

# Reduction identities of the minimum rank on loop graphs

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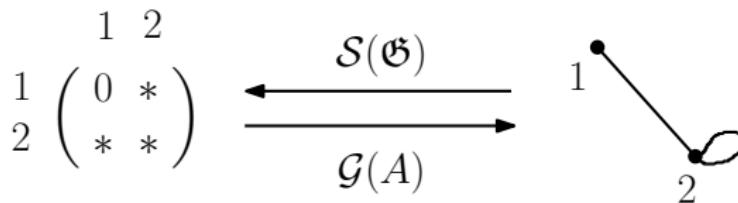
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Leslie Hogben, Gabi Maurer, Kathleen Nowak,  
Aaron Rodriguez, and James Strickland.

# Minimum Rank of Loop Graphs

$$\begin{matrix} & \begin{matrix} 1 & 2 \\ 1 & \left( \begin{matrix} 0 & * \\ * & * \end{matrix} \right) \\ 2 & \end{matrix} \end{matrix} \quad \begin{matrix} \xleftarrow{\mathcal{S}(\mathfrak{G})} & \xrightarrow{\mathcal{G}(A)} \end{matrix} \quad \begin{matrix} 1 \\ 2 \end{matrix}$$


$$\text{mr}(\mathfrak{G}) = \min\{\text{rank}(A) : A \in \mathcal{S}(\mathfrak{G})\}$$

# Example: $K_2$

Loop graph  $\mathfrak{G}$



$$\begin{pmatrix} * & * \\ * & * \end{pmatrix}$$



$$\begin{pmatrix} 0 & * \\ * & * \end{pmatrix}$$



$$\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$$

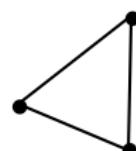
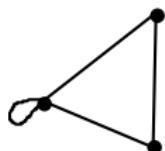
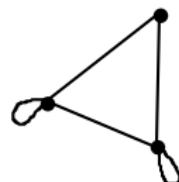
Simple graph  $G$



$$\begin{pmatrix} ? & * \\ * & ? \end{pmatrix}$$

$$\text{mr}(\mathfrak{G}_1) = 1 \quad \text{mr}(\mathfrak{G}_2) = 2 \quad \text{mr}(\mathfrak{G}_3) = 2 \xrightarrow{\min} \text{mr}(G) = 1$$

# Example: Complete Graphs



$$\begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}$$

$$\begin{pmatrix} 0 & * & * \\ * & * & * \\ * & * & * \end{pmatrix}$$

$$\begin{pmatrix} 0 & * & * \\ * & 0 & * \\ * & * & * \end{pmatrix}$$

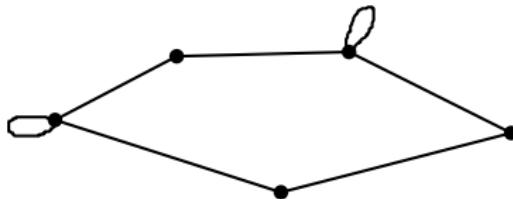
$$\begin{pmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 0 \end{pmatrix}$$

$$\text{mr}(\mathfrak{G}_1) = 1 \quad \text{mr}(\mathfrak{G}_2) = 2 \quad \text{mr}(\mathfrak{G}_3) = 2 \quad \text{mr}(\mathfrak{G}_4) = 3$$

$$\det \begin{pmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{pmatrix} = 2abc \neq 0$$

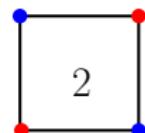
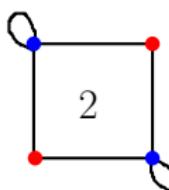
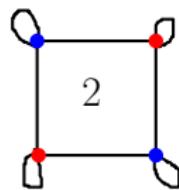
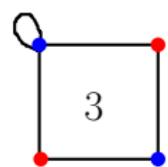
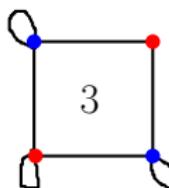
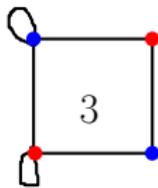
# Paths and Cycles

$$n - 1 = \text{mr}(P_n) \leq \text{mr}(\mathfrak{P}_n) \leq n$$



$$n - 2 = \text{mr}(C_n) \leq \text{mr}(\mathfrak{C}_n) \leq n$$

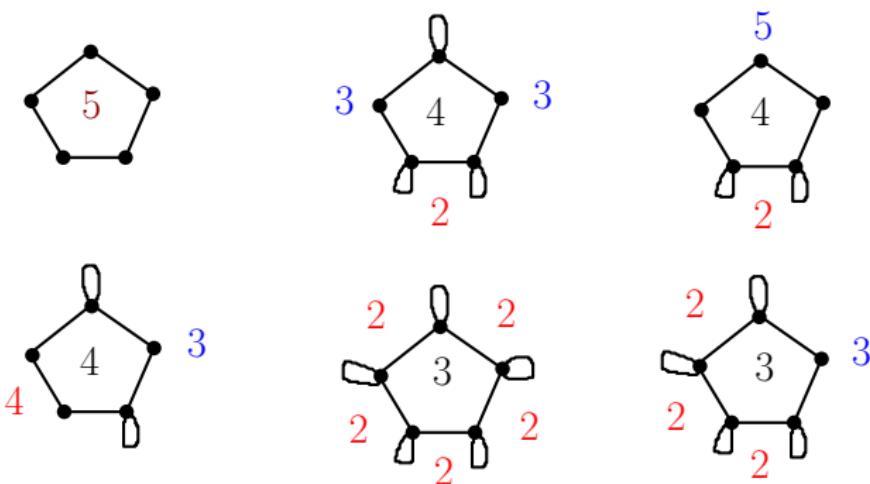
# Even Cycles



$n - 1 \Leftrightarrow$  at least one of blue or red has exactly one loop;

$n - 2 \Leftrightarrow$  otherwise.

# Odd Cycles



$n \Leftrightarrow$  loopless;

$n - 1 \Leftrightarrow$  exactly one even end-loop interval;

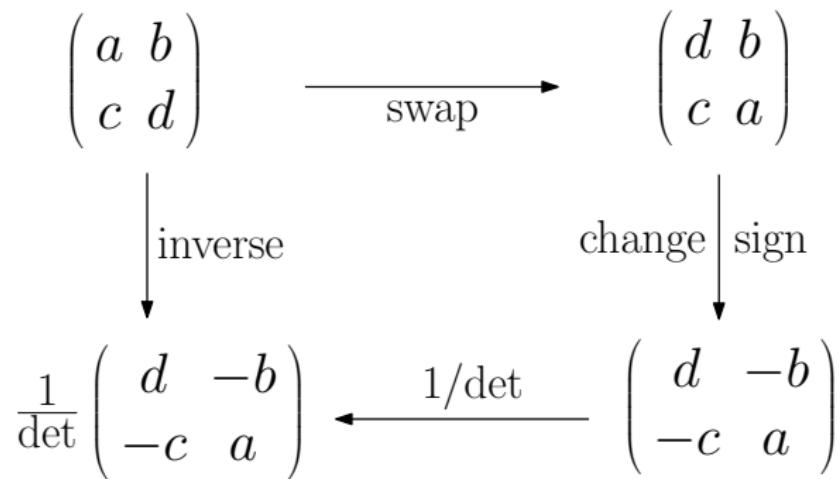
$n - 2 \Leftrightarrow$  otherwise.

# Main Lemma

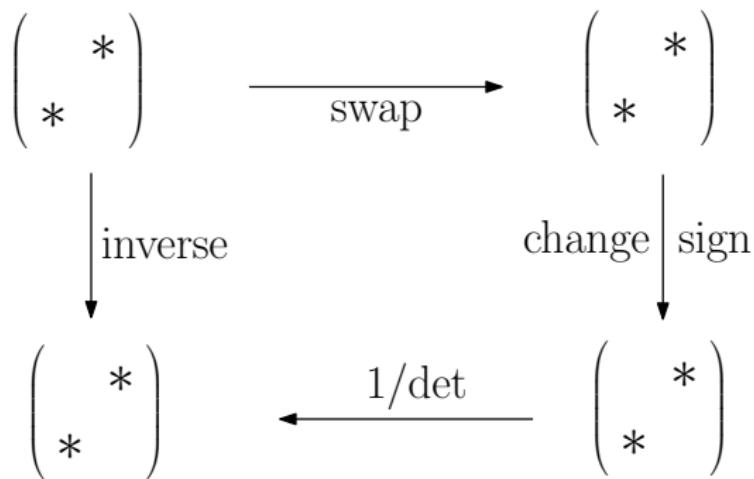
$$\text{mr}\left(\begin{array}{c} \text{graph} \\ \text{with red edges} \end{array}\right) = \text{mr}\left(\begin{array}{c} \text{graph} \\ \text{with red edges} \\ \text{and orange edges} \end{array}\right)$$

$$\text{mr}\left(\begin{array}{c} \text{graph} \\ \text{with red edges} \\ \text{and red loops} \end{array}\right) = \text{mr}\left(\begin{array}{c} \text{graph} \\ \text{with red edges} \\ \text{and orange edges} \\ \text{and orange loops} \end{array}\right)$$

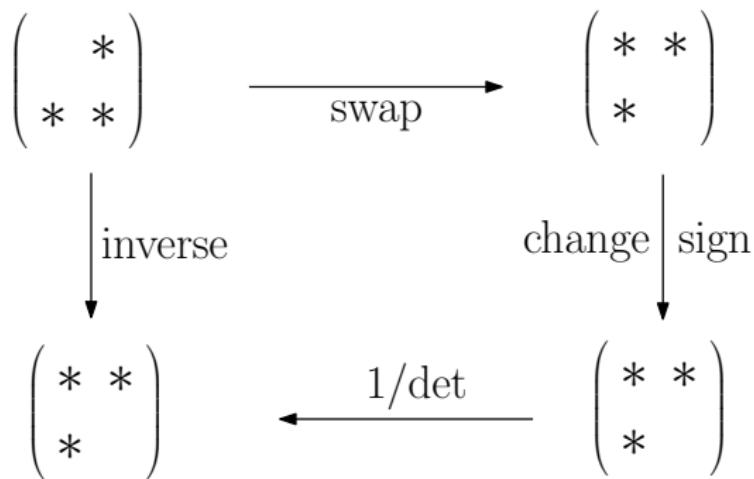
# Inverse of $2 \times 2$ matrices



# Symbolic Inverse



# Symbolic Inverse



# Graph Interpretation

$$\begin{pmatrix} & * & \sim & \sim \\ * & & \sim & \sim \\ \sim & \sim & \sim & \sim \\ \sim & \sim & \sim & \sim \end{pmatrix} \rightarrow \begin{array}{c} \text{graph with blue edges} \\ \text{and no loops} \end{array}$$

$$\begin{pmatrix} & * & \sim & \sim \\ * & * & \sim & \sim \\ \sim & \sim & \sim & \sim \\ \sim & \sim & \sim & \sim \end{pmatrix} \rightarrow \begin{array}{c} \text{graph with green loop} \\ \text{at the top-left vertex} \end{array}$$

# Schur Complement

$$\begin{pmatrix} A & B^\top \\ B & D \end{pmatrix} \xrightarrow{\text{row } 2 - BA^{-1} \text{ row1}} \begin{pmatrix} A & B^\top \\ O & D - BA^{-1}B^\top \end{pmatrix}$$

- If  $A$  is invertible, then

$D - BA^{-1}B^\top$  is called the *Schur complement*.

- Two matrices have the same rank.

# Schur Complement on Graphs

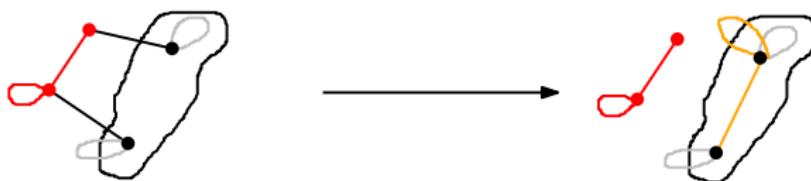
$$\left( \begin{array}{cc|cc} * & * & * & 0 \\ * & & * & 0 \\ \hline * & & & \\ * & * & & \sim \\ 0 & 0 & & \end{array} \right) \longrightarrow \left( \begin{array}{c|cc} * & * & \\ \hline * & & \\ & & * \\ & \sim & \end{array} \right)$$



$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \begin{pmatrix} * & * \\ * & * \end{pmatrix} \begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix} = \begin{pmatrix} * & 0 \\ * & 0 \\ 0 & 0 \end{pmatrix}$$

# Schur Complement on Graphs

$$\left( \begin{array}{cc|cc} * & * & * & 0 \\ * & * & * & 0 \\ \hline * & & & \\ * & * & & \sim \\ 0 & 0 & & \end{array} \right) \longrightarrow \left( \begin{array}{cc|cc} * & * & & \\ * & * & & \\ \hline & & *+* & * \\ & & * & \sim \end{array} \right)$$



$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \begin{pmatrix} * & * \\ * & * \end{pmatrix} \begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix} = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

# Main Lemma

$$\text{mr}\left(\begin{array}{c} \text{graph with red edges and black dots} \\ \text{inside a boundary} \end{array}\right) = \text{mr}\left(\begin{array}{c} \text{graph with red edges and black dots} \\ \text{inside a boundary} \\ \text{with a yellow edge added} \end{array}\right)$$

$$\text{mr}\left(\begin{array}{c} \text{graph with red edges and black dots} \\ \text{inside a boundary} \\ \text{with a red loop} \end{array}\right) = \text{mr}\left(\begin{array}{c} \text{graph with red edges and black dots} \\ \text{inside a boundary} \\ \text{with a red loop} \\ \text{and a yellow loop} \end{array}\right)$$

# Proof of Main Lemma

$$\begin{array}{c} \text{scale row 3} \\ \text{and row 4} \\ \text{realized by} \\ \begin{pmatrix} 0 & 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 3 & 0 \\ 2 & 0 & 0 & 0 & 2 \\ 0 & 3 & 0 & 0 & 3 \\ 0 & 0 & 2 & 3 & 0 \end{pmatrix} \end{array} \xrightarrow{\quad} \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \xrightarrow{\quad \text{Schur complement} \quad} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

(A blue arrow points from the middle matrix to the right matrix.)

$$5 = \text{mr}\left(\begin{array}{c} \text{graph with 5 vertices and 5 edges} \end{array}\right) = \text{mr}\left(\begin{array}{c} \text{graph with 2 vertices and 1 edge} \\ \text{graph with 3 vertices and 2 edges} \end{array}\right) = 3 + 2 = 5$$

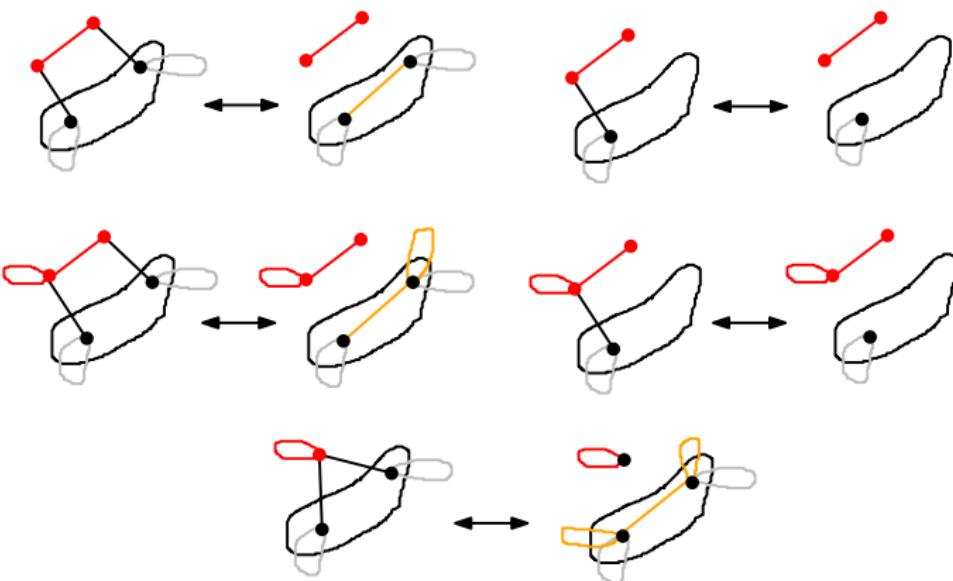
# Proof of Main Lemma

$$\begin{pmatrix} 0 & 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 3 & 0 \\ 2 & 0 & 0 & 0 & 2 \\ 0 & 3 & 0 & 0 & 3 \\ 0 & 0 & 2 & 3 & 0 \end{pmatrix} \xrightarrow{\text{reconstruction}} \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \xrightarrow{\text{realized by}} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

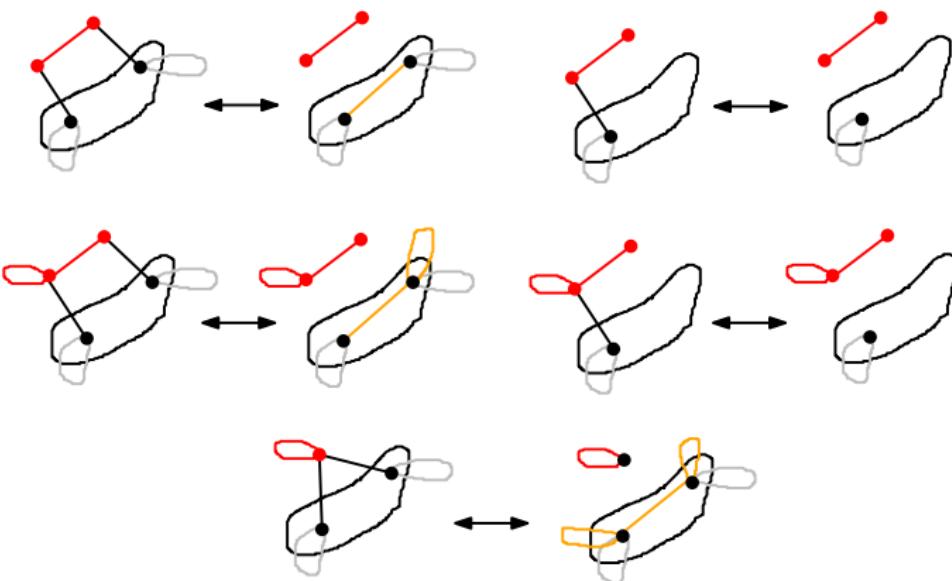
↓

5 =  $\text{mr}(\text{pentagon}) = \text{mr}(\text{triangle}) = 3+2=5$

## Other Results



## Other Results



Thank you.