Zero forcing number, Grundy domination number and variants

Jephian C.-H. Lin

Department of Applied Mathematics, National Sun Yat-sen University

July 12, 2019
2019 Meeting of the International Linear Algebra Society,
Rio de Janeiro, Brazil

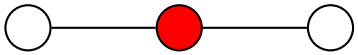
Domination number

Let G be a graph. The domination number $\gamma(G)$ is the minimum cardinality of a set X such that

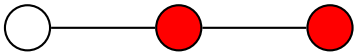
$$\bigcup_{x \in X} N_G[x] = V(G).$$

The total domination number $\gamma^t(G)$ is the minimum cardinality of a set X such that

$$\bigcup_{x \in X} N_G(x) = V(G).$$



$\gamma(P_3) = 1$



$\gamma^t(P_3) = 2$

Greedy algorithm

- Greedy algorithm makes the locally optimal choice at each stage with the hope of finding a global optimum.
- Maze: You may keep going straight at fork. But it might lead you to a dead end.
- Graph coloring: You may keep using the smallest free number to color the next vertex, showing $\chi(G) \leq \Delta(G) + 1$.
- Greedy algorithm for domination number: When X are chosen and not yet dominate the whole graph, pick a vertex v such that

$$N_G[v] \setminus \bigcup_{x \in X} N_G[x] \neq \emptyset.$$

Greedy algorithm

- Greedy algorithm makes the locally optimal choice at each stage with the hope of finding a global optimum.
- Maze: You may keep going straight at fork. But it might lead you to a dead end.
- Graph coloring: You may keep using the smallest free number to color the next vertex, showing $\chi(G) \leq \Delta(G) + 1$.
- Greedy algorithm for domination number: When X are chosen and not yet dominate the whole graph, pick a vertex v such that

$$N_G[v] \setminus \bigcup_{x \in X} N_G[x] \neq \emptyset.$$

Greedy algorithm

- Greedy algorithm makes the locally optimal choice at each stage with the hope of finding a global optimum.
- Maze: You may keep going straight at fork. But it might lead you to a dead end.
- Graph coloring: You may keep using the smallest free number to color the next vertex, showing $\chi(G) \leq \Delta(G) + 1$.
- Greedy algorithm for domination number: When X are chosen and not yet dominate the whole graph, pick a vertex v such that

$$N_G[v] \setminus \bigcup_{x \in X} N_G[x] \neq \emptyset.$$

Greedy algorithm

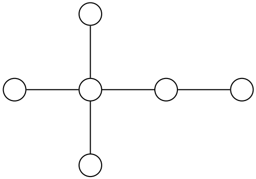
- Greedy algorithm makes the locally optimal choice at each stage with the hope of finding a global optimum.
- Maze: You may keep going straight at fork. But it might lead you to a dead end.
- Graph coloring: You may keep using the smallest free number to color the next vertex, showing $\chi(G) \leq \Delta(G) + 1$.
- Greedy algorithm for domination number: When X are chosen and not yet dominate the whole graph, pick a vertex v such that

$$N_G[v] \setminus \bigcup_{x \in X} N_G[x] \neq \emptyset.$$

Grundy domination number

The Grundy domination number $\gamma_{\mathrm{gr}}(G)$ is the length of the longest sequence $(v_1, v_2, \ldots, v_k)$ such that

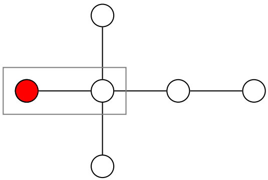
$$N_G[v_i] \setminus \bigcup_{j=1}^{i-1} N_G[v_j] \neq \emptyset.$$



Grundy domination number

The Grundy domination number $\gamma_{\mathrm{gr}}(G)$ is the length of the longest sequence $(v_1, v_2, \ldots, v_k)$ such that

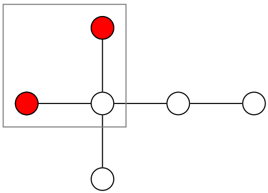
$$N_G[v_i] \setminus \bigcup_{j=1}^{i-1} N_G[v_j] \neq \emptyset.$$



Grundy domination number

The Grundy domination number $\gamma_{\mathrm{gr}}(G)$ is the length of the longest sequence $(v_1, v_2, \ldots, v_k)$ such that

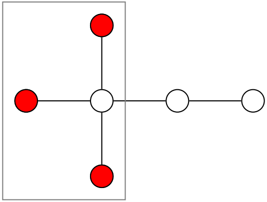
$$N_G[v_i] \setminus \bigcup_{j=1}^{i-1} N_G[v_j] \neq \emptyset.$$



Grundy domination number

The Grundy domination number $\gamma_{\mathrm{gr}}(G)$ is the length of the longest sequence $(v_1, v_2, \ldots, v_k)$ such that

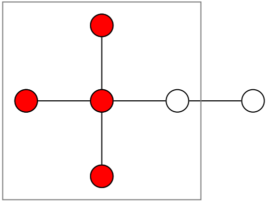
$$N_G[v_i] \setminus \bigcup_{j=1}^{i-1} N_G[v_j] \neq \emptyset.$$



Grundy domination number

The Grundy domination number $\gamma_{\mathrm{gr}}(G)$ is the length of the longest sequence $(v_1, v_2, \ldots, v_k)$ such that

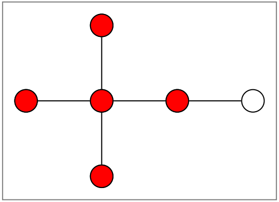
$$N_G[v_i] \setminus \bigcup_{j=1}^{i-1} N_G[v_j] \neq \emptyset.$$



Grundy domination number

The Grundy domination number $\gamma_{\rm gr}(G)$ is the length of the longest sequence $(v_1, v_2, \ldots, v_k)$ such that

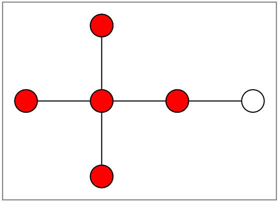
$$N_G[v_i] \setminus \bigcup_{j=1}^{i-1} N_G[v_j] \neq \emptyset.$$



Grundy domination number

The Grundy domination number $\gamma_{\mathrm{gr}}(G)$ is the length of the longest sequence $(v_1, v_2, \ldots, v_k)$ such that

$$N_G[v_i] \setminus \bigcup_{j=1}^{i-1} N_G[v_j] \neq \emptyset.$$

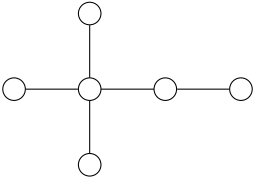


So $\gamma_{\mathrm{gr}}(G) = 5$.

Grundy total domination number

The Grundy total domination number $\gamma_{\mathrm{gr}}^{t}(G)$ is the length of the longest sequence $(v_1, v_2, \ldots, v_k)$ such that

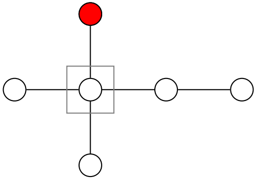
$$N_G(v_i) \setminus \bigcup_{j=1}^{i-1} N_G(v_j) \neq \emptyset.$$



Grundy total domination number

The Grundy total domination number $\gamma_{\mathrm{gr}}^t(G)$ is the length of the longest sequence $(v_1, v_2, \ldots, v_k)$ such that

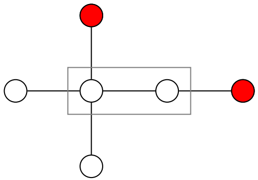
$$N_G(v_i) \setminus \bigcup_{j=1}^{i-1} N_G(v_j) \neq \emptyset.$$



Grundy total domination number

The Grundy total domination number $\gamma_{\mathrm{gr}}^{t}(G)$ is the length of the longest sequence $(v_1, v_2, \ldots, v_k)$ such that

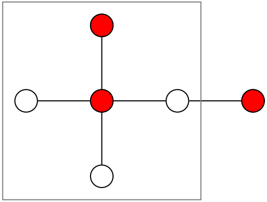
$$N_G(v_i) \setminus \bigcup_{j=1}^{i-1} N_G(v_j) \neq \emptyset.$$



Grundy total domination number

The Grundy total domination number $\gamma_{\mathrm{gr}}^{t}(G)$ is the length of the longest sequence $(v_1, v_2, \ldots, v_k)$ such that

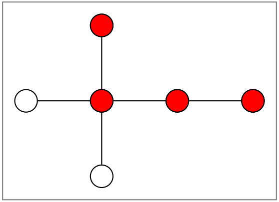
$$N_G(v_i) \setminus \bigcup_{j=1}^{i-1} N_G(v_j) \neq \emptyset.$$



Grundy total domination number

The Grundy total domination number $\gamma_{\mathrm{gr}}^{t}(G)$ is the length of the longest sequence $(v_1, v_2, \ldots, v_k)$ such that

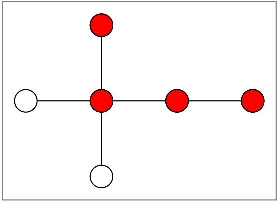
$$N_G(v_i) \setminus \bigcup_{j=1}^{i-1} N_G(v_j) \neq \emptyset.$$



Grundy total domination number

The Grundy total domination number $\gamma_{\mathrm{gr}}^{t}(G)$ is the length of the longest sequence $(v_1, v_2, \ldots, v_k)$ such that

$$N_G(v_i) \setminus \bigcup_{j=1}^{i-1} N_G(v_j) \neq \emptyset.$$



So $\gamma_{\mathrm{gr}}^{t}(G) = 4$.

Other dominating sequences (Brešar et al. 2017)

The *Z*-Grundy domination number $\gamma_{\rm gr}^{Z}(G)$ is the length of the longest sequence $(v_1, v_2, \ldots, v_k)$ such that

$$N_G(v_i) \setminus \bigcup_{j=1}^{i-1} N_G[v_j] \neq \emptyset.$$

The *L*-Grundy domination number $\gamma_{\rm gr}^{L}(G)$ is the length of the longest sequence $(v_1, v_2, \ldots, v_k)$ such that

$$N_G[v_i] \setminus \bigcup_{j=1}^{i-1} N_G(v_j) \neq \emptyset.$$

[Note: $v_1, \ldots, v_k$ have to be distinct vertices.]

Theorem (Brešar et al. 2017)

For any graph G,

$$Z(G) = n - \gamma_{\mathrm{gr}}^{Z}(G),$$

where $Z(G)$ is the zero forcing number.

Main theorem

Theorem (L 2019)

Let G be a graph and $|V(G)| = n$. Then

1. $Z(G) = n - \gamma_{\mathrm{gr}}^{Z}(G)$,
2. $Z_{\dot{\ell}}(G) = n - \gamma_{\mathrm{gr}}(G)$,
3. $Z_{-}(G) = n - \gamma_{\mathrm{gr}}^{t}(G)$,
4. $Z_{L}(G) = n - \gamma_{\mathrm{gr}}^{L}(G)$.

Maximum nullity and minimum rank

For a graph G, define $\mathcal{S}(G)$ as the collection of all real symmetric matrices whose

$$i,j\text{-entry} = \begin{cases} \neq 0 & \text{if } ij \in E(G),\ i \neq j; \\ = 0 & \text{if } ij \notin E(G),\ i \neq j; \\ \in \mathbb{R} & \text{if } i = j; \end{cases}$$

- minimum rank $\text{mr}(G)$ = smallest possible rank among matrices in $\mathcal{S}(G)$
- maximum nullity $M(G)$ = largest possible nullity among matrices in $\mathcal{S}(G)$
- $M(G) = n - \text{mr}(G)$ for any graph G on n vertices.

Rank bound

Theorem (L 2017)

Let G be a graph. Then

$$\gamma_{\mathrm{gr}}(G) \leq \mathrm{rank}(A)$$

for any $A \in \mathcal{S}(G)$ with diagonal entries all nonzero; and

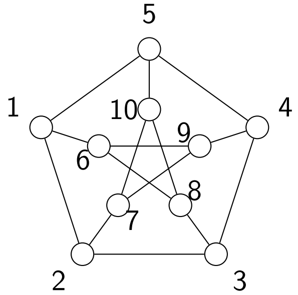
$$\gamma_{\mathrm{gr}}^{t}(G) \leq \mathrm{rank}(A)$$

for any $A \in \mathcal{S}(G)$ with zero diagonal.

Let P be the Petersen graph. Consider

$$A = \begin{bmatrix} C - I & I_5 \\ I_5 & C' - I \end{bmatrix} \text{ and } B = \begin{bmatrix} C & I_5 \\ I_5 & -C' \end{bmatrix},$$

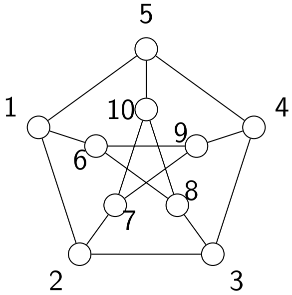
where C and C' are the adjacency matrix of C_5 and $\overline{C_5}$, respectively. Then $\gamma_{\mathrm{gr}}(P) \leq \mathrm{rank}(A) = 5$ and the sequence $(1, 2, 3, 4, 5)$ is optimal.



Let P be the Petersen graph. Consider

$$A = \begin{bmatrix} C - I & I_5 \\ I_5 & C' - I \end{bmatrix} \text{ and } B = \begin{bmatrix} C & I_5 \\ I_5 & -C' \end{bmatrix},$$

where C and C' are the adjacency matrix of C_5 and $\overline{C_5}$, respectively. Then $\gamma_{\mathrm{gr}}^t(G) \leq \mathrm{rank}(B) = 6$ and the sequence $(9, 1, 2, 3, 4, 5)$ is optimal.



Proof of the rank bound

- Goal: Show $\gamma_{\mathrm{gr}}(G) \leq \operatorname{rank}(A)$ for all $A \in \mathcal{S}(G)$ with nonzero diagonal entries.
- Key: Permutation does not change the rank, and the dominating sequence gives an echelon form.

Pick an optimal sequence $(v_1, \ldots, v_k)$ and a matrix A. Let N_i be the vertices dominated by v_i but not any vertex before v_i.

$$
\begin{array}{c} \\ v_1 \\ v_2 \\ \vdots \\ v_k \\ \text{other vertices} \end{array}
\begin{array}{c}
\begin{array}{cccc} N_1 & N_2 & \cdots & N_k \end{array} \\
\begin{bmatrix}
* \cdots * & 0 & \cdots & 0 \\
? & * \cdots * & 0 & \vdots \\
? & ? & \ddots & 0 \\
? & \cdots & ? & * \cdots * \\
? & ? & ? & ?
\end{bmatrix}
\end{array}
$$

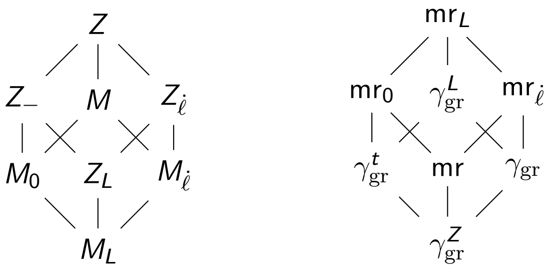
Proof of the rank bound

- Goal: Show $\gamma_{\mathrm{gr}}(G) \leq \mathrm{rank}(A)$ for all $A \in \mathcal{S}(G)$ with nonzero diagonal entries.
- Key: Permutation does not change the rank, and the dominating sequence gives an echelon form.

Pick an optimal sequence $(v_1, \ldots, v_k)$ and a matrix A. Let N_i be the vertices dominated by v_i but not any vertex before v_i.

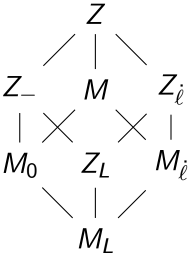
$$\begin{array}{c} \\ v_1 \\ v_2 \\ \vdots \\ v_k \\ \text{other vertices} \end{array} \begin{array}{c} \begin{array}{cccc} N_1 & N_2 & \cdots & N_k \end{array} \\ \begin{bmatrix} * \cdots * & 0 & \cdots & 0 \\ ? & * \cdots * & 0 & \vdots \\ ? & ? & \ddots & 0 \\ ? & \cdots & ? & * \cdots * \\ ? & ? & ? & ? \end{bmatrix} \end{array}$$

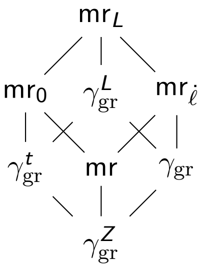
$M < Z$ and $\gamma_{\mathrm{gr}} < \mathrm{mr}$



Thanks!

$M < Z$ and $\gamma_{\rm gr} < \mathrm{mr}$





Thanks!

References I

B. Brešar, Cs. Bujtás, T. Gologranc, S. Klavžar, G. Košmrlj, B. Patkós, Zs. Tuza, and M. Vizer.
Grundy dominating sequences and zero forcing sets.
Discrete Optim., 26:66–77, 2017.

B. Brešar, T. Gologranc, M. Milanič, D. F. Rall, and R. Rizzi.
Dominating sequences in graphs.
Discrete Math., 336:22–36, 2014.

B. Brešar, M. A. Henning, and D. F. Rall.
Total dominating sequences in graphs.
Discrete Math., 339:1665–1676, 2016.

References II

J. C.-H. Lin.
Zero forcing number, Grundy domination number, and their variants.
Linear Algebra Appl., 563:240–254, 2019.