Z(K_{3,3,3}) = 7.$$

Example: $K_{3,3,3}$

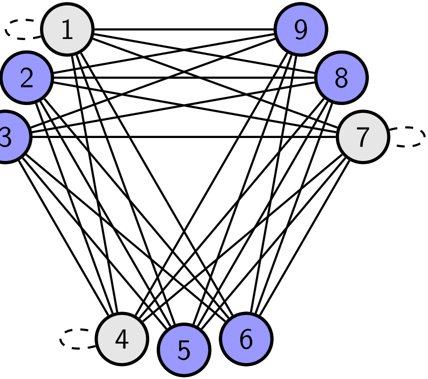
1,2,3 have loops
others are unknown



$$\widehat{Z}_{oc}(K_{3,3,3}) = 6 \text{ and } \widehat{Z}(K_{3,3,3}) = Z(K_{3,3,3}) = 7.$$

Example: $K_{3,3,3}$

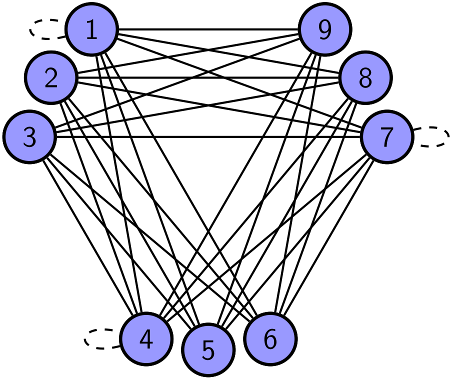
1,4,7 have no loops
others are unknown



$$\widehat{Z}_{oc}(K_{3,3,3}) = 6 \text{ and } \widehat{Z}(K_{3,3,3}) = Z(K_{3,3,3}) = 7.$$

Example: $K_{3,3,3}$

1,4,7 have no loops
others are unknown



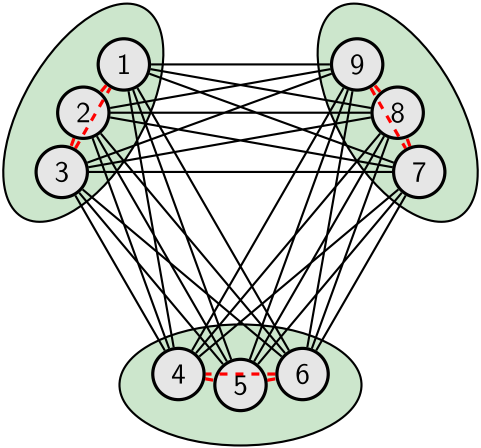
$$\widehat{Z}_{oc}(K_{3,3,3}) = 6 \text{ and } \widehat{Z}(K_{3,3,3}) = Z(K_{3,3,3}) = 7.$$

Field matters

- Let A be the adjacency matrix.
- $\text{null}(A) = 6 = M(K_{3,3,3}) = \widehat{Z}_{oc}(K_{3,3,3})$.
- $\text{null}^{\mathbb{F}_2}(A) = 7 = M^{\mathbb{F}_2}(K_{3,3,3}) = Z(K_{3,3,3})$.

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

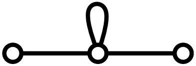
$\mathfrak{C}_3^0$ vs $K_{3,3,3}$



$\widehat{Z}_{oc}(K_{3,3,3}) = 6$ and $\widehat{Z}(K_{3,3,3}) = Z(K_{3,3,3}) = 7.$

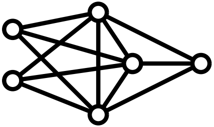
Graph & Matrix blowups

loop graph $\mathfrak{G}$



(2, 3, 1)-blowup

simple graph H



$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 4 & 7 \\ 0 & 7 & 0 \end{bmatrix} \xrightarrow{(2,3,1)\text{-blowup}} \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 4 & 4 & 4 & 7 \\ 1 & 1 & 4 & 4 & 4 & 7 \\ 1 & 1 & 4 & 4 & 4 & 7 \\ 0 & 0 & 7 & 7 & 7 & 0 \end{bmatrix}$$

$A \in \mathcal{S}^F(\mathfrak{G})$ $\qquad$ $A' \in \mathcal{S}^F(H)$

References I

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