Comparability and cocomparability bigraphs

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Joint work with

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Simon Fraser U

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Ross McConnell
Colorado State U

(photos from department websites and personal websites)

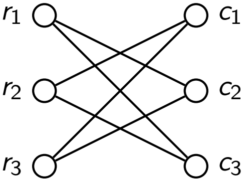
$\diagup$-free

$$\diagup = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

A $0, 1$-matrix is $\diagup$-free if the rows and columns can be permuted (independently) so that the resulting matrix does not contain $\diagup$ as a submatrix.

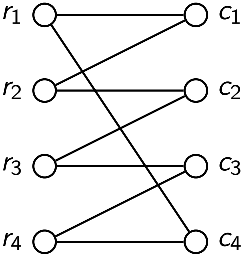
- How to recognize a $\diagup$-free matrix?
- If a matrix is $\diagup$-free, how to find the correct permutations?

C_6 is $\diagup$-free



$$\begin{bmatrix} \color{red}{0} & \color{red}{1} & 1 \\ \color{red}{1} & \color{red}{0} & 1 \\ 1 & 1 & 0 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} 1 & \color{red}{0} & \color{red}{1} \\ 0 & 1 & 1 \\ 1 & \color{red}{1} & \color{red}{0} \end{bmatrix} \xrightarrow{c_2 \leftrightarrow c_3} \begin{bmatrix} 1 & 1 & 0 \\ \color{red}{0} & \color{red}{1} & 1 \\ \color{red}{1} & \color{red}{0} & 1 \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

C_8 is not ╱-free



$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

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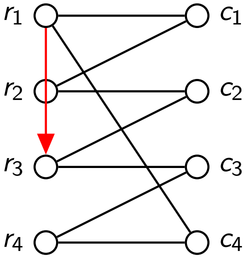

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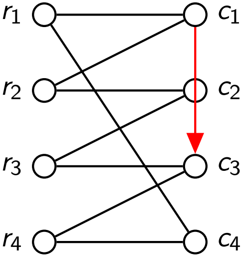

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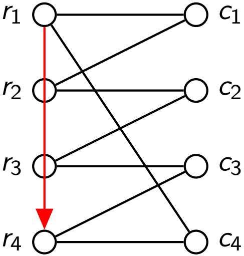

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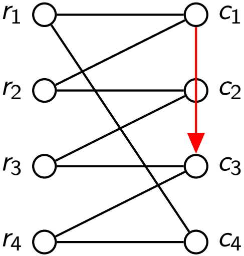

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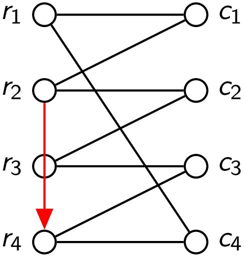

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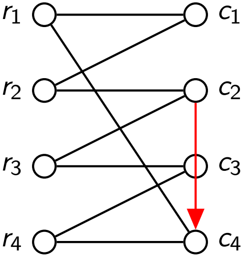

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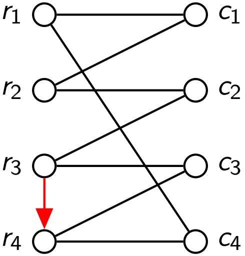
C_8 is not /-free



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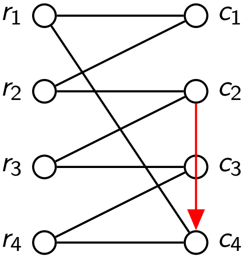

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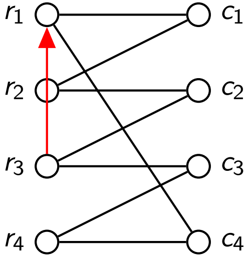


$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Submatrix avoiding problem

- A: a $0, 1$-matrix (usually the biadjacency matrix)
- S: a small matrix (usually 2×2)
- A is S-free if rows and columns can be permuted independently to avoid S.

How to recognize? How to realize?

- A: a square $0, 1$-matrix (usually the adjacency matrix)
- S: a small matrix (usually 2×2)
- A is symmetrically S-free if rows and columns can be permuted symmetrically to avoid S.

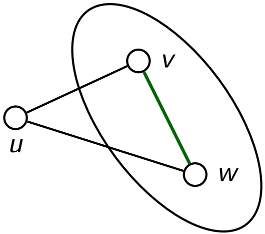
Chordal graph

- $R(G) = A(G) + I$, the neighborhood matrix
- principal $\Gamma = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ with one of 1 embedded on the diagonal of the target matrix.

A graph G is called a chordal graph if one of the following holds.

- G has no long (≥ 4) cycle.
- $R(G)$ is symmetrically principal Γ-free.

$$\begin{array}{c} \\ u \\ v \\ w \end{array} \begin{array}{c} \begin{array}{ccc} u & v & w \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$$

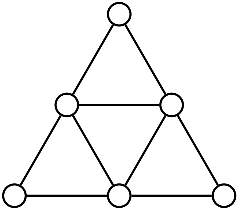


Strongly chordal graph

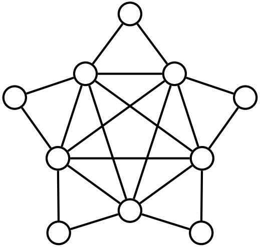
Theorem (Chang 1982; Farber 1983)

A graph G is called a strongly chordal graph if one of the following holds.

- *G has no long cycle and no induced trampoline.*
- *$R(G)$ is symmetrically ~~principal~~ Γ-free.*



trampolines



Chordal bigraph

Theorem

A bipartite graph G is called a chordal bigraph if one of the following holds.

- *G has no long (≥ 6) cycle.*
- *The biadjacency matrix of G is ~~symmetrically~~ Γ-free.*

Comparability graph

A comparability graph is the graph of a poset, where two vertices are adjacent if and only if they are comparable.

poset

a comparability graph

On an induced P_3, the orientations of its two edges are related.

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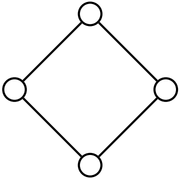
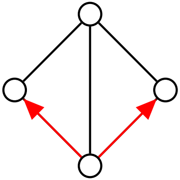


poset



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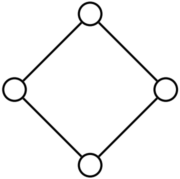
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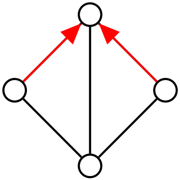
a comparability graph

On an induced P_3, the orientations of its two edges are related.

Cocomparability graph

- principal $\diagup = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ with one of 1 embedded on the diagonal of the target matrix.

A graph G is called a cocomparability graph if one of the following holds.

- $\overline{G}$ is a comparability graph.
- $R(G) = A(G) + I$ is symmetrically principal $\diagup$-free.
- G has no invertible pair.
- G has no vertex asteroid. [Gilmore and Hoffman 1964]

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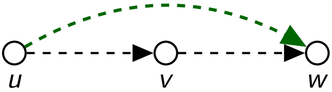
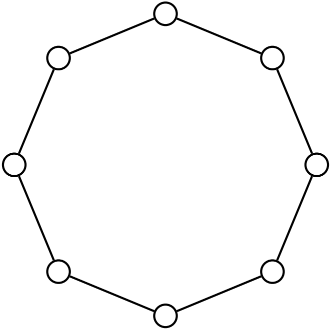


Invertible pair: C_8 is not a cocomparability graph



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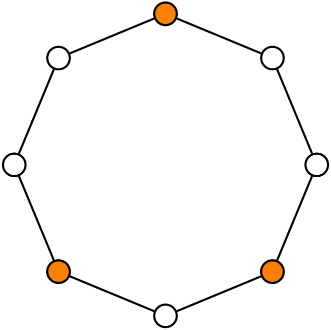

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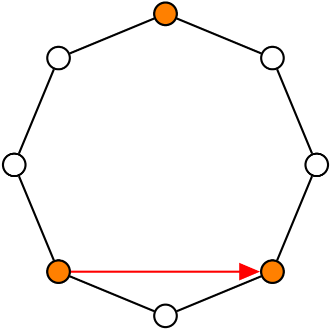

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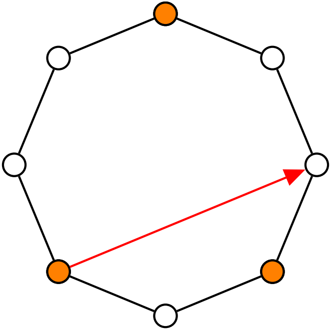

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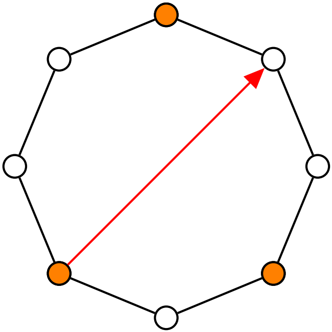

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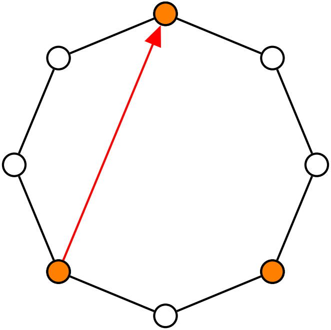

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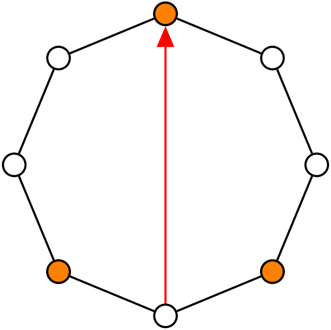

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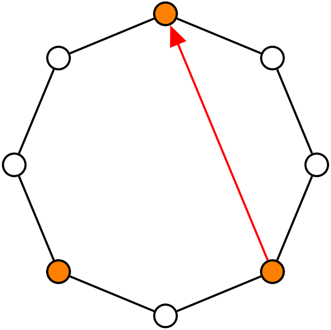

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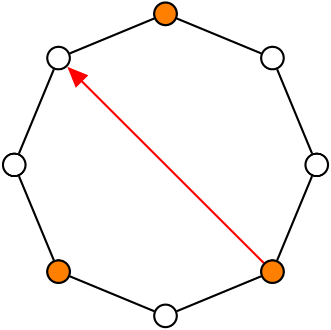

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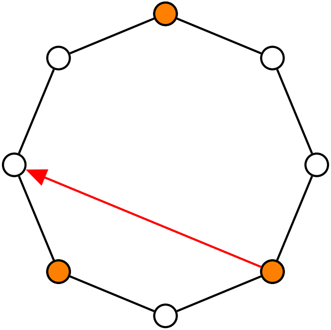

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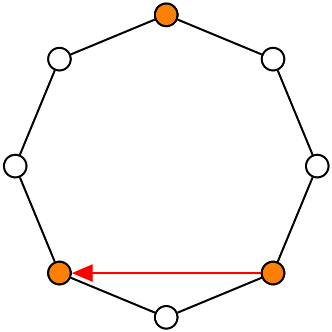

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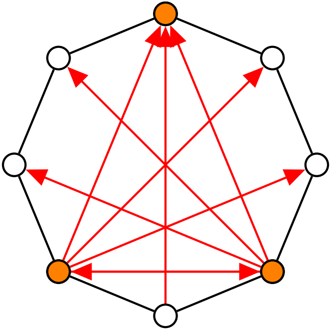
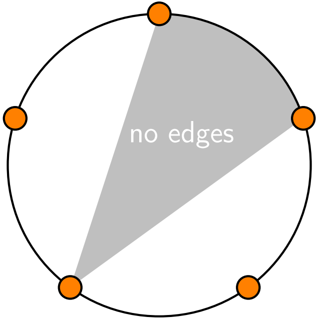


Vertex asteroid

Vertex asteroid: a set of vertices $v_0, \ldots, v_{2k}$ of odd numbers such that

- there is a path P_i from v_i to v_{i+1}, and
- v_i is not adjacent to any vertex on P_{i+k}.



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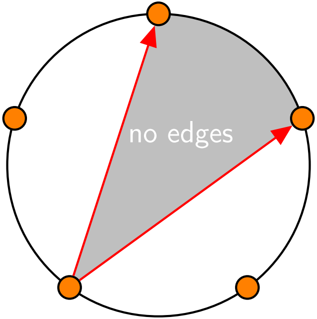

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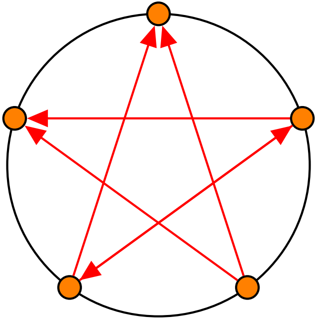
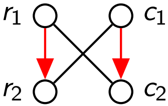


Cocomparability bigraph

A bipartite graph G is called a cocomparability bigraph if one of the following holds.

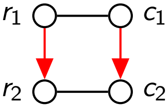
- the biadjacency matrix of G is ~~symmetrically principal~~ $\diagup$-free.
- G has no invertible pair.
- G has no edge asteroid.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



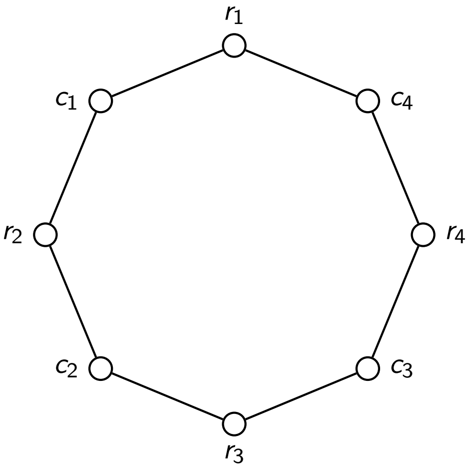
not allowed

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



allowed

Invertible pair: C_8 is not ⁄-free (revisited)



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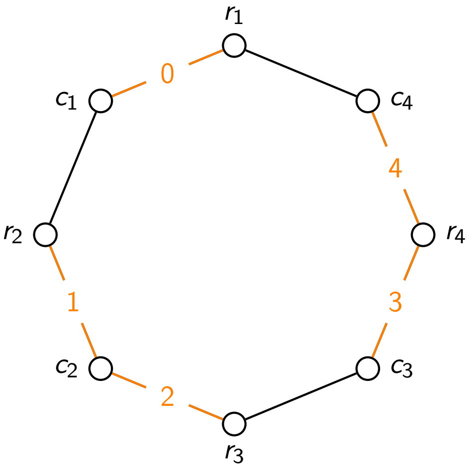

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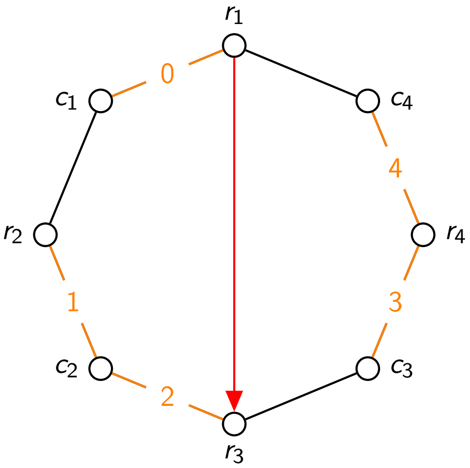

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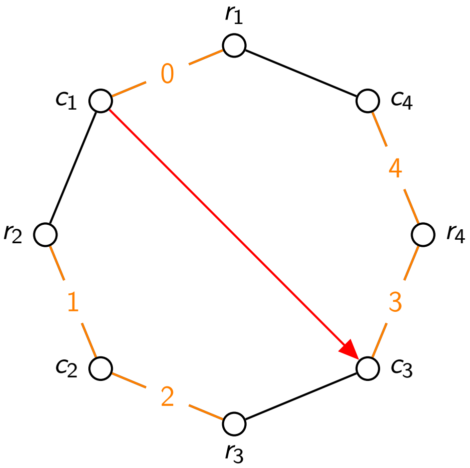

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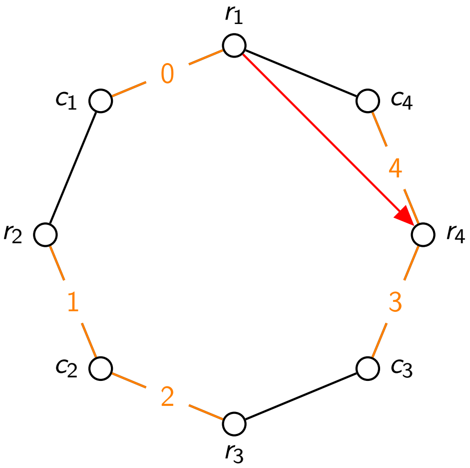

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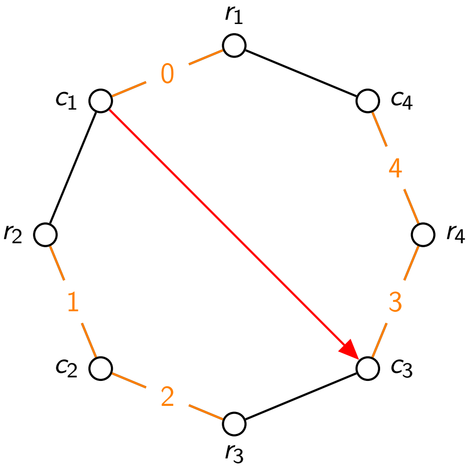
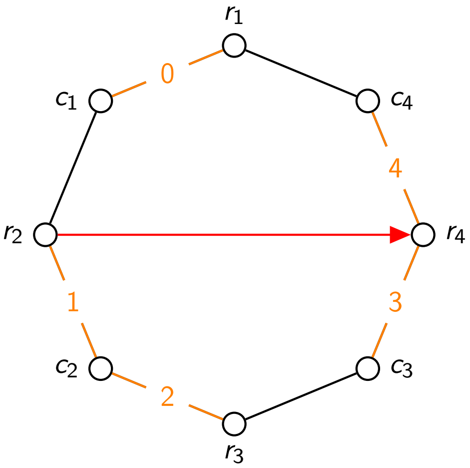


Invertible pair: C_8 is not $\diagup$-free (revisited)



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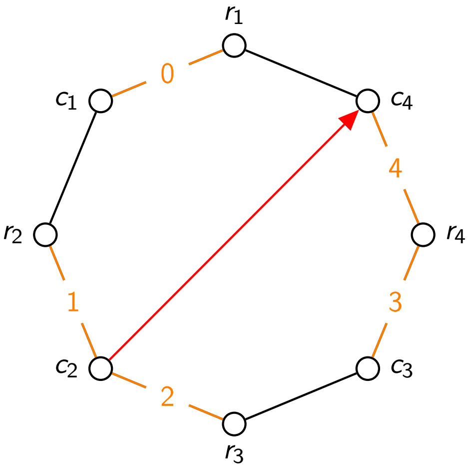

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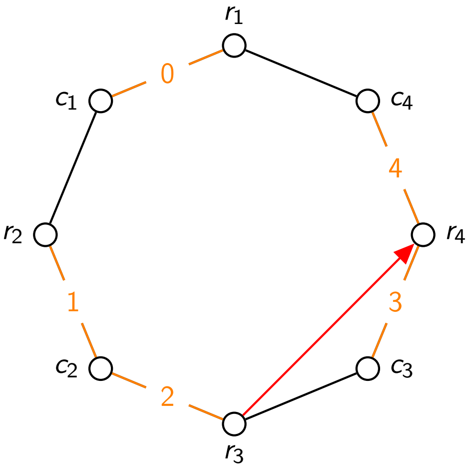

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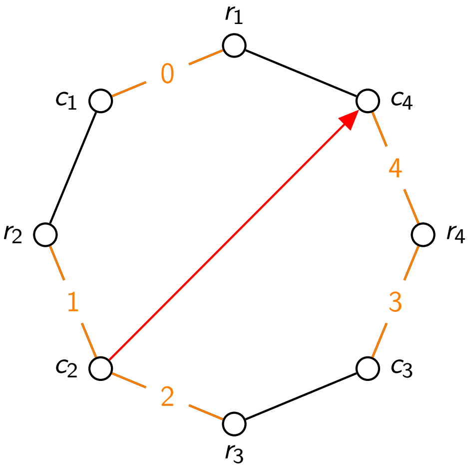

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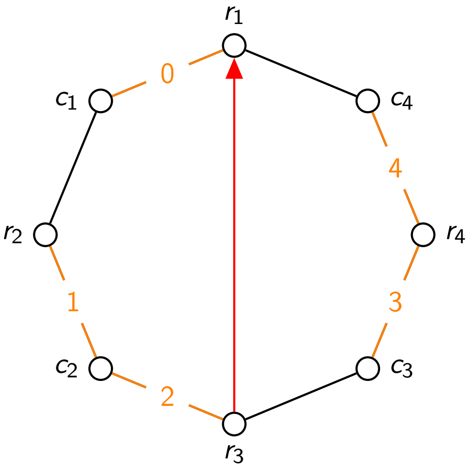

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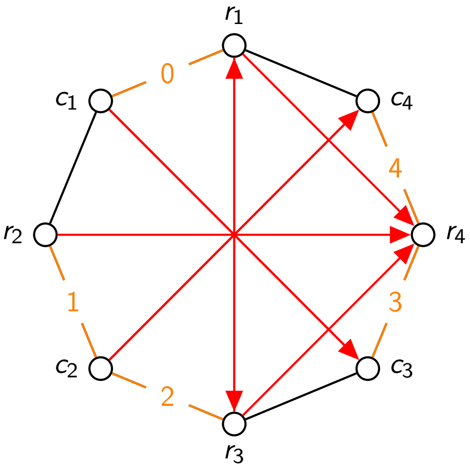
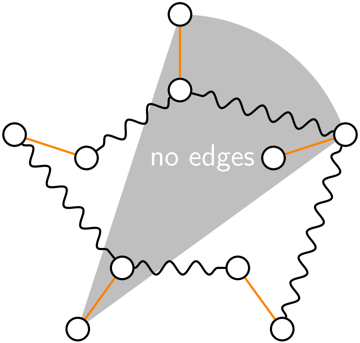


Edge asteroid

Edge asteroid: a set of edges $e_0, \ldots, e_{2k}$ of odd numbers such that

- there is a path P_i from e_i to e_{i+1}, and
- v_i is not adjacent to any vertex on P_{i+k} (including all vertices on e_{i+k} and e_{i+k+1}).



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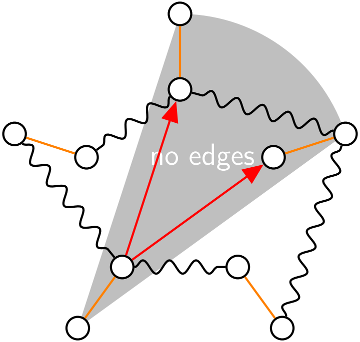

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- there is a path P_i from e_i to e_{i+1}, and
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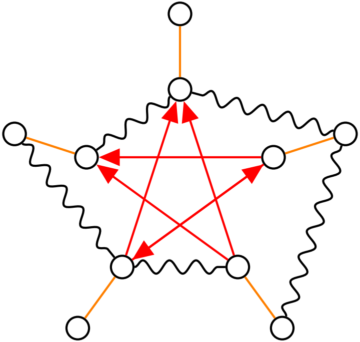

Algorithms

Theorem (Hell, Huang, JL, and McConnell 2020)

There is a polynomial-time algorithm to recognize a $\diagup$-free matrix.

Theorem (Hell, Huang, JL, and McConnell 2020)

There is a polynomial-time algorithm to find permutations of rows and columns of $\diagup$-free matrix to avoid $\diagup$.

Thank you!

Algorithms

Theorem (Hell, Huang, JL, and McConnell 2020)

There is a polynomial-time algorithm to recognize a $\diagup$*-free matrix.*

Theorem (Hell, Huang, JL, and McConnell 2020)

There is a polynomial-time algorithm to find permutations of rows and columns of $\diagup$*-free matrix to avoid* $\diagup$*.*

Thank you!

Algorithms

Theorem (Hell, Huang, JL, and McConnell 2020)

There is a polynomial-time algorithm to recognize a $\diagup$-free matrix.

Theorem (Hell, Huang, JL, and McConnell 2020)

There is a polynomial-time algorithm to find permutations of rows and columns of $\diagup$-free matrix to avoid $\diagup$.

Thank you!

References I

A. Brandsädt, V. B. Le, and J. Spinrad.
Graph Classes: A Survey (Monographs on Discrete Mathematics and Applications).
SIAM, 1999.

G. J. Chang.
k-domination and graph covering problems.
PhD thesis, School of OR and IE, Cornell University, Ithaca, N.Y., 1982.

M. Farber.
Characterizations of strongly chordal graphs.
Discrete Math., 43:173–189, 1983.

References II

P. C. Gilmore and A. J. Hoffman.
A characterization of comparability graphs and interval graphs.
Canad. J. Math., 16:539–548, 1964.

M. C. Golumbic.
Algorithmic Graph Theory and Perfect Graph.
Academic Press, 1980.

P. Hell, J. Huang, J. C.-H. Lin, and R. M. McConnell.
Bipartite analogues of comparability and cocomparability graphs.
SIAM J. Discrete Math., 34:1969–1983, 2020.