Engaging students with collaborative tasks in linear algebra

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Driver's training in Taiwan

Source: Driving-Coach

Driver's training in Taiwan

Only the exam matters.

Source: Driving-Coach

English training in Taiwan

Source: Facebook

English training in Taiwan

Only the standard answer matters.

Source: Facebook

Consequence in higher education in Taiwan

A function $f(x)$ is continuous at a point c means:

Textbook version

For any any $\epsilon > 0$, there is $\delta > 0$ such that $|x - c| < \delta$ implies $|f(x) - f(c)| < \epsilon$.

Student version

$\epsilon > 0$, $\delta > 0$, $|x - c| < \delta$, $|f(x) - f(c)| < \epsilon$

For students, mathematics is more about calculation, but not too much about the logic.

Potential solutions...

More exploratory exercises.

- Learn to find answers in different ways.
- Learn the beauty of open answers.

→ Finding the determinant in a tedious way

More intuition deliveries.

- Learn the meaning beyond the symbols.
- Learn the logic of mathematical settings.

→ Feeling the vector representation

Potential solutions...

More exploratory exercises.

- Learn to find answers in different ways.
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$\rightarrow$ Finding the determinant in a tedious way

More intuition deliveries.

- Learn the meaning beyond the symbols.
- Learn the logic of mathematical settings.

$\rightarrow$ Feeling the vector representation

Finding the determinant in a tedious way

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

$$\det(A) = \cdots + \det \overset{\text{permutation: } 4321}{\begin{bmatrix} 0 & 0 & 0 & 4 \\ 0 & 0 & 7 & 0 \\ 0 & 10 & 0 & 0 \\ 13 & 0 & 0 & 0 \end{bmatrix}} + \cdots$$

$$= \text{sum of } 24 \text{ terms}$$

$$= 0.$$

$\det(A)$ = sum of 24 terms

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

permutation	1234	1243	1324	1342	1423	1432
value	1056					
permutation	2134	2143	2314	2341	2413	2431
value						
permutation	3124	3142	3214	3241	3412	3421
value						
permutation	4123	4132	4213	4231	4312	4321
value						

$\det(A)$ = sum of 24 terms

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

permutation	1234	1243	1324	1342	1423	1432
value	1056	−1080				
permutation	2134	2143	2314	2341	2413	2431
value						
permutation	3124	3142	3214	3241	3412	3421
value						
permutation	4123	4132	4213	4231	4312	4321
value						

$\det(A)$ = sum of 24 terms

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

permutation	1234	1243	1324	1342	1423	1432
value	1056	-1080	-1120			

permutation	2134	2143	2314	2341	2413	2431
value						

permutation	3124	3142	3214	3241	3412	3421
value						

permutation	4123	4132	4213	4231	4312	4321
value						

$\det(A)$ = sum of 24 terms

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

permutation	1234	1243	1324	1342	1423	1432
value	1056	-1080	-1120	1176		

permutation	2134	2143	2314	2341	2413	2431
value						

permutation	3124	3142	3214	3241	3412	3421
value						

permutation	4123	4132	4213	4231	4312	4321
value						

$\det(A)$ = sum of 24 terms

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

permutation	1234	1243	1324	1342	1423	1432
value	1056	-1080	-1120	1176	1200	

permutation	2134	2143	2314	2341	2413	2431
value						

permutation	3124	3142	3214	3241	3412	3421
value						

permutation	4123	4132	4213	4231	4312	4321
value						

$\det(A)$ = sum of 24 terms

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

permutation	1234	1243	1324	1342	1423	1432
value	1056	-1080	-1120	1176	1200	-1232

permutation	2134	2143	2314	2341	2413	2431
value						

permutation	3124	3142	3214	3241	3412	3421
value						

permutation	4123	4132	4213	4231	4312	4321
value						

$\det(A)$ = sum of 24 terms

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

permutation	1234	1243	1324	1342	1423	1432
value	1056	-1080	-1120	1176	1200	-1232

permutation	2134	2143	2314	2341	2413	2431
value	-1760					

permutation	3124	3142	3214	3241	3412	3421
value						

permutation	4123	4132	4213	4231	4312	4321
value						

$\det(A)$ = sum of 24 terms

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

permutation	1234	1243	1324	1342	1423	1432
value	1056	-1080	-1120	1176	1200	-1232

permutation	2134	2143	2314	2341	2413	2431
value	-1760	1800				

permutation	3124	3142	3214	3241	3412	3421
value						

permutation	4123	4132	4213	4231	4312	4321
value						

$\det(A)$ = sum of 24 terms

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permutation	1234	1243	1324	1342	1423	1432
value	1056	-1080	-1120	1176	1200	-1232

permutation	2134	2143	2314	2341	2413	2431
value	-1760	1800	2016			

permutation	3124	3142	3214	3241	3412	3421
value						

permutation	4123	4132	4213	4231	4312	4321
value						

$\det(A)$ = sum of 24 terms

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permutation	1234	1243	1324	1342	1423	1432
value	1056	-1080	-1120	1176	1200	-1232

permutation	2134	2143	2314	2341	2413	2431
value	-1760	1800	2016	-2184		

permutation	3124	3142	3214	3241	3412	3421
value						

permutation	4123	4132	4213	4231	4312	4321
value						

$\det(A)$ = sum of 24 terms

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

permutation	1234	1243	1324	1342	1423	1432
value	1056	-1080	-1120	1176	1200	-1232
permutation	2134	2143	2314	2341	2413	2431
value	-1760	1800	2016	-2184	-2160	

permutation	3124	3142	3214	3241	3412	3421
value						
permutation	4123	4132	4213	4231	4312	4321
value						

$\det(A)$ = sum of 24 terms

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

permutation	1234	1243	1324	1342	1423	1432
value	1056	-1080	-1120	1176	1200	-1232
permutation	2134	2143	2314	2341	2413	2431
value	-1760	1800	2016	-2184	-2160	2288
permutation	3124	3142	3214	3241	3412	3421
value						
permutation	4123	4132	4213	4231	4312	4321
value						

$\det(A)$ = sum of 24 terms

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

permutation	1234	1243	1324	1342	1423	1432
value	1056	−1080	−1120	1176	1200	−1232
permutation	2134	2143	2314	2341	2413	2431
value	−1760	1800	2016	−2184	−2160	2288
permutation	3124	3142	3214	3241	3412	3421
value	2400					
permutation	4123	4132	4213	4231	4312	4321
value						

$\det(A)$ = sum of 24 terms

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

permutation	1234	1243	1324	1342	1423	1432
value	1056	−1080	−1120	1176	1200	−1232
permutation	2134	2143	2314	2341	2413	2431
value	−1760	1800	2016	−2184	−2160	2288
permutation	3124	3142	3214	3241	3412	3421
value	2400	−2520				
permutation	4123	4132	4213	4231	4312	4321
value						

$\det(A)$ = sum of 24 terms

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

permutation	1234	1243	1324	1342	1423	1432
value	1056	−1080	−1120	1176	1200	−1232
permutation	2134	2143	2314	2341	2413	2431
value	−1760	1800	2016	−2184	−2160	2288
permutation	3124	3142	3214	3241	3412	3421
value	2400	−2520	−2592			
permutation	4123	4132	4213	4231	4312	4321
value						

$\det(A)$ = sum of 24 terms

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

permutation	1234	1243	1324	1342	1423	1432
value	1056	−1080	−1120	1176	1200	−1232
permutation	2134	2143	2314	2341	2413	2431
value	−1760	1800	2016	−2184	−2160	2288
permutation	3124	3142	3214	3241	3412	3421
value	2400	−2520	−2592	2808		
permutation	4123	4132	4213	4231	4312	4321
value						

$\det(A)$ = sum of 24 terms

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

permutation	1234	1243	1324	1342	1423	1432
value	1056	-1080	-1120	1176	1200	-1232
permutation	2134	2143	2314	2341	2413	2431
value	-1760	1800	2016	-2184	-2160	2288
permutation	3124	3142	3214	3241	3412	3421
value	2400	-2520	-2592	2808	3024	
permutation	4123	4132	4213	4231	4312	4321
value						

$\det(A)$ = sum of 24 terms

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

permutation	1234	1243	1324	1342	1423	1432
value	1056	-1080	-1120	1176	1200	-1232
permutation	2134	2143	2314	2341	2413	2431
value	-1760	1800	2016	-2184	-2160	2288
permutation	3124	3142	3214	3241	3412	3421
value	2400	-2520	-2592	2808	3024	-3120

permutation	4123	4132	4213	4231	4312	4321
value						

$\det(A)$ = sum of 24 terms

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

permutation	1234	1243	1324	1342	1423	1432
value	1056	−1080	−1120	1176	1200	−1232
permutation	2134	2143	2314	2341	2413	2431
value	−1760	1800	2016	−2184	−2160	2288
permutation	3124	3142	3214	3241	3412	3421
value	2400	−2520	−2592	2808	3024	−3120
permutation	4123	4132	4213	4231	4312	4321
value	−3000					

$\det(A)$ = sum of 24 terms

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

permutation	1234	1243	1324	1342	1423	1432
value	1056	−1080	−1120	1176	1200	−1232
permutation	2134	2143	2314	2341	2413	2431
value	−1760	1800	2016	−2184	−2160	2288
permutation	3124	3142	3214	3241	3412	3421
value	2400	−2520	−2592	2808	3024	−3120
permutation	4123	4132	4213	4231	4312	4321
value	−3000	3080				

$\det(A)$ = sum of 24 terms

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

permutation	1234	1243	1324	1342	1423	1432
value	1056	−1080	−1120	1176	1200	−1232
permutation	2134	2143	2314	2341	2413	2431
value	−1760	1800	2016	−2184	−2160	2288
permutation	3124	3142	3214	3241	3412	3421
value	2400	−2520	−2592	2808	3024	−3120
permutation	4123	4132	4213	4231	4312	4321
value	−3000	3080	3240			

$\det(A)$ = sum of 24 terms

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

permutation	1234	1243	1324	1342	1423	1432
value	1056	-1080	-1120	1176	1200	-1232
permutation	2134	2143	2314	2341	2413	2431
value	-1760	1800	2016	-2184	-2160	2288
permutation	3124	3142	3214	3241	3412	3421
value	2400	-2520	-2592	2808	3024	-3120
permutation	4123	4132	4213	4231	4312	4321
value	-3000	3080	3240	-3432		

$\det(A)$ = sum of 24 terms

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

permutation	1234	1243	1324	1342	1423	1432
value	1056	−1080	−1120	1176	1200	−1232
permutation	2134	2143	2314	2341	2413	2431
value	−1760	1800	2016	−2184	−2160	2288
permutation	3124	3142	3214	3241	3412	3421
value	2400	−2520	−2592	2808	3024	−3120
permutation	4123	4132	4213	4231	4312	4321
value	−3000	3080	3240	−3432	−3528	

$\det(A)$ = sum of 24 terms

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

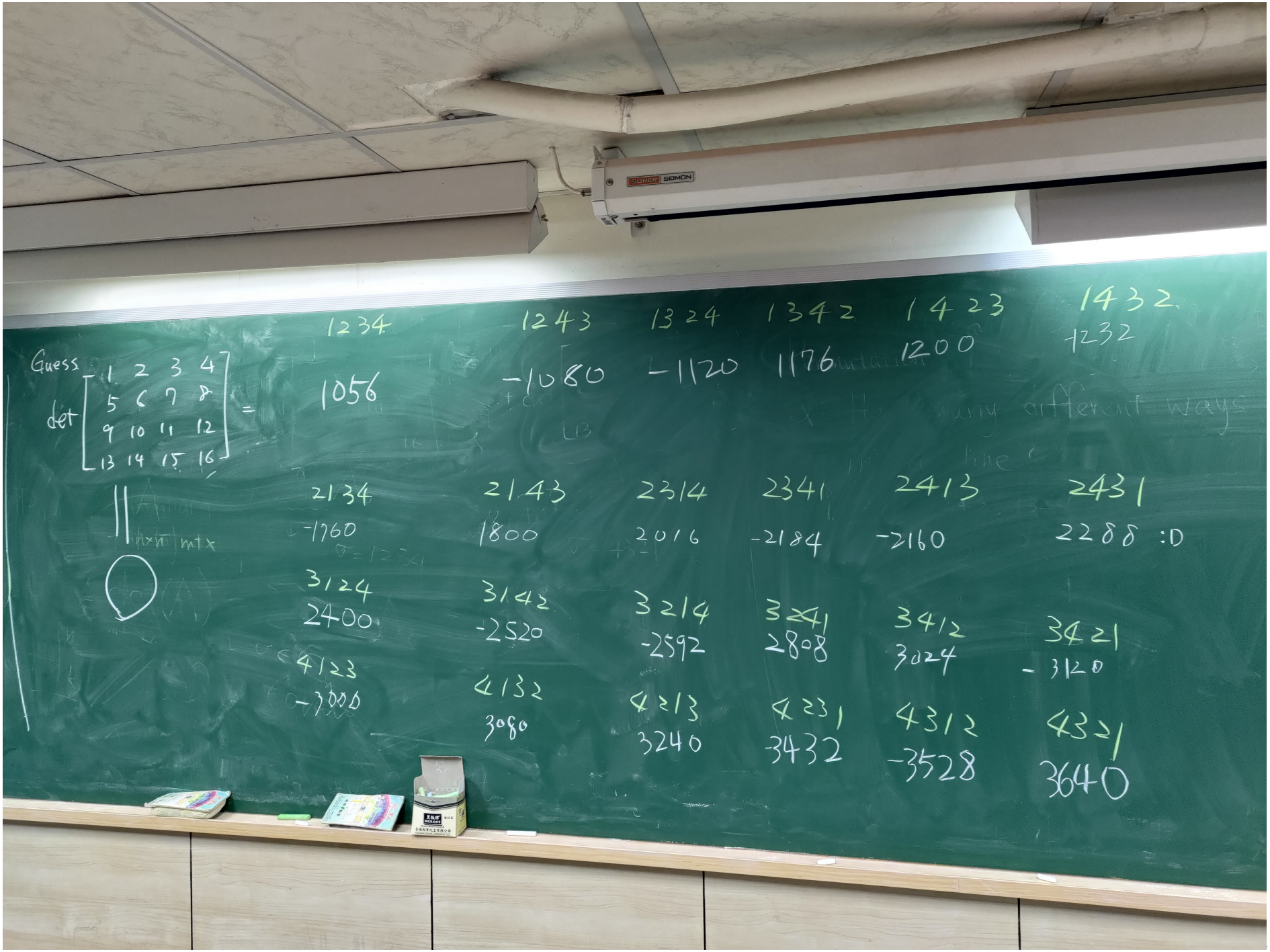
permutation	1234	1243	1324	1342	1423	1432
value	1056	−1080	−1120	1176	1200	−1232
permutation	2134	2143	2314	2341	2413	2431
value	−1760	1800	2016	−2184	−2160	2288
permutation	3124	3142	3214	3241	3412	3421
value	2400	−2520	−2592	2808	3024	−3120
permutation	4123	4132	4213	4231	4312	4321
value	−3000	3080	3240	−3432	−3528	3640

$\det(A)$ = sum of 24 terms

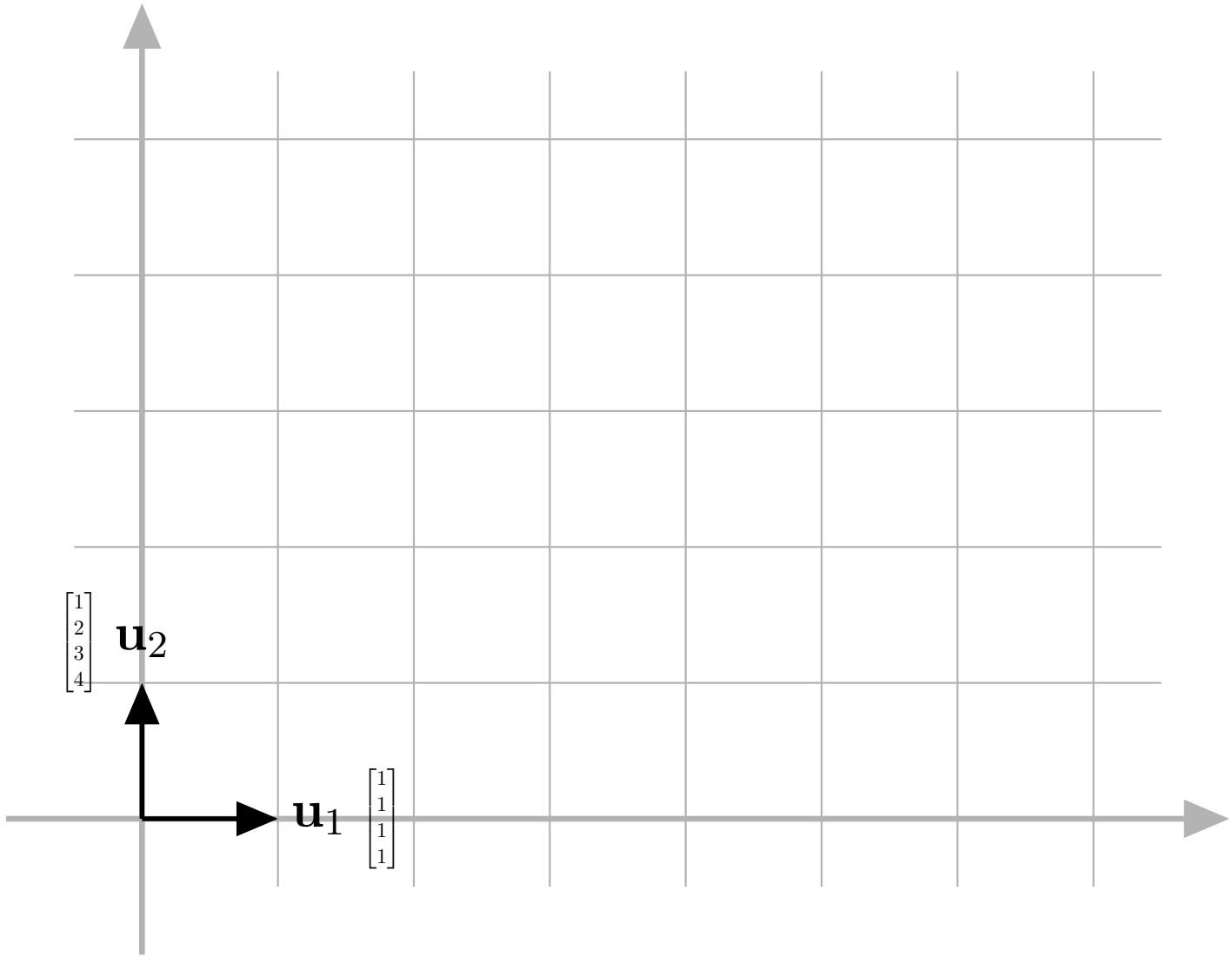
$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

permutation	1234	1243	1324	1342	1423	1432
value	1056	-1080	-1120	1176	1200	-1232
permutation	2134	2143	2314	2341	2413	2431
value	-1760	1800	2016	-2184	-2160	2288
permutation	3124	3142	3214	3241	3412	3421
value	2400	-2520	-2592	2808	3024	-3120
permutation	4123	4132	4213	4231	4312	4321
value	-3000	3080	3240	-3432	-3528	3640

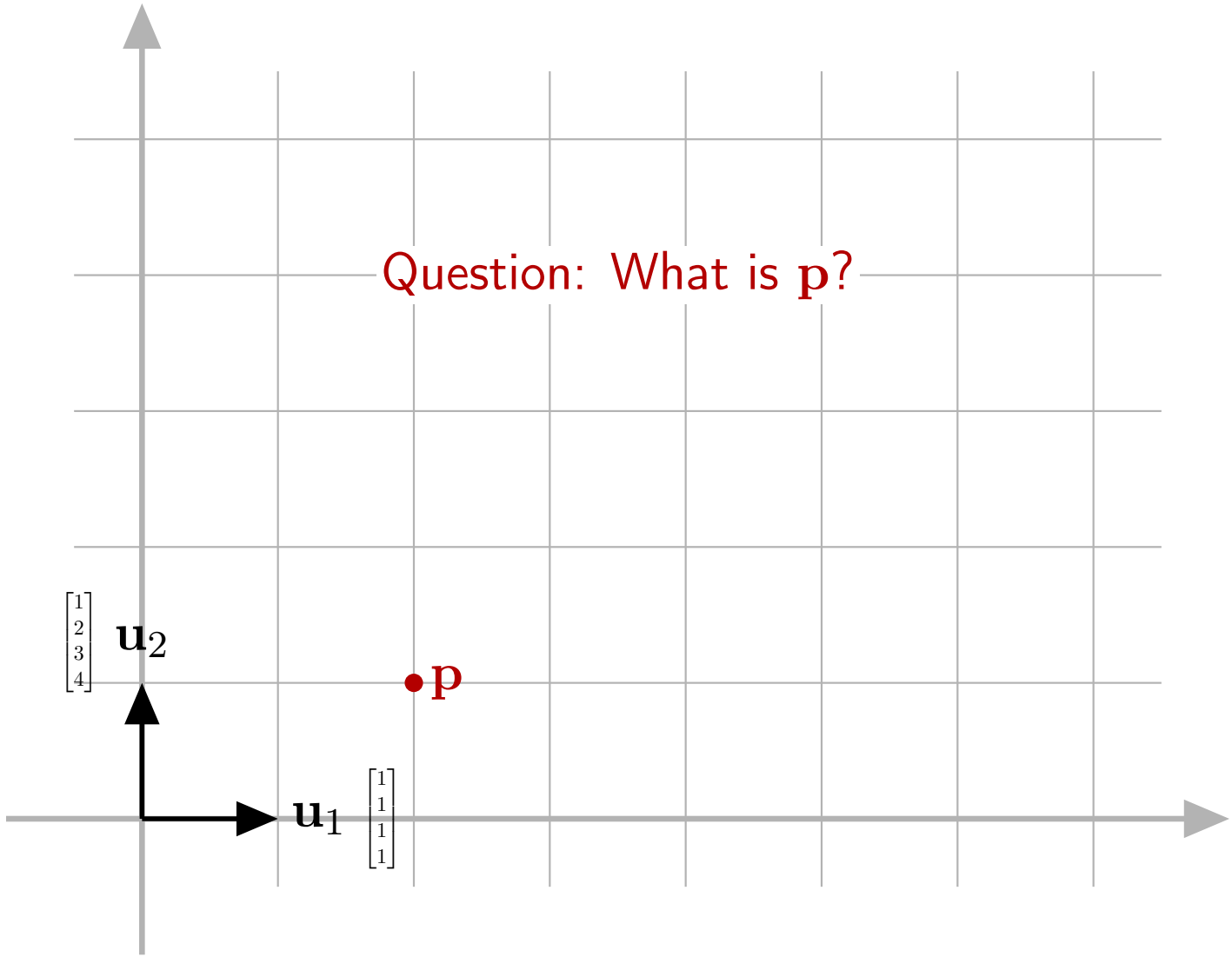
Do they sum up to be 0?


Guess
det
1 2 3 4
5 6 7 8
9 10 11 12
13 14 15 16
=
nxn mtx
1234
1056
1243
-1080
1324
-1120
1342
1176
1423
1200
1432
1232
2134
-1760
2143
1800
2314
2016
2341
-2184
2413
-2160
2431
2288 :D
3124
2400
3142
-2520
3214
-2592
3241
2808
3412
3024
3421
-3120
4123
-3000
4132
3080
4213
3240
4231
3432
4312
-3528
4321
3640

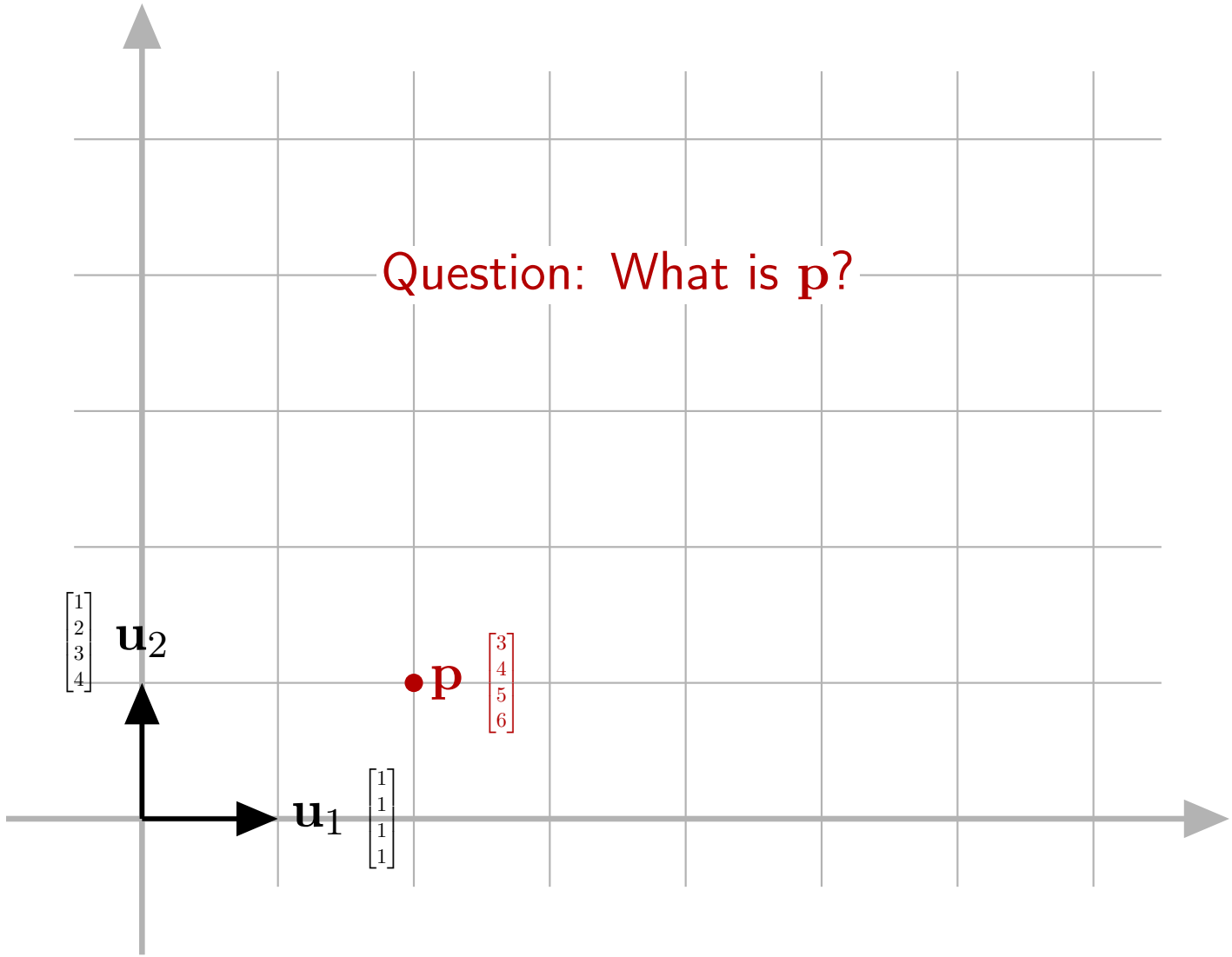
Feeling the vector representation



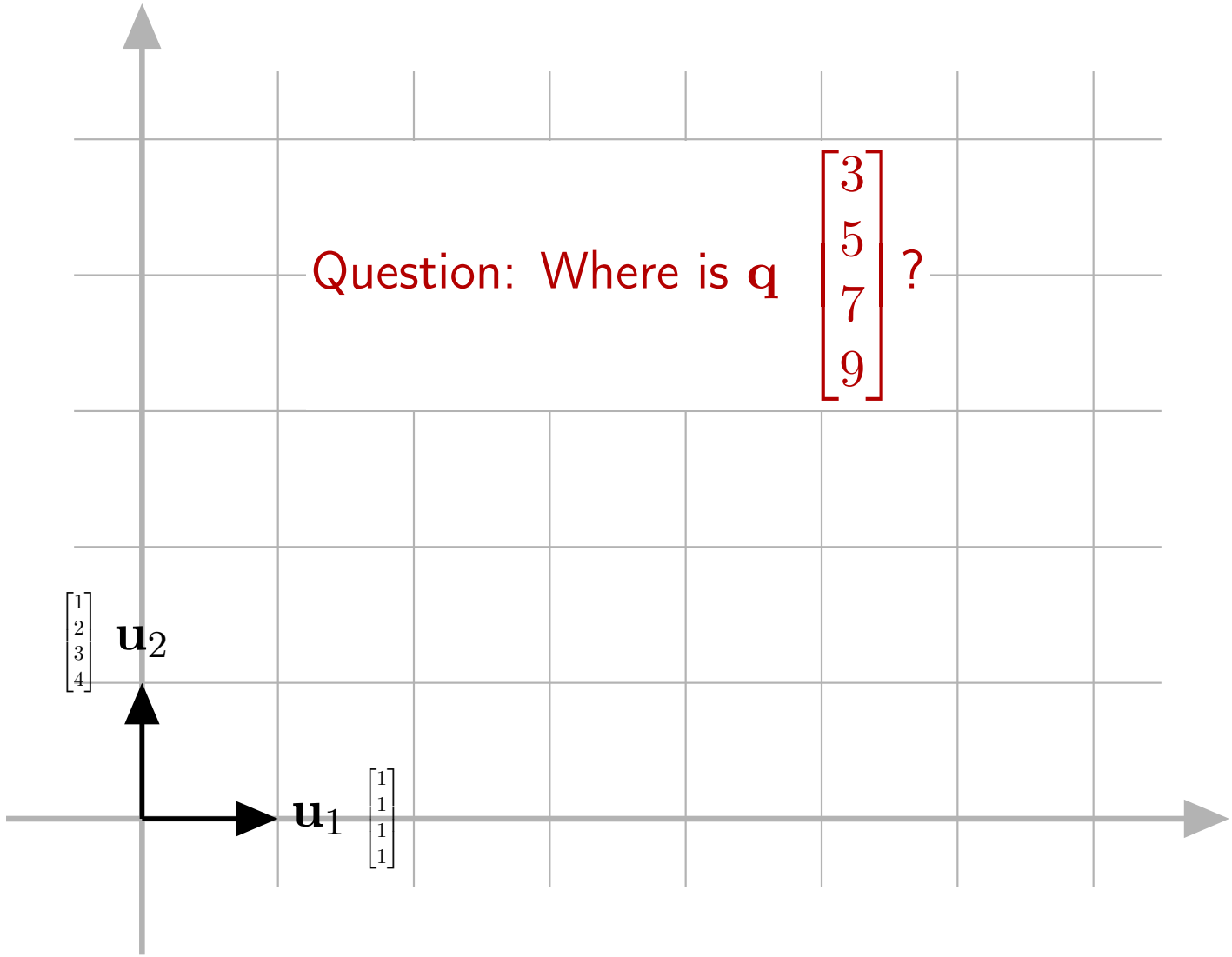
Feeling the vector representation



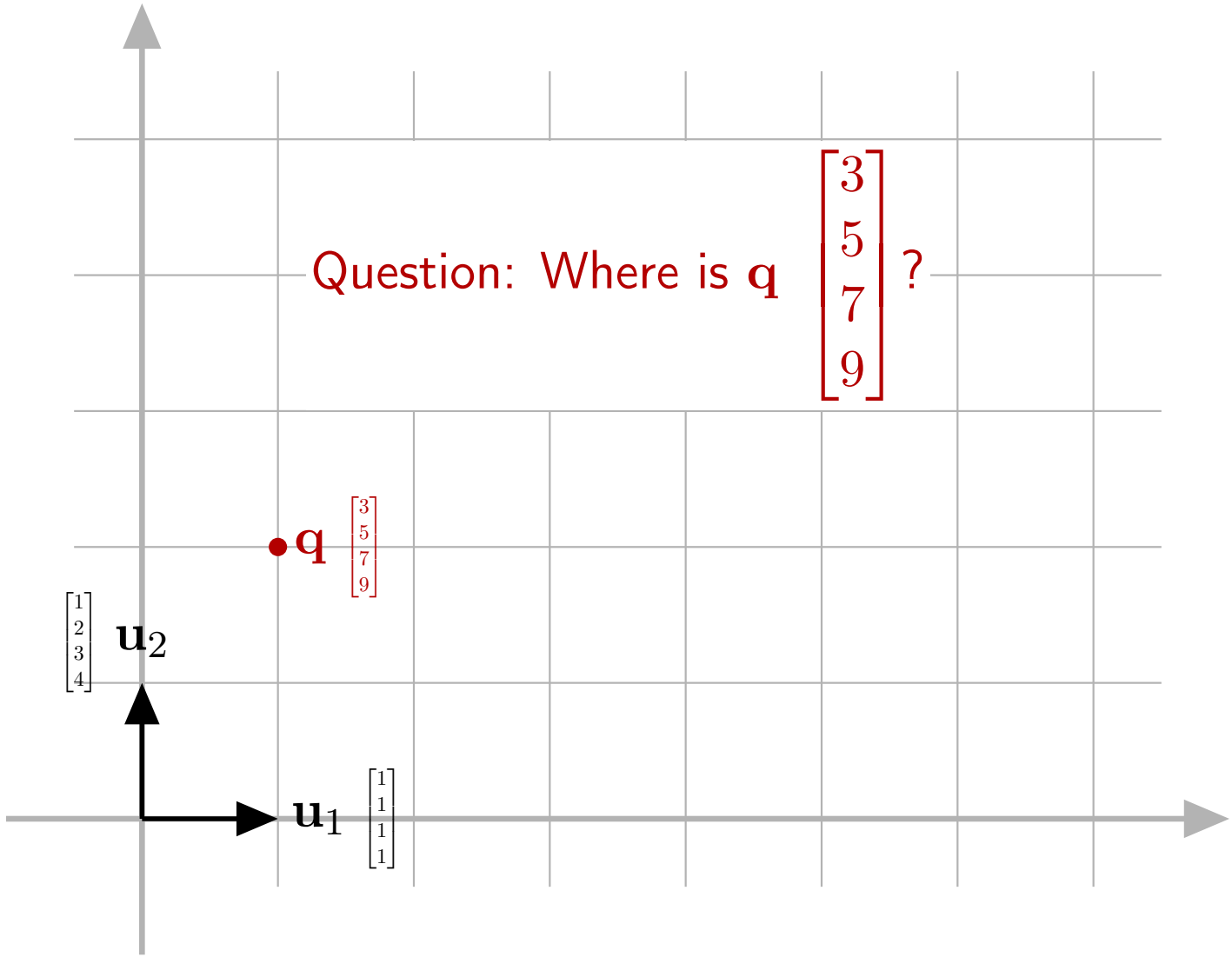
Feeling the vector representation



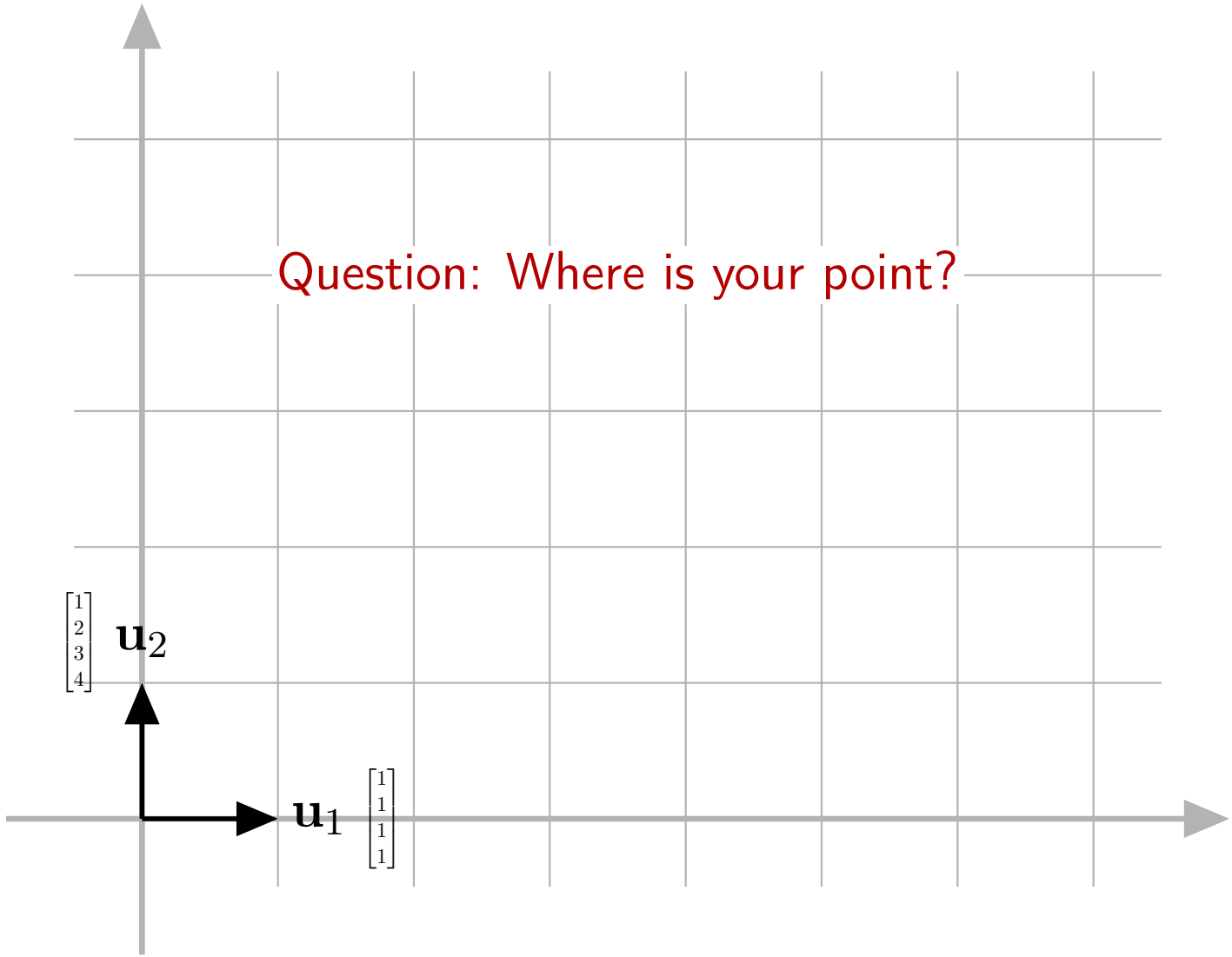
Feeling the vector representation



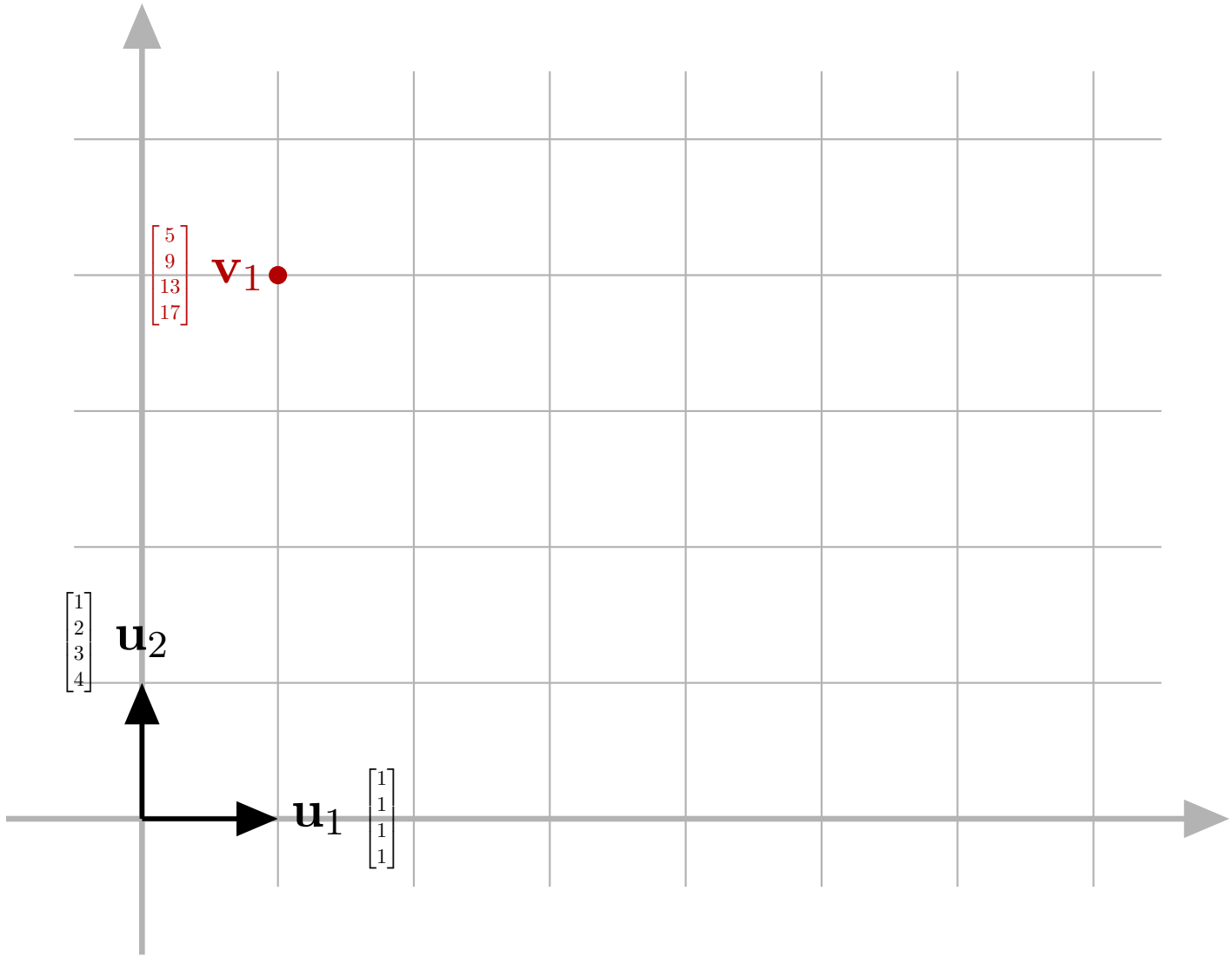
Feeling the vector representation



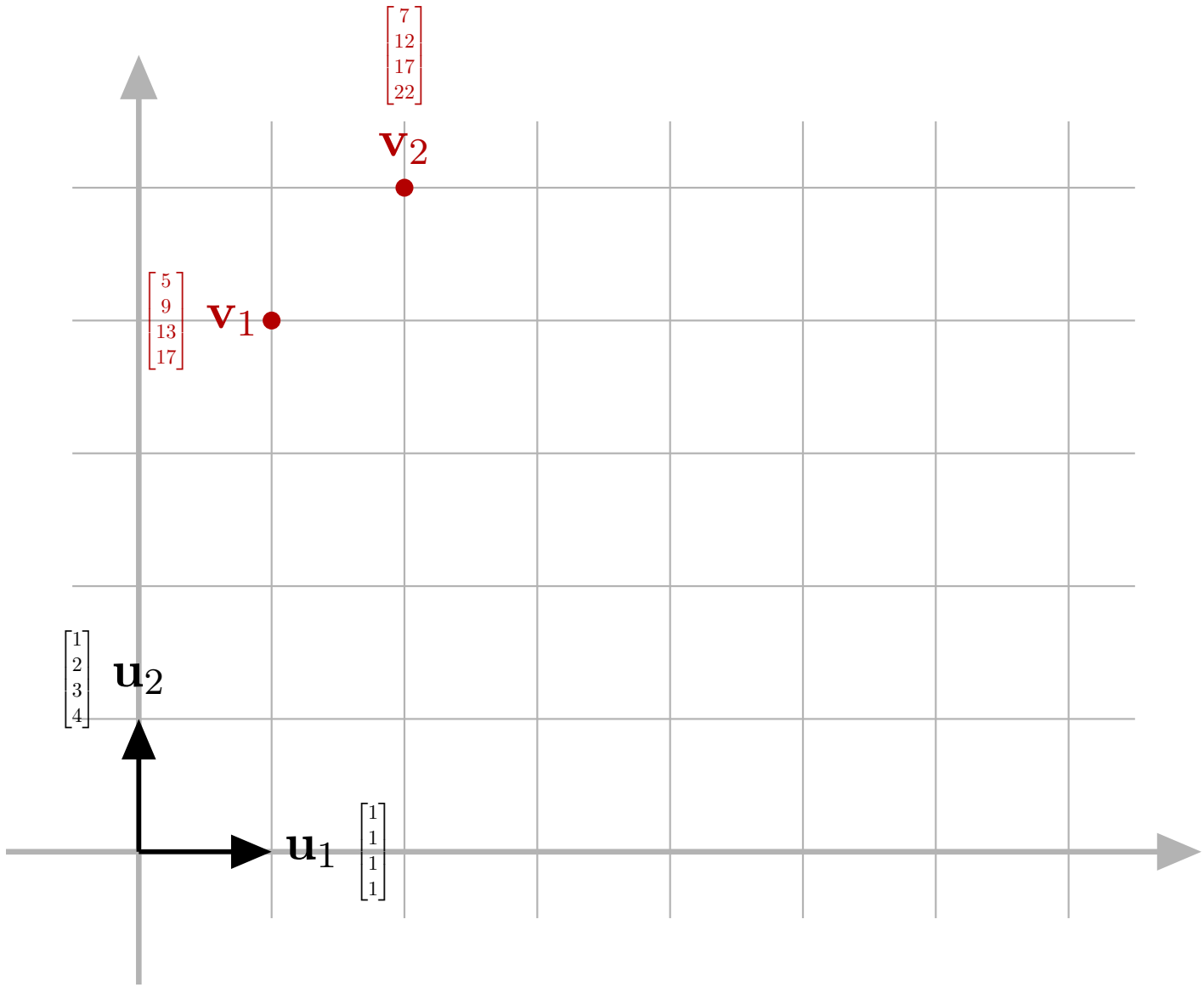
Feeling the vector representation



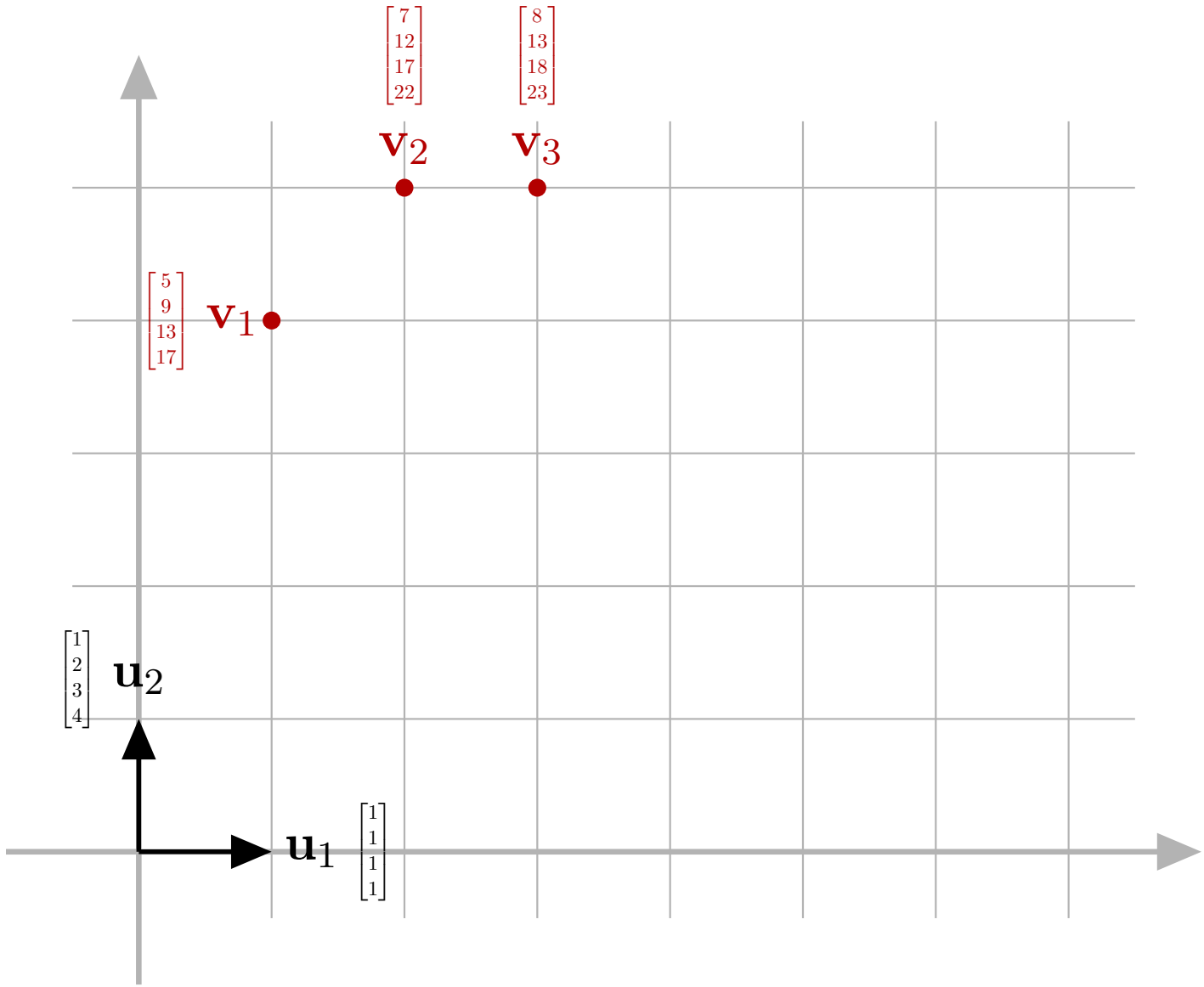
Feeling the vector representation



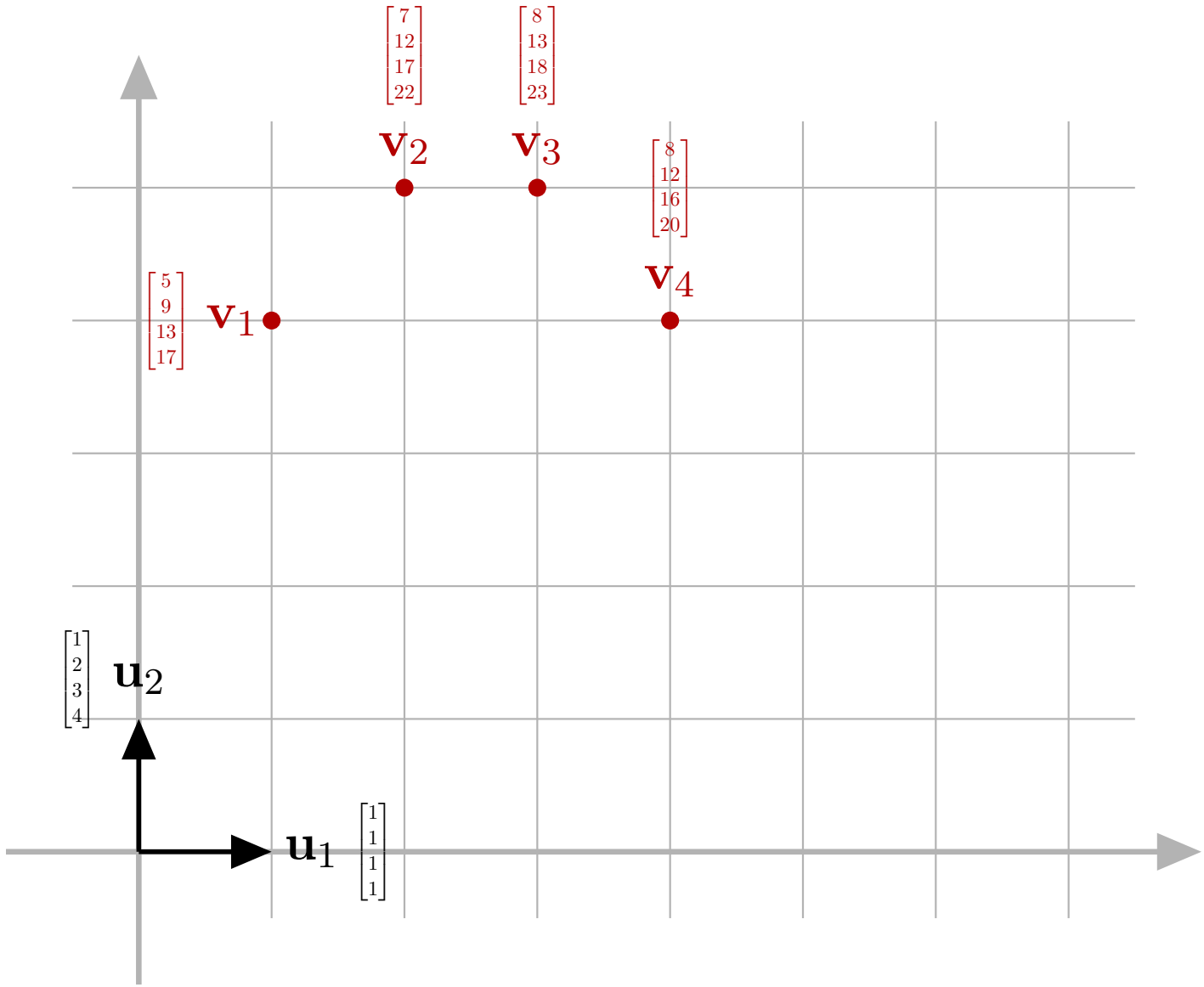
Feeling the vector representation



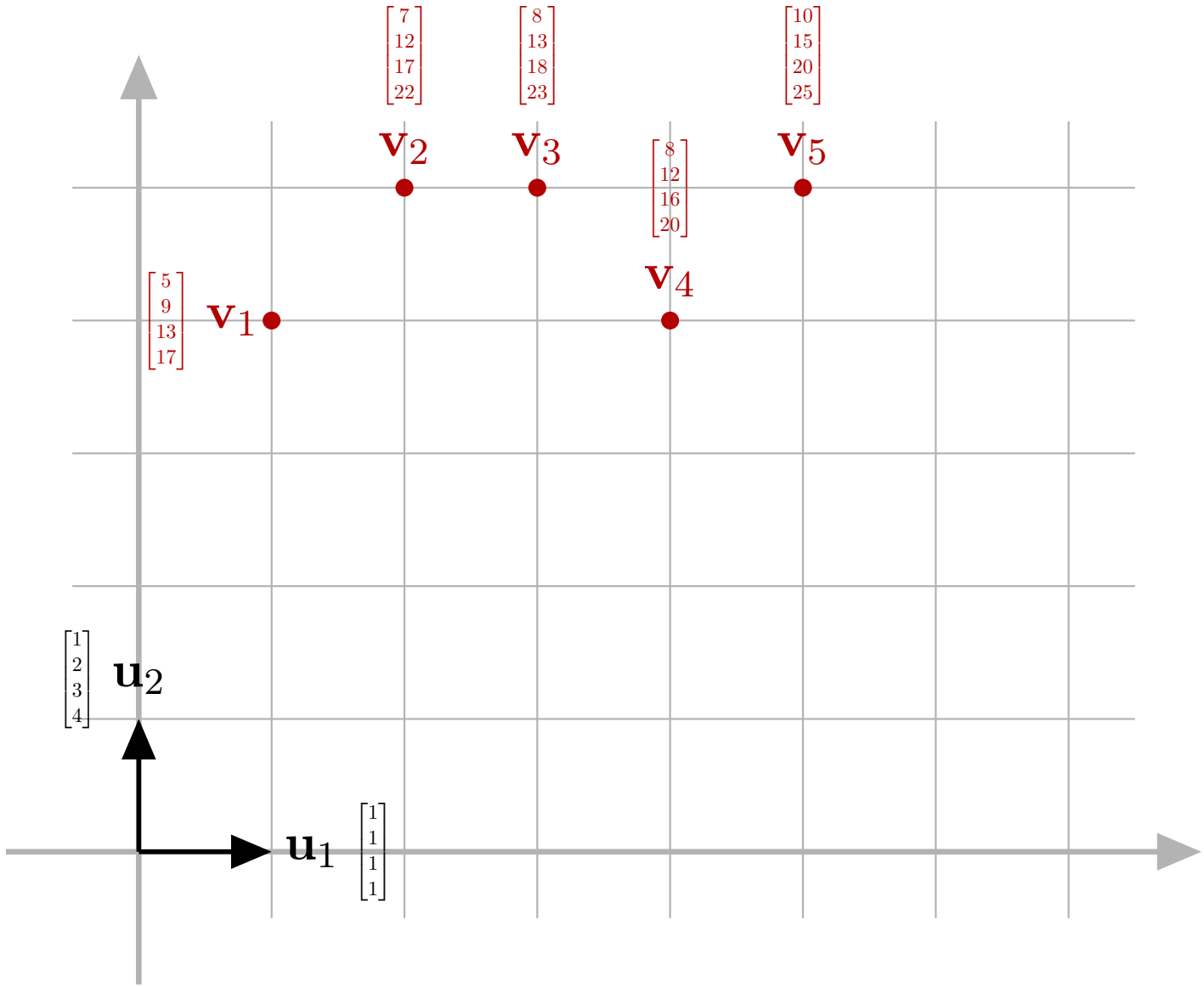
Feeling the vector representation



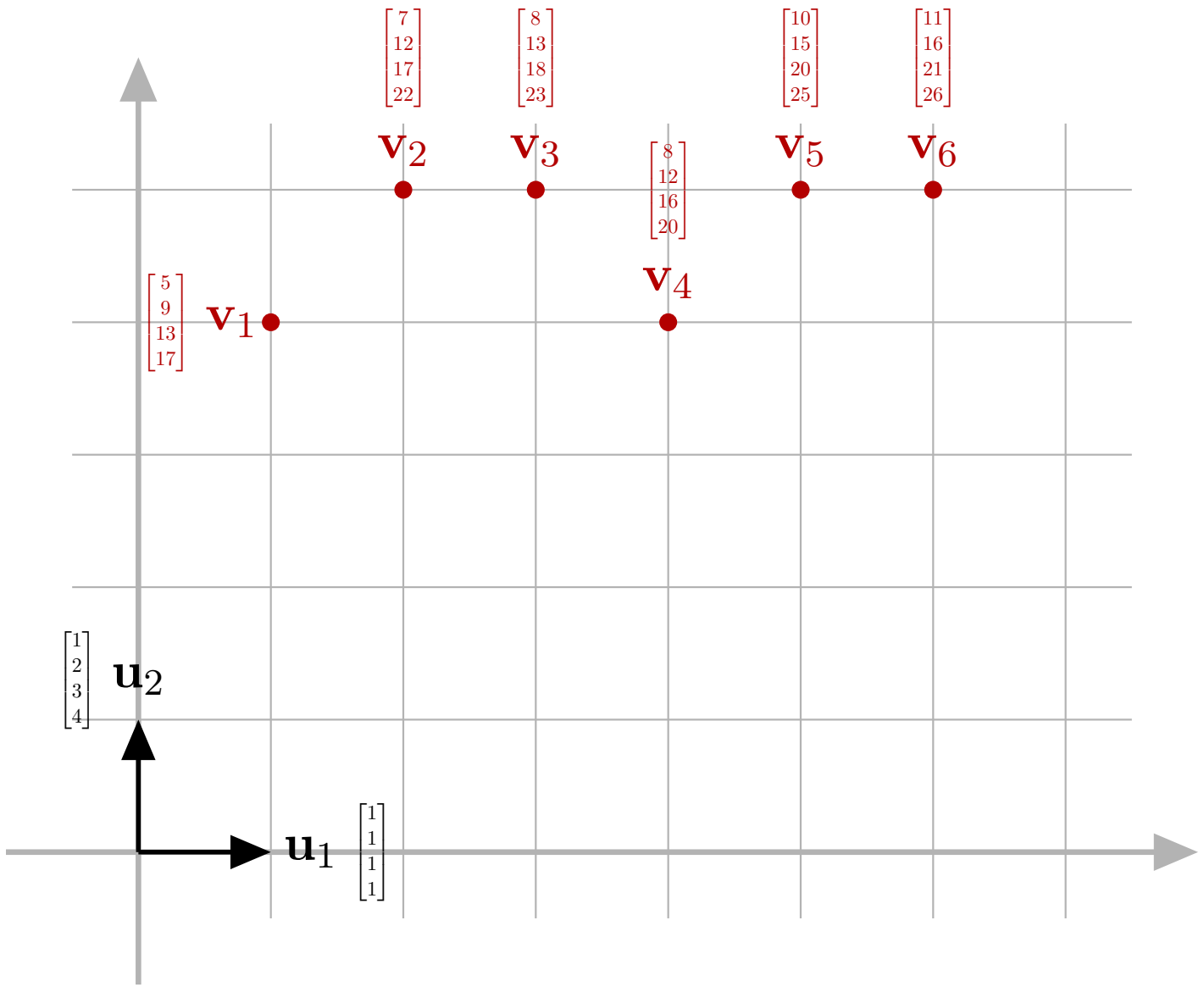
Feeling the vector representation



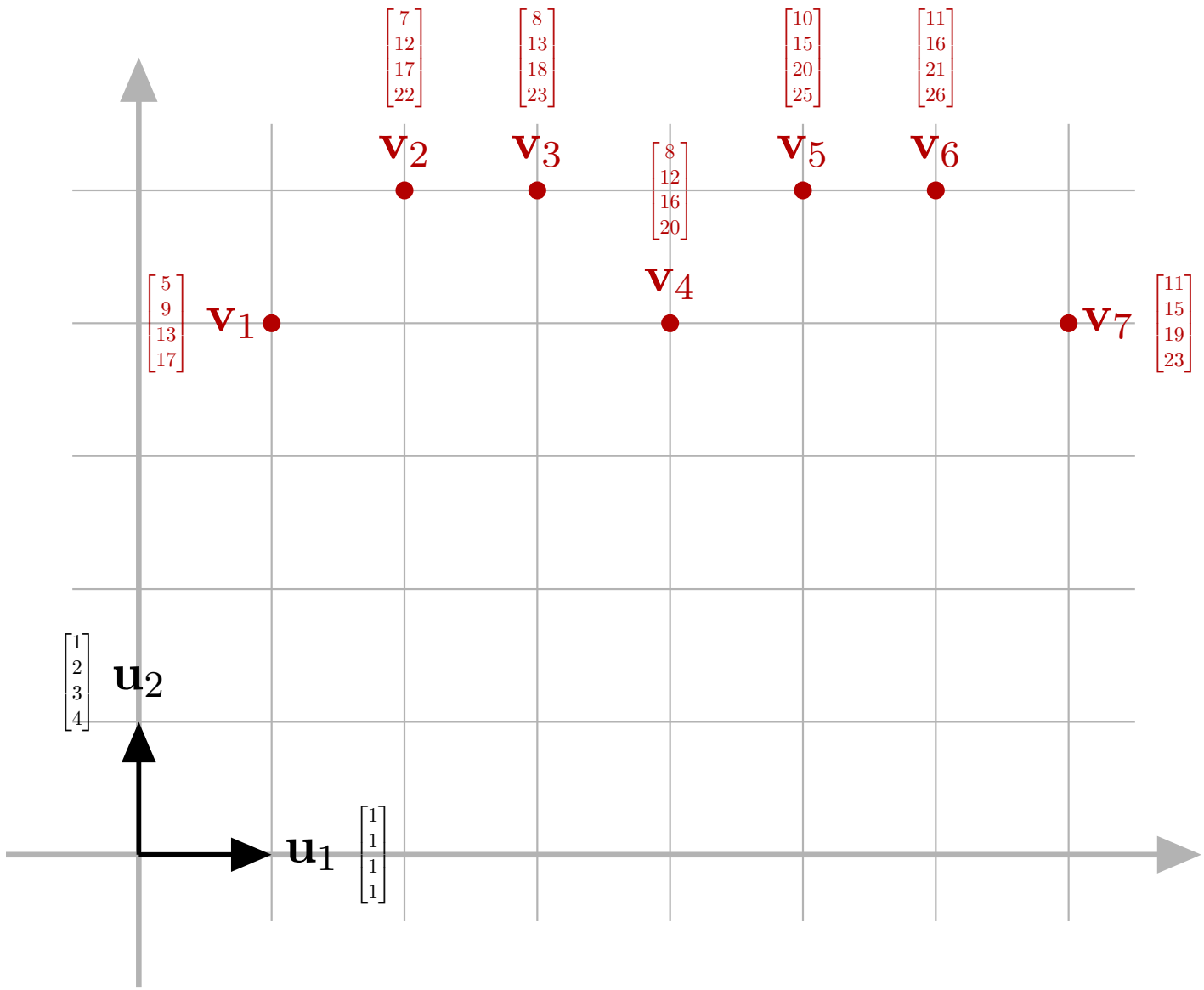
Feeling the vector representation



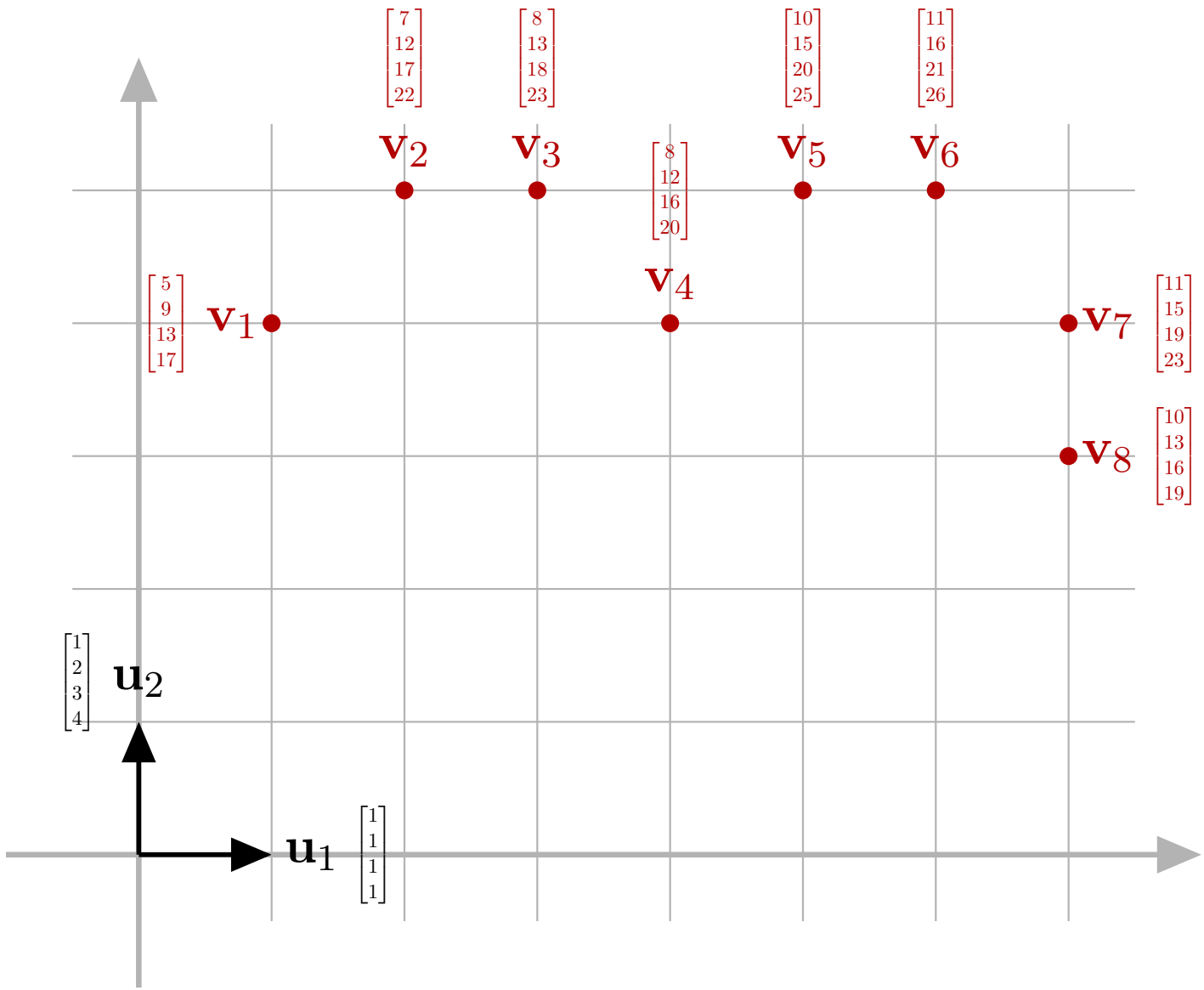
Feeling the vector representation



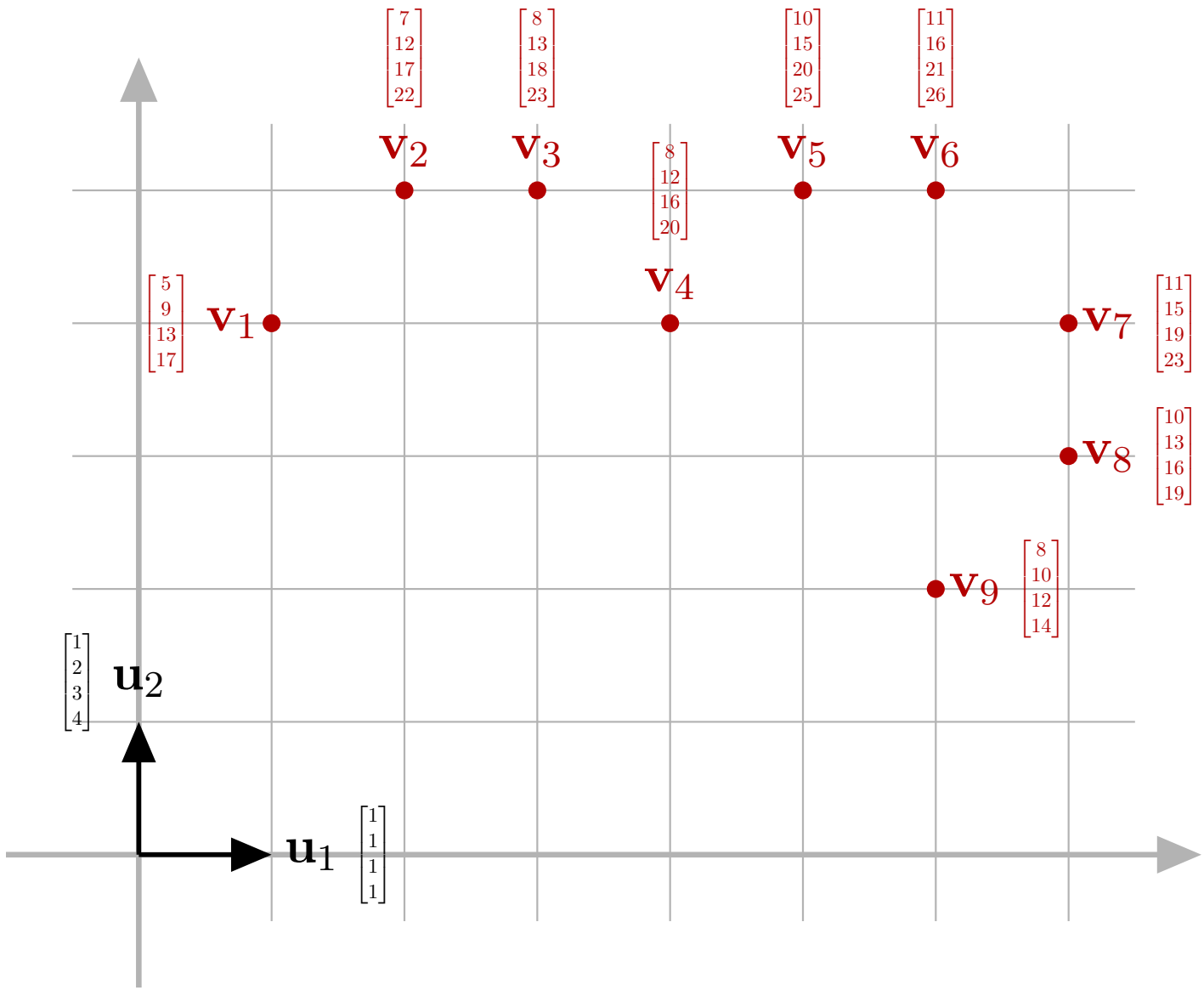
Feeling the vector representation



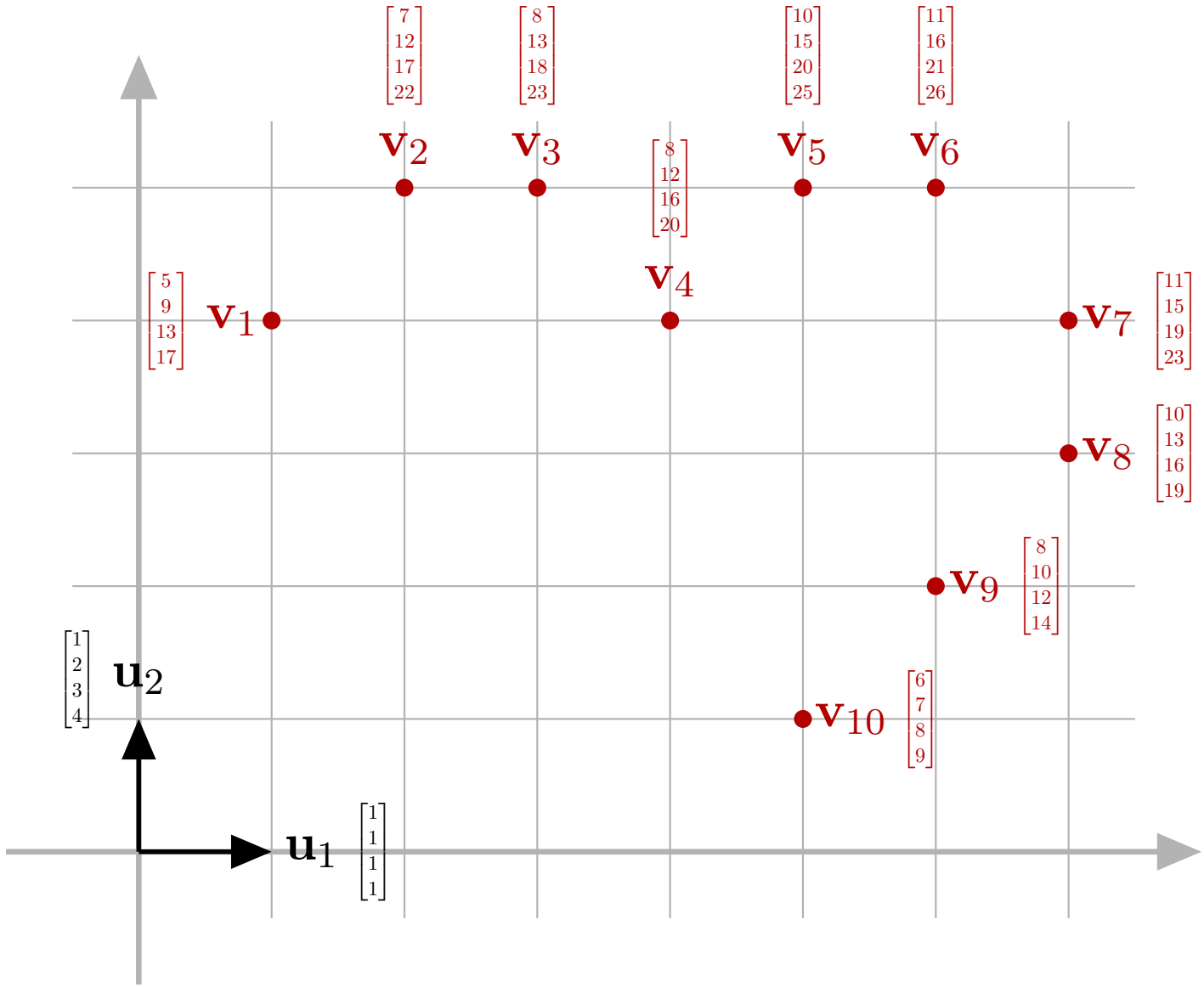
Feeling the vector representation



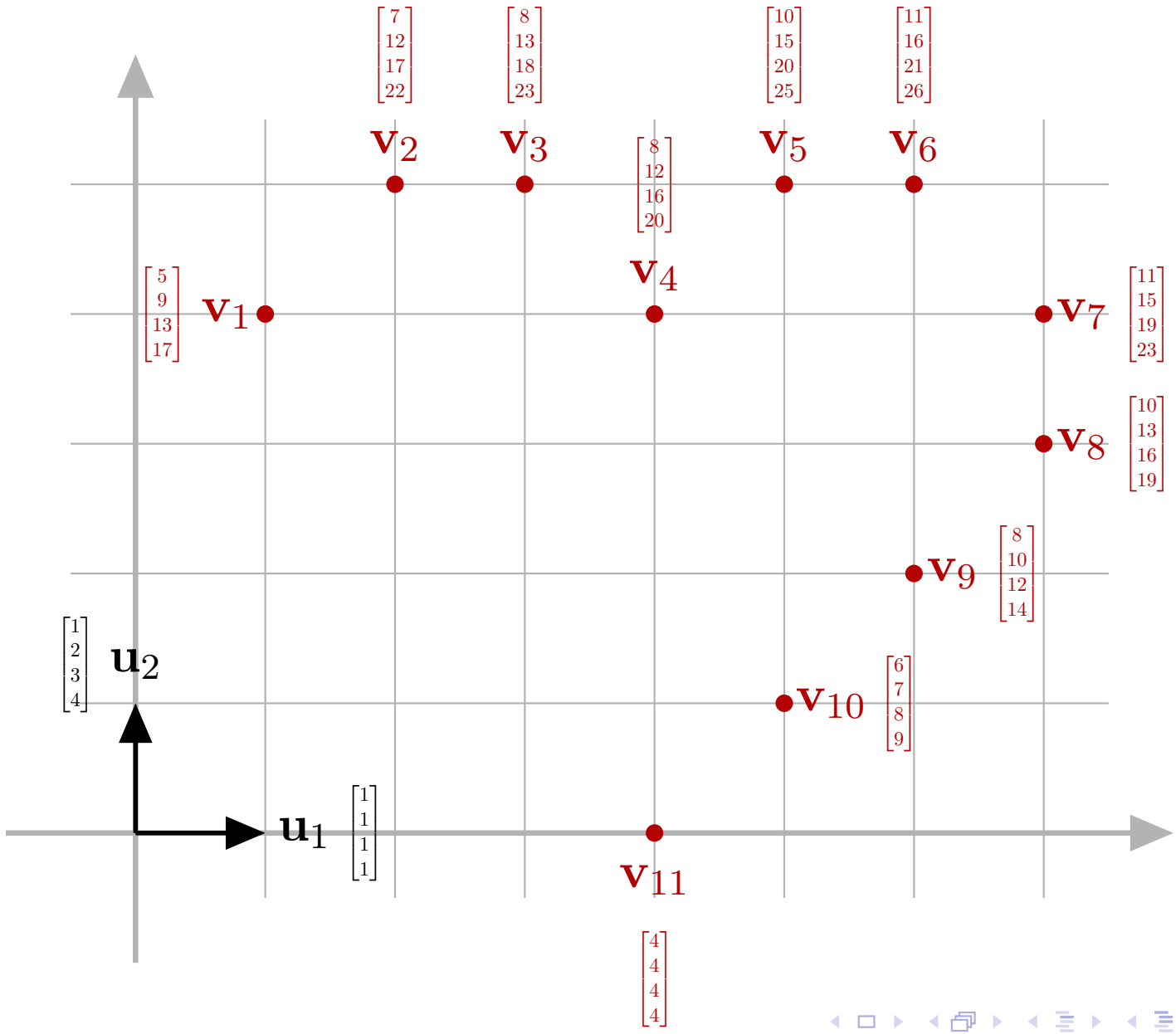
Feeling the vector representation



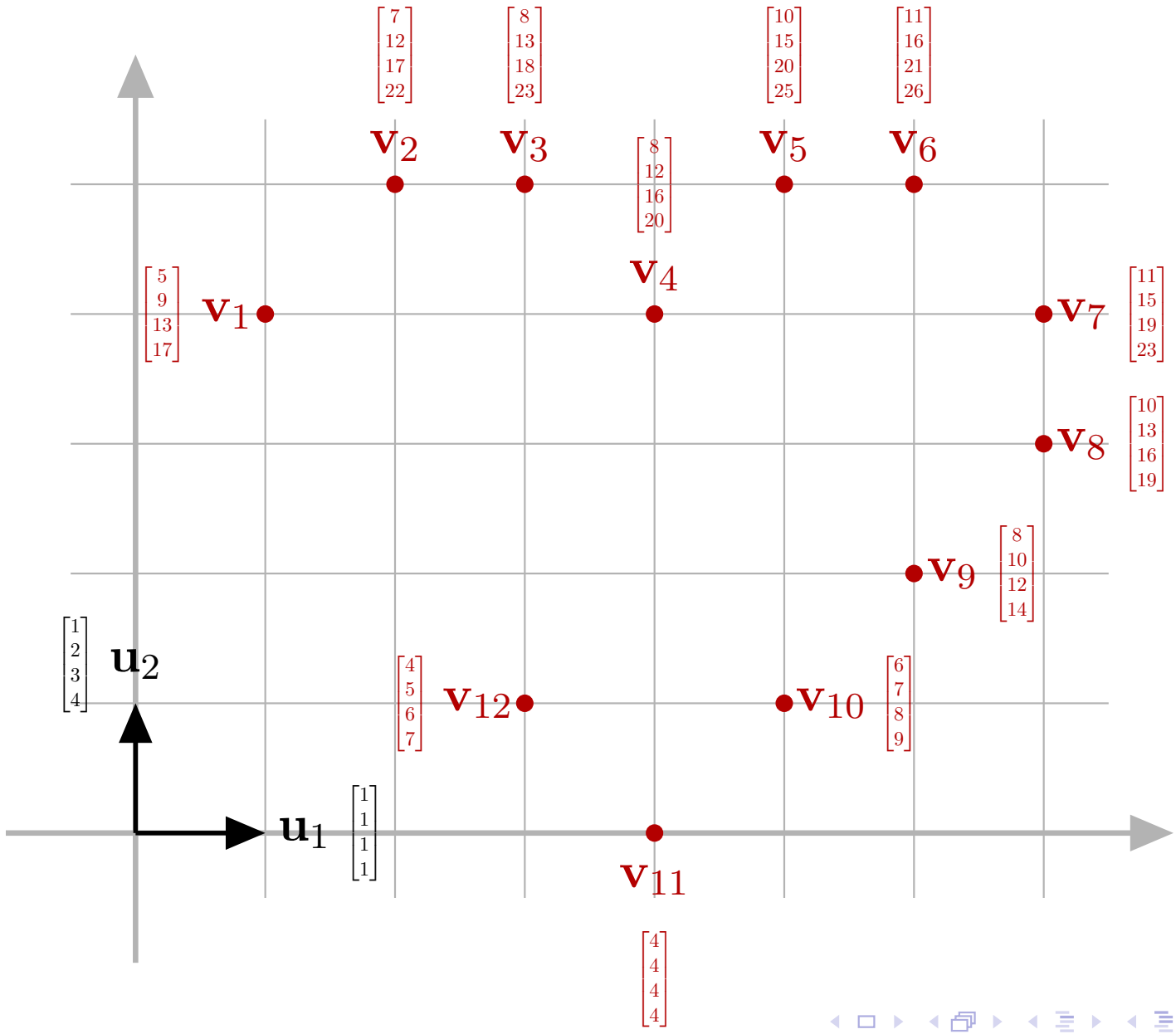
Feeling the vector representation



Feeling the vector representation



Feeling the vector representation



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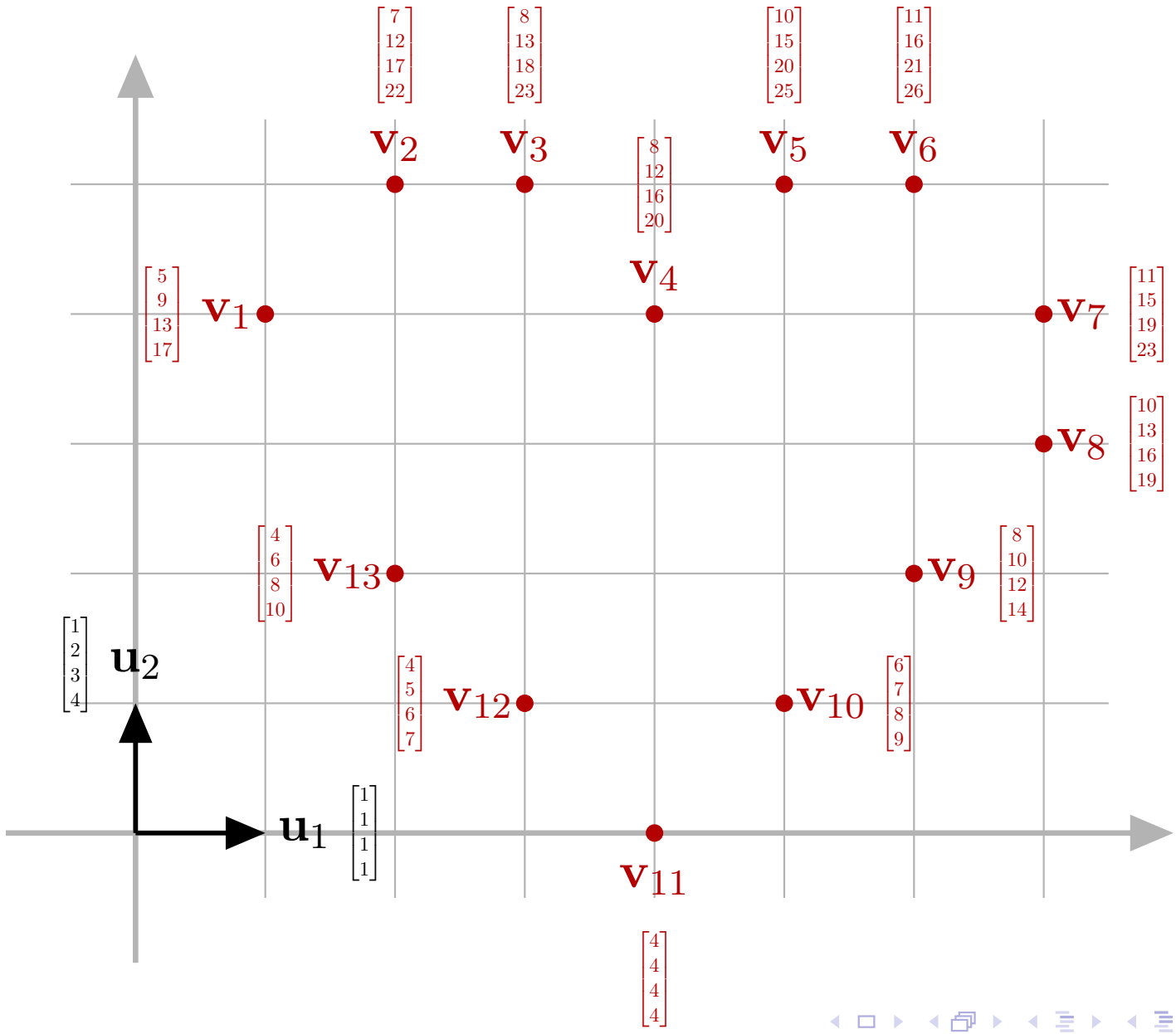
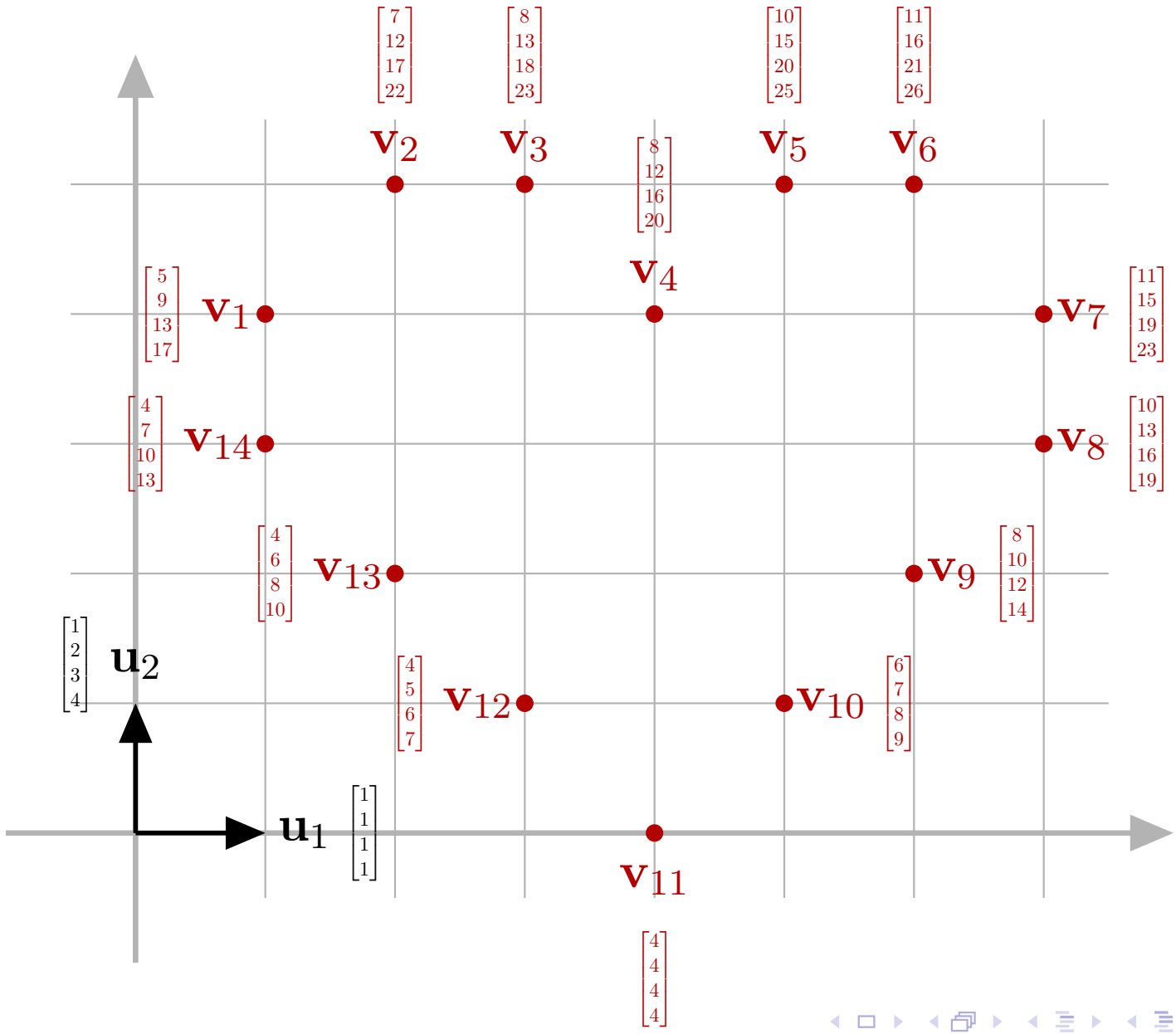
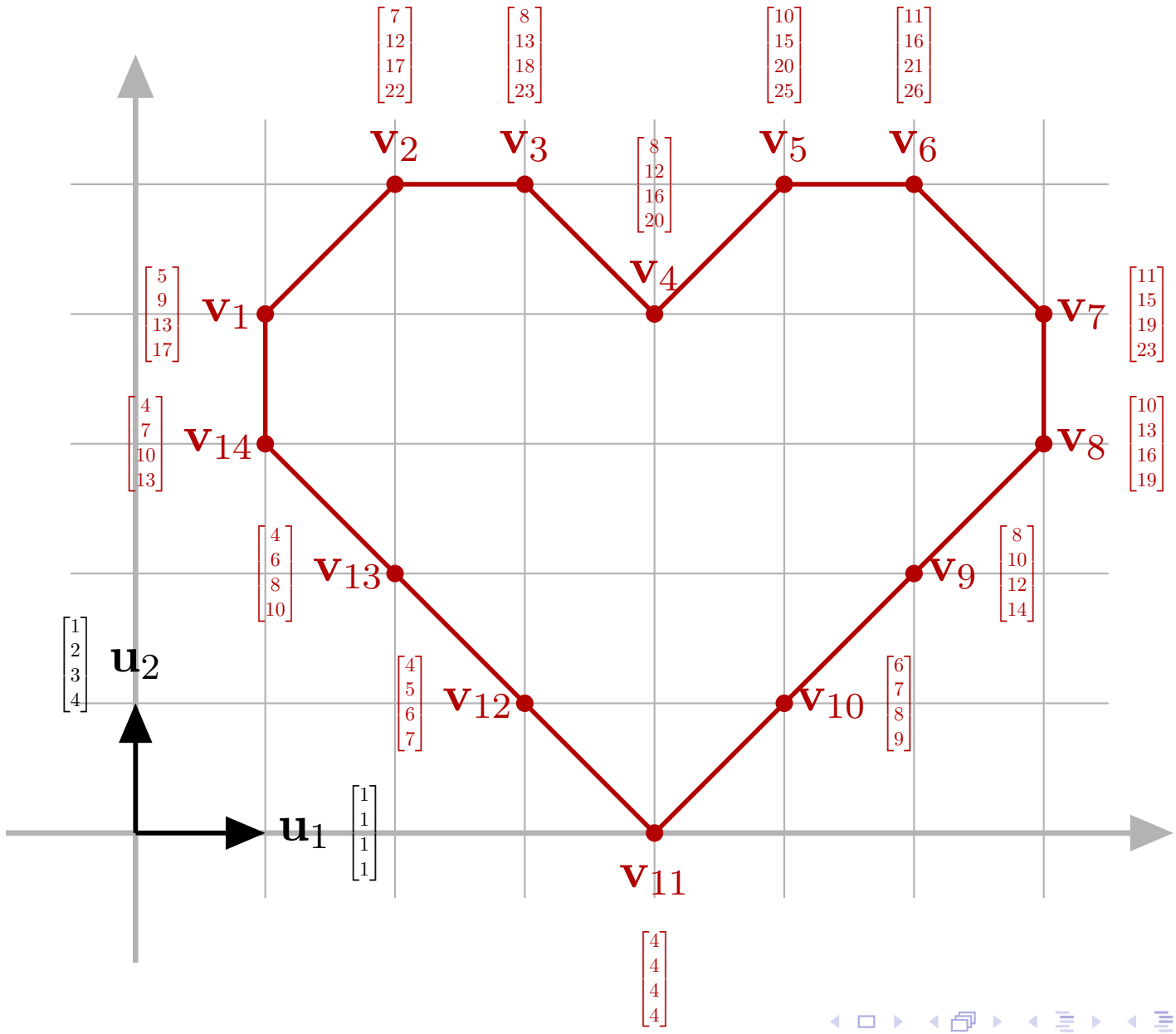
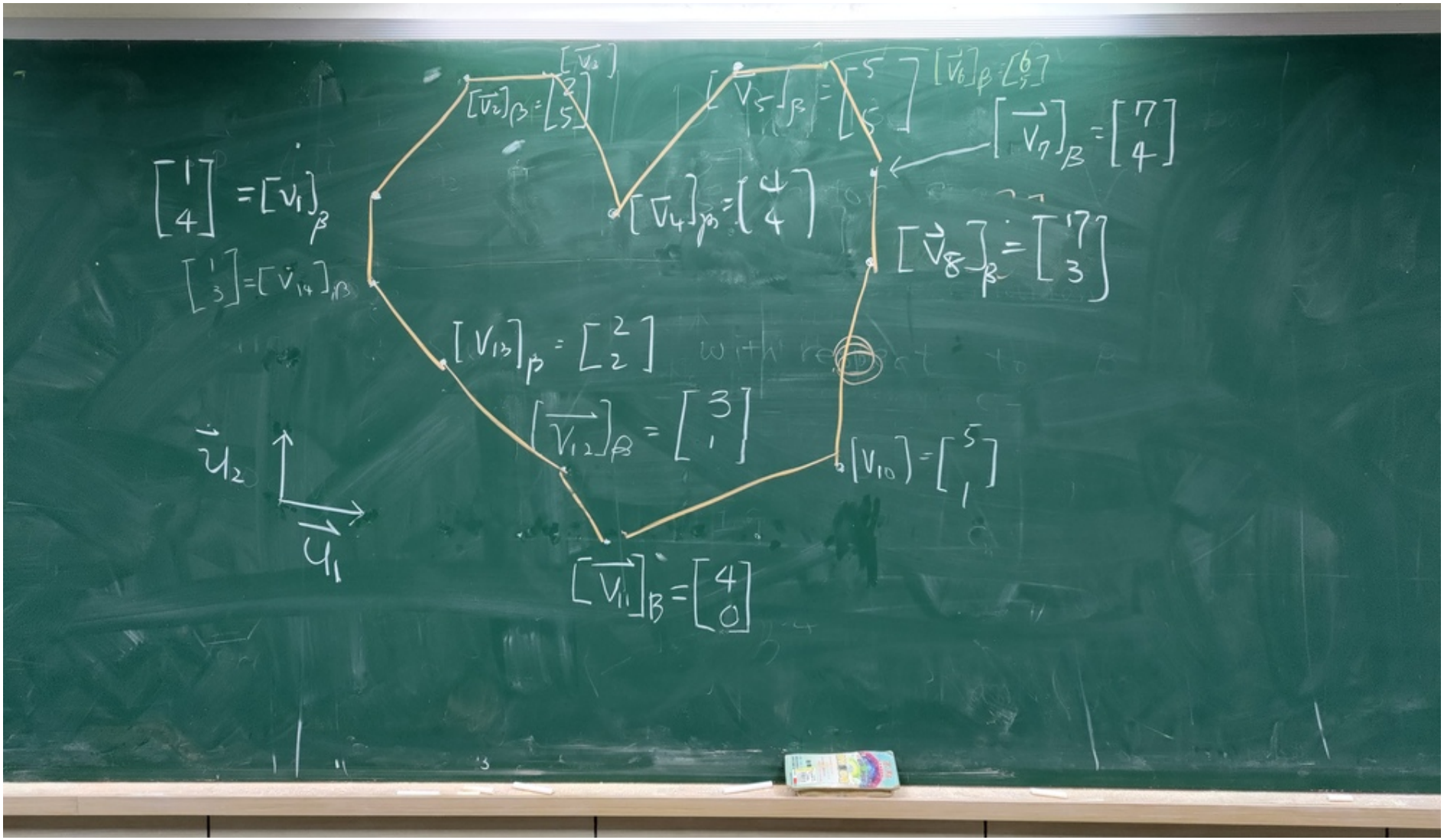


Feeling the vector representation

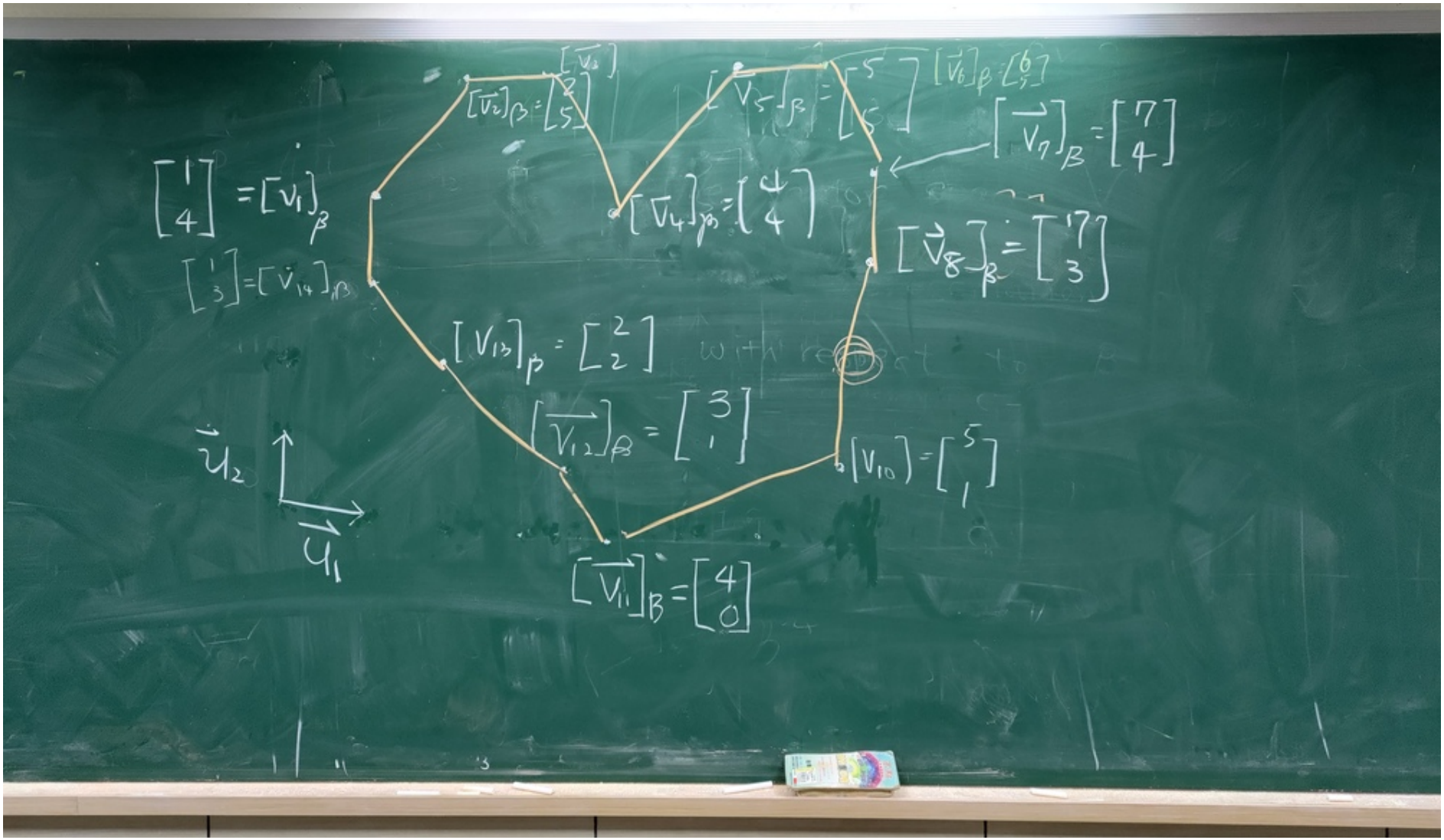


Feeling the vector representation




$\begin{bmatrix}1\\4\end{bmatrix} = [v_1]_\beta$
$\begin{bmatrix}1\\3\end{bmatrix} = [v_{14}]_\beta$
$[\vec{v}_2]_\beta = \begin{bmatrix}2\\5\end{bmatrix}$
$[\vec{v}_5]_\beta = \begin{bmatrix}5\\5\end{bmatrix}$
$[\vec{v}_6]_\beta = \begin{bmatrix}6\\5\end{bmatrix}$
$[\vec{v}_7]_\beta = \begin{bmatrix}7\\4\end{bmatrix}$
$[\vec{v}_4]_\beta = \begin{bmatrix}4\\4\end{bmatrix}$
$[\vec{v}_8]_\beta = \begin{bmatrix}7\\3\end{bmatrix}$
$[v_{13}]_\beta = \begin{bmatrix}2\\2\end{bmatrix}$
$[\vec{v}_{12}]_\beta = \begin{bmatrix}3\\1\end{bmatrix}$
$[v_{10}] = \begin{bmatrix}5\\1\end{bmatrix}$
$\vec{u}_2$
$\vec{u}_1$
$[\vec{v}_{11}]_\beta = \begin{bmatrix}4\\0\end{bmatrix}$

Thank you!


$\begin{bmatrix}1\\4\end{bmatrix} = [v_1]_\beta$
$\begin{bmatrix}1\\3\end{bmatrix} = [v_{14}]_\beta$
$[\vec{v}_2]_\beta = \begin{bmatrix}2\\5\end{bmatrix}$
$[\vec{v}_3]_\beta = \begin{bmatrix}5\\5\end{bmatrix}$
$[\vec{v}_6]_\beta = \begin{bmatrix}6\\5\end{bmatrix}$
$[\vec{v}_7]_\beta = \begin{bmatrix}7\\4\end{bmatrix}$
$[\vec{v}_4]_\beta = \begin{bmatrix}4\\4\end{bmatrix}$
$[\vec{v}_8]_\beta = \begin{bmatrix}7\\3\end{bmatrix}$
$[v_{13}]_\beta = \begin{bmatrix}2\\2\end{bmatrix}$
$[\vec{v}_{12}]_\beta = \begin{bmatrix}3\\1\end{bmatrix}$
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$\vec{u}_2$
$\vec{u}_1$
$[\vec{v}_{11}]_\beta = \begin{bmatrix}4\\0\end{bmatrix}$

Thank you!