# Inverse Fiedler vector problem of a graph

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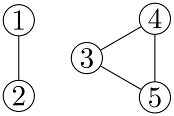
# Laplacian matrix

## Definition

Let $G$ be a graph on $n$ vertices. The Laplacian matrix of $G$ is the $n \times n$ matrix $L(G) = [\ell_{i,j}]$ such that

$$\ell_{i,j} = \begin{cases} -1 & \text{if } \{i,j\} \in E(G), \\ \deg_G(i) & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$




$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$

# Algebraic connectivity and Fiedler vector

**Definition**

Let $G$ be a grpah and $L$ its Laplacian matrix. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of $L$ and $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ the corresponding eigenbasis. Then $\lambda_2$ is the algebraic connectivity and $\mathbf{v}_2$ is the Fiedler vector of $G$.

- $\lambda_1 = 0$ and $\mathbf{v}_1 = \mathbf{1}$ for any graph.
- $L$ is PSD.
- $\mathrm{null}(L) = \#$ of components of $G$, so $\lambda_2 > 0 \iff G$ is connected.
- $\lambda_2(G) \leq \kappa(G)$, the vertex connectivity.

**Notes**

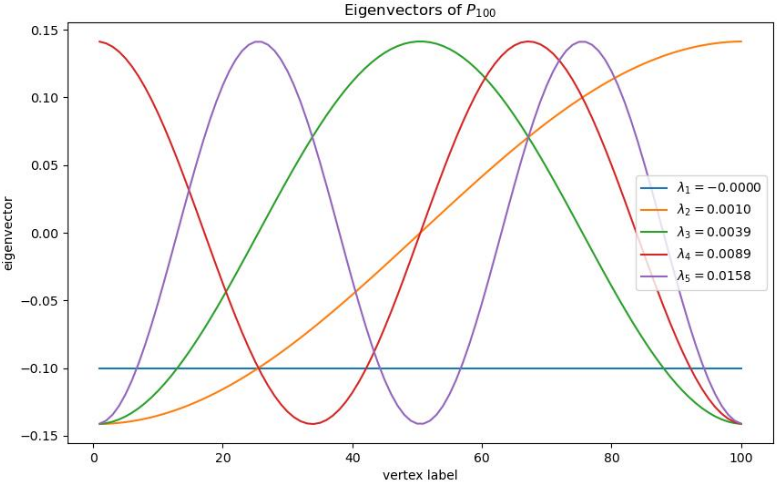
- Fiedler introduced the algebraic connectivity in 1973.
- Fiedler called $\mathbf{v}_2$ as characteristic valuation in 1975.
- Fiedler visited NSYSU for WONRA2012.

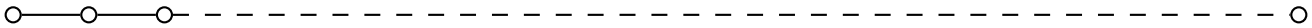
Eleventh Workshop on Numerical Ranges and Numerical Radii (WONRA 2012)

July 9-12, 2012, National Sun Yat-sen University, Kaohsiung, Taiwan

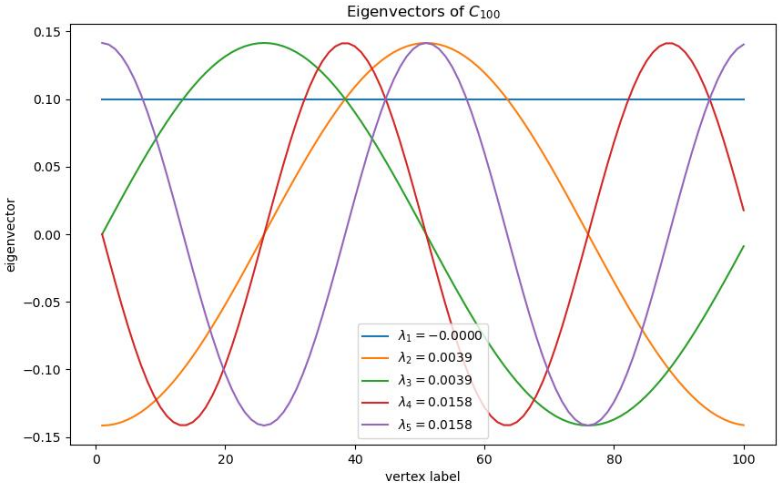
Source: `https://www-math.nsysu.edu.tw/~wong/WONRA2012/`

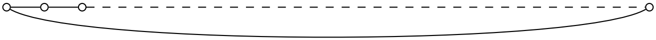
# Why Fiedler vector? $P_n$



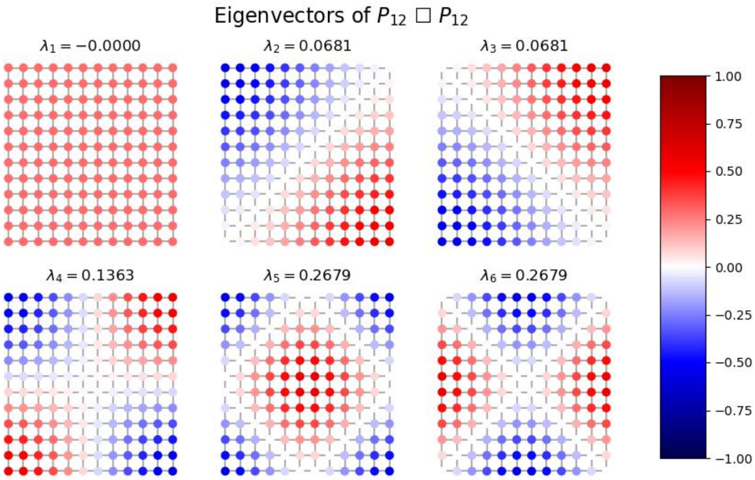



# Why Fiedler vector? $C_n$

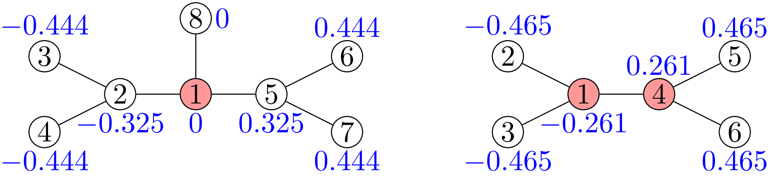


# Why Fiedler vector? $P_m \square P_n$



# Characteristic set: Fiedler vector on a tree



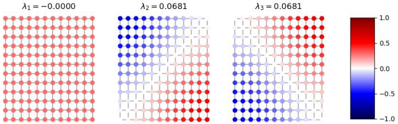

**Theorem (Fiedler 1975)**

*Let $T$ be a tree and $\mathbf{v}_2 = (x_i)$ its Fiedler vector. Then either*

1. *there is a unique vertex $i$ with $x_i = 0$ that is incident to some $j$ with $x_j \neq 0$ (**Type I**), or*
2. *there is a unique edge $\{i, j\}$ such that $x_i x_j < 0$ (**Type II**).*

Either $\{i\}$ or $\{i, j\}$ is called the characteristic set, which is independent of the choice of $\mathbf{v}_2$.

# Courant nodal domain theorem: Laplacian eigenvector on a graph




**Theorem (Courant nodal domain theorem; BGLT 2001)**

*Let $G$ be a connected graph and $\mathbf{v}_2 = (x_i)$ its Fiedler vector. Let*

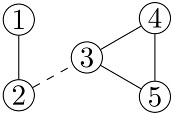
$$N_{\geq 0} = \{i : x_i \geq 0\} \text{ and } N_{\leq 0} = \{i : x_i \leq 0\}.$$

*Then*

$$\# \text{ of components in } G[N_{\geq 0}] + \# \text{ of components in } G[N_{\leq 0}] \leq 2.$$

When $\mathbf{v}_2$ is nowhere zero, we say $\{N_{\geq 0}, N_{\leq 0}\}$ is a spectral bipartition of $G$.

# Weighted Laplacian matrix




$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1.1 & -0.1 & 0 & 0 \\ 0 & -0.1 & 2.1 & -1 & -1 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$

## Definition

Let $G$ be a weighted graph on $n$ vertices with weights $w_{i,j}$. The weighted Laplacian matrix of $G$ is the $n \times n$ matrix $L(G) = \left[\ell_{i,j}\right]$ such that

$$\ell_{i,j} = \begin{cases} -w_{i,j} & \text{if } \{i,j\} \in E(G), \\ \sum_{k:k\sim i} w_{i,k} & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

# Weighted Laplacian matrix

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Let $G$ be a grpah and $L$ its weighted Laplacian matrix. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of $L$ and $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ the corresponding eigenbasis. Then $\lambda_2$ is the algebraic connectivity and $\mathbf{v}_2$ is the Fiedler vector of the weighted graph.

- $\lambda_1 = 0$ and $\mathbf{v}_1 = \mathbf{1}$ for any graph.
- $L$ is PSD.
- $\mathrm{null}(L) = \#$ of components of $G$, so $\lambda_2 > 0 \iff G$ is connected.
- ~~$\lambda_2(G) \leq \kappa(G)$, the vertex connectivity.~~

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- Characteristic set is still valid.
- Courant nodal domain theorem is still valid.
- Colin de Verdière parameter $\mu(G) \sim$ maximum multiplicity of the algebraic connectivity over all "good" discrete Schrödinger operators.

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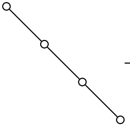
# If we consider all possible weight assignments ...

- For a tree, can the characterisitc set be anywhere?
- For a graph, can the spectral bipartition be any partition of the vertex set?
- For a graph, can any vector be the Fiedler vector of some weighted Laplacian matrix?

# Inverse Fiedler vector problem of Laplacian matrices (IFPL)

Let $G$ be a graph. Define $\mathcal{S}_L(G)$ as the family of all weighted Laplacian matrices $A = \left[a_{ij}\right]$.



$$\longrightarrow \begin{bmatrix} a & -a & 0 & 0 \\ -a & a+b & -b & 0 \\ 0 & -b & b+c & -c \\ 0 & 0 & -c & c \end{bmatrix} \underset{?}{\overset{\text{spectral decomposition}}{\rightleftarrows}} \lambda_2,\ \mathbf{v}_2$$

IFPL: What are the possible $\lambda_2, \mathbf{v}_2$ of a matrix in $\mathcal{S}_L(G)$?

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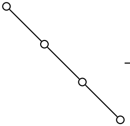

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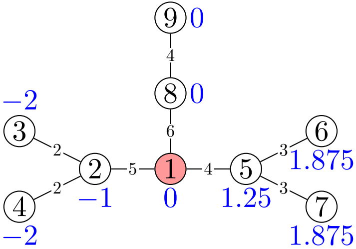
IFPL: What are the possible $\lambda_2, \mathbf{v}_2$ of a matrix in $\mathcal{S}_L(G)$?

Let $T$ be a tree and $L$ its weighted Laplacian matrix corresponding to the weight assignment $\mathbf{w}$. Let $\mathbf{x}$ the Fiedler vector. Then exactly one of the following two cases will occur:

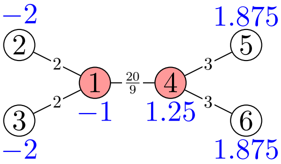
**Type I** Some entries of $\mathbf{x}$ are zero. In this case, there is exactly a vertex $i$ with $x_i = 0$ that is adjacent some vertex $j$ with $x_j \neq 0$. Moreover, for any path in $T$ starting at $i$, the values of $\mathbf{x}$ along the path is either strictly increasing, strictly decreasing, or constantly zero. We say $\{i\}$ is the characteristic set of $(T, \mathbf{w})$.



# Fiedler's theorem

Let $T$ be a tree and $L$ its weighted Laplacian matrix corresponding to the weight assignment $\mathbf{w}$. Let $\mathbf{x}$ the Fiedler vector. Then exactly one of the following two cases will occur:

**Type II** No entry of $\mathbf{x}$ is zero. In this case, there is exactly an edge $\{i,j\}$ with $x_ix_j < 0$, say $x_i < 0 < x_j$. Moreover, for any path starting at $i$ without passing $j$, the values of $\mathbf{x}$ along the path is strictly decreasing; for any path starting at $j$ without passing $i$, the values of $\mathbf{x}$ along the path is strictly increasing. We say $\{i,j\}$ is the characteristic set of $(T,\mathbf{w})$.



# Fiedler-like vector

## Definition

Let $T$ be a tree on $n$ vertices. A vector $\mathbf{x} = [x_i] \in \mathbb{R}^{V(T)}$ is said to be Fiedler-like with respect to $T$ if $\mathbf{1}^\top \mathbf{x} = 0$ and one of the following two conditions holds:

- **Type I** There is exactly a vertex $i$ with $x_i = 0$ that is adjacent to some vertex $j$ with $x_j \neq 0$. And for any path in $T$ starting at $i$, the values of $\mathbf{x}$ along the path is either strictly increasing, strictly decreasing, or constantly zero.
- **Type II** There is exactly an edge $\{i, j\}$ with $x_i x_j < 0$, say $x_i < 0 < x_j$. And for any path starting at $i$ without passing $j$, the values of $\mathbf{x}$ along the path is strictly decreasing; for any path starting at $j$ without passing $i$, the values of $\mathbf{x}$ along the path is strictly increasing.

# General observations

- Given $G$ and $\mathbf{x}$, we want to solve $A\mathbf{x} = \lambda\mathbf{x}$ for some $A \in \mathcal{S}_L(G)$.
- By rescaling $A$, we may assume $\lambda = 1$.
- Every $A \in \mathcal{S}_L(G)$ can be written as $A = NWN^\top$, where $N$ is the vertex-edge incidence matrix.
- To solve $NWN^\top\mathbf{x} = \mathbf{x}$,
  1. Compute $N^\top\mathbf{x}$.
  2. Solve $N\mathbf{y} = \mathbf{x}$ for $\mathbf{y}$.
  3. Solve $W(N^\top\mathbf{x}) = \mathbf{y}$ and get diagonal entries of $W$ to be the entrywise division $\mathbf{y} \oslash (N^\top\mathbf{x})$.
  4. Check if $\lambda = 1$ is the second smallest eigenvalue.

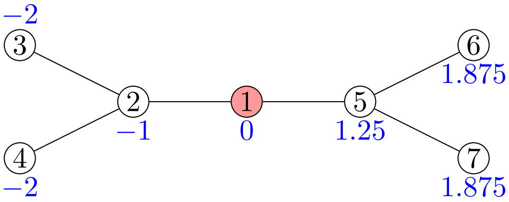
- $N\mathbf{y} = \mathbf{x}$ is solvable if and only if $\mathbf{1}^\top\mathbf{x} = 0$.
- When $G$ is a tree, columns of $N$ are independent and $N\mathbf{y} = \mathbf{x}$ has a unique solusion whenever solvable.

# Inverse Fiedler vector problem of a tree

**Theorem (L and Shirazi 2025+)**

*Let $T$ be a tree. Then $\mathbf{x}$ is a Fiedler vector of $T$ if and only if $\mathbf{x}$ is Fiedler-like with respect to $T$.*

- Recall: weights $= \mathbf{y} \oslash (N^\top \mathbf{x})$.
- $N^\top \mathbf{x}$ is the difference of $\mathbf{x}$ (outer $-$ inner).
- $\mathbf{y}$ is the sum of the branch.



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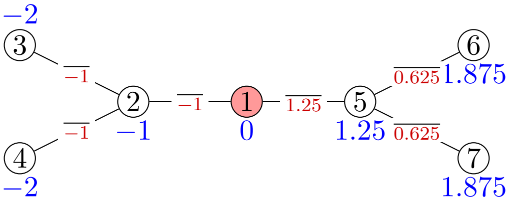

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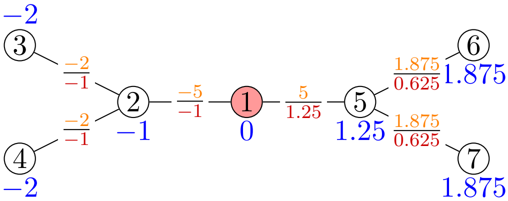

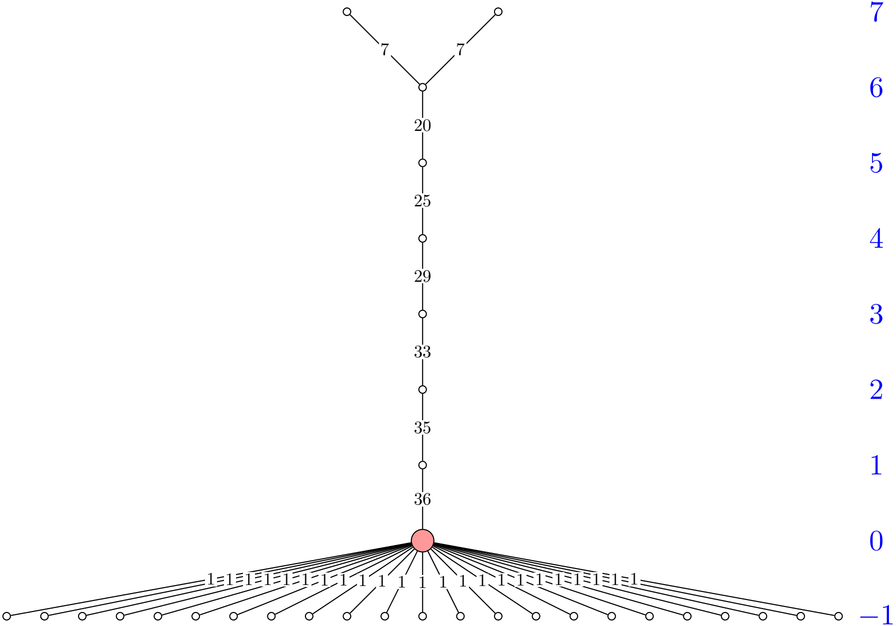

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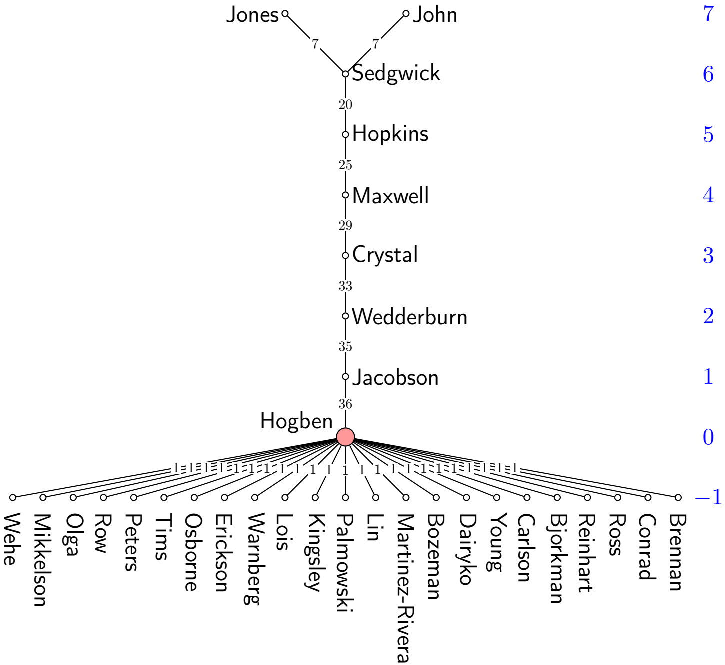

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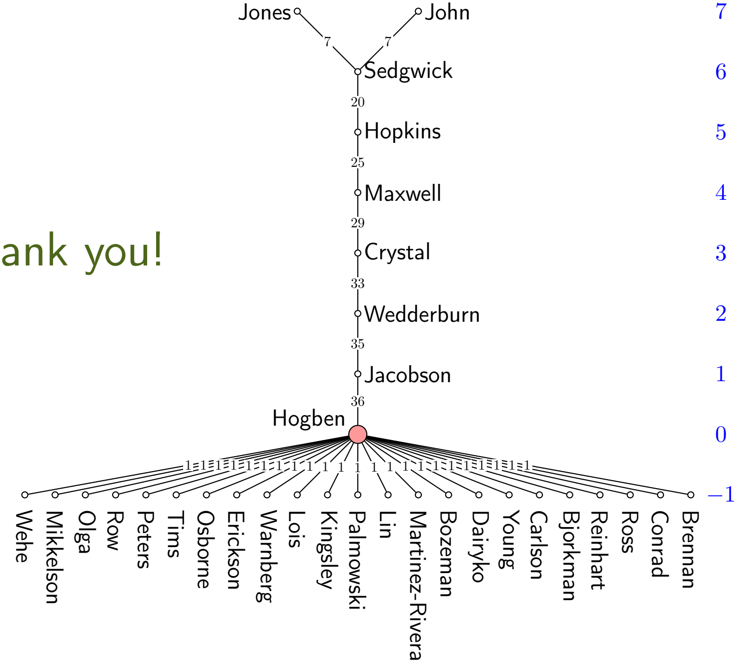
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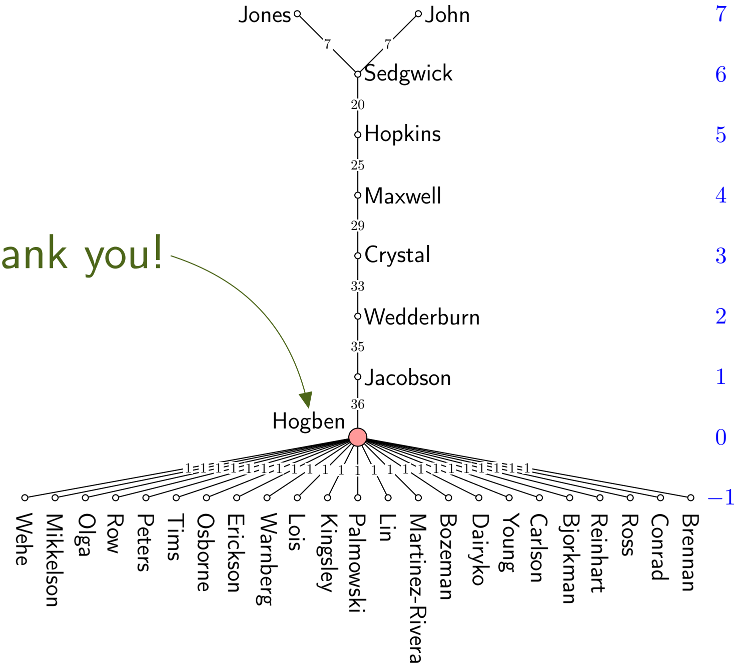

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