The bifurcation lemma for strong properties in the inverse eigenvalue problem of a graph

Jephian C.-H. Lin 林晉宏

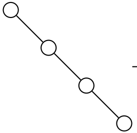
Department of Applied Mathematics, National Sun Yat-sen University

June 16, 2023
24th Conference of the International Linear Algebra Society, Madrid, Spain

Inverse eigenvalue problem of a graph (IEP-G)

Let G be a graph. Define $\mathcal{S}(G)$ as the family of all real symmetric matrices $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ such that

$$a_{ij} \begin{cases} \neq 0 & \text{if } ij \in E(G), i \neq j; \\ = 0 & \text{if } ij \notin E(G), i \neq j; \\ \in \mathbb{R} & \text{if } i = j. \end{cases}$$



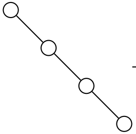
$$\longrightarrow \begin{bmatrix} ? & * & 0 & 0 \\ * & ? & * & 0 \\ 0 & * & ? & * \\ 0 & 0 & * & ? \end{bmatrix} \underset{?}{\overset{\text{spec}}{\rightleftarrows}} \{1, 2, 3, 4\}$$

IEP-G: What are the possible spectra of a matrix in $\mathcal{S}(G)$?

Inverse eigenvalue problem of a graph (IEP-G)

Let G be a graph. Define $\mathcal{S}(G)$ as the family of all real symmetric matrices $A = \left[a_{ij}\right]$ such that

$$a_{ij} \begin{cases} \neq 0 & \text{if } ij \in E(G), i \neq j; \\ = 0 & \text{if } ij \notin E(G), i \neq j; \\ \in \mathbb{R} & \text{if } i = j. \end{cases}$$



$$\longrightarrow \begin{tikzcd} \begin{bmatrix} ? & * & 0 & 0 \\ * & ? & * & 0 \\ 0 & * & ? & * \\ 0 & 0 & * & ? \end{bmatrix} \arrow[r, bend left, "\text{spec}"] & \{1,2,3,4\} \arrow[l, bend left, "?"] \end{tikzcd}$$

IEP-G: What are the possible spectra of a matrix in $\mathcal{S}(G)$?

Supergraph lemma

Lemma (BFHHLS 2017)

Let G and H' be two graphs with $V(G) = V(H')$ and $E(G) \subseteq E(H')$. If $A \in \mathcal{S}(G)$ has the SSP, then there is a matrix $A' \in \mathcal{S}(H')$ such that

- $\operatorname{spec}(A') = \operatorname{spec}(A)$,
- *A' has the SSP, and*
- *$\|A' - A\|$ can be chosen arbitrarily small.*

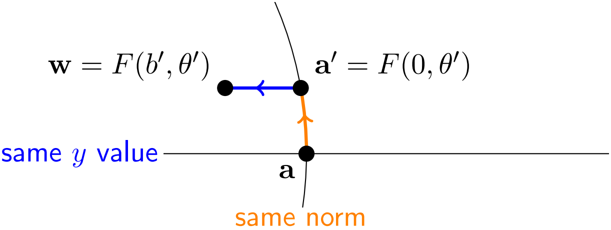
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} \sim 1 & \epsilon & 0 & 0 \\ \epsilon & \sim 2 & \epsilon & 0 \\ 0 & \epsilon & \sim 3 & \epsilon \\ 0 & 0 & \epsilon & \sim 4 \end{bmatrix}$$

SSP will be defined later

Inverse function theorem in $\mathbb{R}^2$

Fix a point $\mathbf{a} \in \mathbb{R}^2$. Combine two perturbations:

$$\begin{cases} \mathbf{a} + b\mathbf{e}_1 & \text{(same } y\text{)} \\ R_\theta \mathbf{a} & \text{(same norm)} \end{cases} \implies F(b, \theta) = R_\theta \mathbf{a} + b\mathbf{e}_1$$



$\frac{dF}{db,\theta}$ invertible $\implies$ any nearby $\mathbf{w}$ can be written as $\mathbf{w} = F(b', \theta')$

For whatever y value nearby, there is $\mathbf{a}'$ with $\|\mathbf{a}'\| = \|\mathbf{a}\|$.

Inverse function theorem

Theorem (Inverse function theorem)

Let $F : U \to W$ be a smooth function. If $\dot{F}$ at a point $\mathbf{u}_0 \in U$ is invertible, then F is locally invertible around $\mathbf{u}_0$.

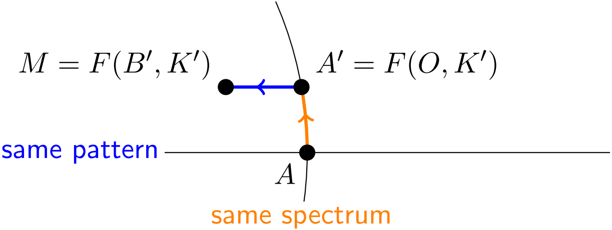
Theorem (FHLS 2022)

Let $F : U \to W$ be a smooth function. If $\dot{F}$ at a point $\mathbf{u}_0 \in U$ is surjective, then F is locally surjective around $\mathbf{u}_0$.

Inverse function theorem in $\mathrm{Sym}_n(\mathbb{R})$

Fix a point $A \in \mathcal{S}(G)$. Combine two perturbations:

$$\begin{cases} A + B & \text{(same pattern)} \\ e^{-K}Ae^{K} & \text{(same spectrum)} \end{cases} \implies F(B, K) = e^{-K}Ae^{K} + B$$



$\dot{F}$ surjective $\implies$ any nearby M can be written as $M = F(B', K')$

For whatever pattern nearby, there is A' with $\mathrm{spec}(A') = \mathrm{spec}(A)$.

Pattern perturbation

$$F(B,K) = e^{-K}Ae^{K} + B$$

Define $\mathcal{S}^{\text{cl}}(G)$ as the topological closure of $\mathcal{S}(G)$:

$$\mathcal{S}^{\text{cl}}(G) = \{A = \begin{bmatrix} a_{i,j} \end{bmatrix} \in \text{Sym}_n(\mathbb{R}) : a_{i,j} = 0 \iff \{i,j\} \in E(\overline{G})\}.$$

Let $A \in \mathcal{S}(G)$. Then $A + B \in \mathcal{S}(G)$ when $\|B\|$ is small enough.

> The tangent space of $F(B,K)$ at (O,O) with respect to B is $\mathcal{S}^{\text{cl}}(G)$.

Isospectral perturbation

$$F(B,K) = e^{-K}Ae^{K} + B$$

The function e^{K} is a bijection between

$$\{\text{skew-symmetric matrices nearby } O\} \to \{\text{orthogonal matrices nearby } I\}$$

for real matrices.

The tangent space of $F(B,Q)$ at (O,O) with respect to Q is

$$\{-KA + AK : K \in \mathrm{Skew}_n(\mathbb{R})\}.$$

Strong spectral property, transversality, and surjectivity

Definition

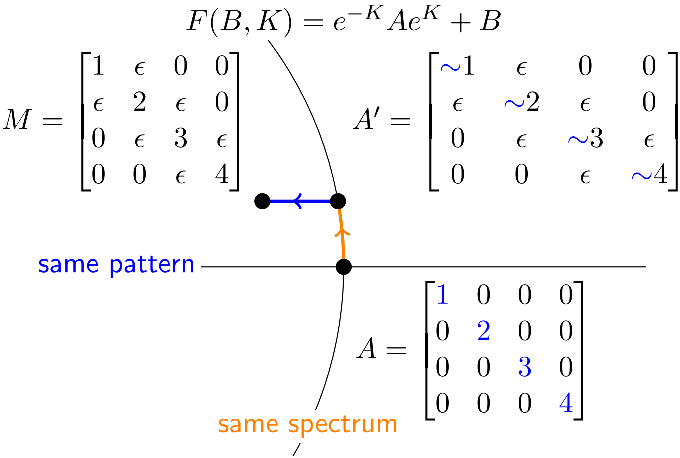
Let A be a real symmetric matrix. Then A has the strong spectral property (SSP) if $X = O$ is the only real symmetric matrix that satisfies

$$A \circ X = I \circ X = [A, X] = O.$$

Let $F(B, K) = e^{-K} A e^{K} + B$. Then the following are equivalent:

1. A has the SSP.
2. $\mathcal{S}^{\mathrm{cl}}(G) + \{-KA + AK : K \in \mathrm{Skew}_n(\mathbb{R})\} = \mathrm{Sym}_n(\mathbb{R})$.
3. The derivative $\dot{F}$ is surjective.

Illustration of the supergraph lemma



For whatever pattern nearby, there is A' with $\mathrm{spec}(A') = \mathrm{spec}(A)$.

They must be true, right?

Let $A \in \mathcal{S}(G)$ with the SSP. People *believed* that ...

- For any set of real numbers Λ' nearby $\operatorname{spec}(A)$, there is a matrix $A' \in \mathcal{S}(G)$ with $\operatorname{spec}(A') = \Lambda'$.
- For any refinement $\mathbf{m}'$ of $\mathbf{m}(A)$, there is a matrix $A' \in \mathcal{S}(G)$ with $\mathbf{m}(A') = \mathbf{m}'$.
- For any $k > q(A)$, there is a matrix $A' \in \mathcal{S}(G)$ with $q(A') = k$.

Let $A \in \mathcal{Q}(P)$ be a nilpotent matrix with the nSSP. People *knew* that ...

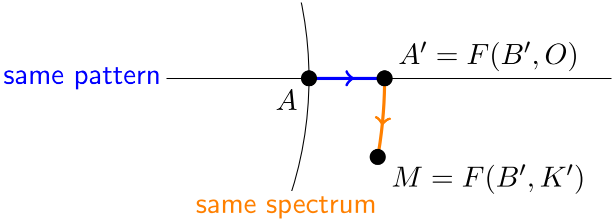
- For any set of complex numbers Λ' (invariant under conjugation) nearby $\{0, \ldots, 0\}$, there is a matrix $A' \in \mathcal{Q}(P)$ with $\operatorname{spec}(A') = \Lambda'$.

nSSP = the condition of the nilpotent-centralizer method

Bifurcation lemma

Theorem (FHLS 2022)

Let $A \in \mathcal{S}(G)$ with the SSP. Then for any set of real numbers Λ' nearby $\operatorname{spec}(A)$, there is a matrix A' with $\operatorname{spec}(A') = \Lambda'$.



$$F(B, K) = e^{-K}(A + B)e^{K}$$

The nSSP

Definition

Let A be a real matrix. Then A has the non-symmetric strong spectral property (nSSP) if $X = O$ is the only real matrix that satisfies

$$A \circ X = [A, X^\top] = O.$$

Let $\mathcal{Q}^{\mathrm{v}}(P)$ be the set of matrices with the same zero entries as P.

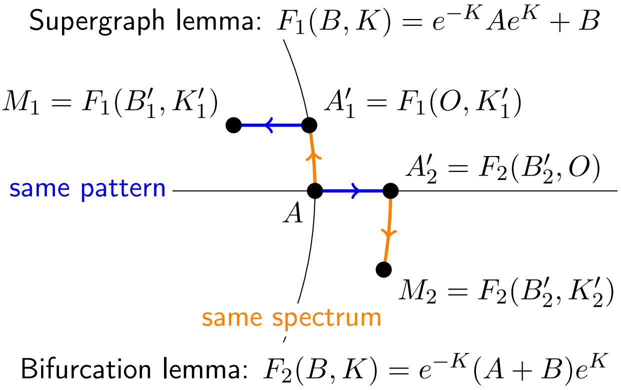
Let $F(B, Q) = Q^{-1}(A + B)Q$, where $B \in \mathcal{Q}^{\mathrm{v}}(P)$. Then the following are equivalent:

1. A has the nSSP.
2. $\mathcal{Q}^{\mathrm{v}}(P) + \{-LA + AL : L \in \mathrm{Mat}_n(\mathbb{R})\} = \mathrm{Mat}_n(\mathbb{R})$.
3. The derivative $\dot{F}$ is surjective.

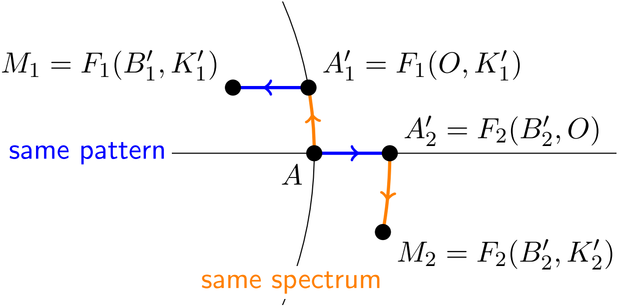
Bifurcation lemma for non-symmetric matrices

Theorem (FHLS 2022)

Let $A \in \mathcal{Q}(P)$ *with the nSSP for some sign pattern* P*. Then for any set of complex numbers* Λ' *(invariant under conjugation) nearby* $\operatorname{spec}(A)$*, there is a matrix* $A' \in \mathcal{Q}(P)$ *with* $\operatorname{spec}(A') = \Lambda'$*.*

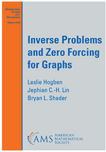

Supergraph lemma: $F_1(B,K) = e^{-K}Ae^{K} + B$
$M_1 = F_1(B_1', K_1')$
$A_1' = F_1(O, K_1')$
$A_2' = F_2(B_2', O)$
same pattern
A
$M_2 = F_2(B_2', K_2')$
same spectrum
Bifurcation lemma: $F_2(B,K) = e^{-K}(A+B)e^{K}$

Supergraph lemma: $F_1(B,K) = e^{-K}Ae^{K} + B$



Bifurcation lemma: $F_2(B,K) = e^{-K}(A+B)e^{K}$

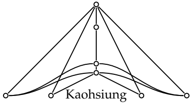
For more IEP-G...



Thanks!

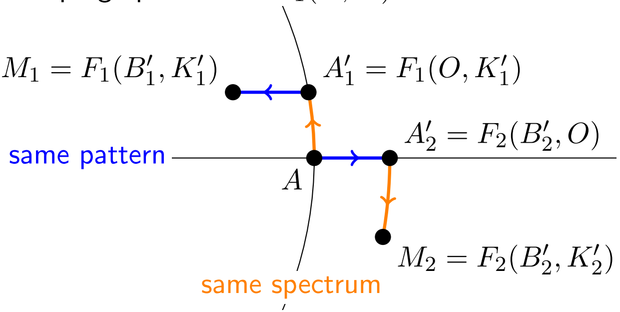
ILAS2025
https://ilas2025.tw/

ILAS2025 in Taiwan



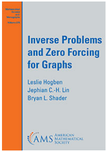
https://ilas2025.tw/

Supergraph lemma: $F_1(B, K) = e^{-K} A e^{K} + B$



Bifurcation lemma: $F_2(B, K) = e^{-K}(A + B)e^{K}$

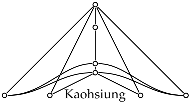
For more IEP-G...



Thanks!

ILAS2025
https://ilas2025.tw/

ILAS2025 in Taiwan



https://ilas2025.tw/

References I

W. Barrett, S. M. Fallat, H. T. Hall, L. Hogben, J. C.-H. Lin, and B. Shader.
Generalizations of the Strong Arnold Property and the minimum number of distinct eigenvalues of a graph.
Electron. J. Combin., 24:#P2.40, 2017.

L. Hogben, J. C.-H. Lin, and B. Shader.
Inverse Problems and Zero Forcing for Graphs.
American Mathematical Society, Providence, 2022.