The minimum rank problem on loop graphs

Jephian C.-H. Lin

Department of Mathematics, Iowa State University

Jan 7, 2017
2017 Joint Mathematics Meetings, Atlanta, GA

The minimum rank problem

- The minimum rank problem refers to finding the minimum rank or the maximum nullity of matrices under certain restrictions.
- The restrictions can be the zero-nonzero pattern, the inertia, or other properties of a matrix.
- The minimum rank problem is motivated by
  - the inverse eigenvalue problem — Matrix theory, Engineering
  - Colin de Verdière parameter, Lovász's orthogonal representation — Graph theory

Example of the maximum nullity

$*$ =nonzero

$$\begin{bmatrix} 0 & * & * & 0 \\ * & * & * & 0 \\ * & * & * & * \\ 0 & 0 & * & 0 \end{bmatrix}$$

Any matrix following this pattern is always nonsingular, meaning the maximum nullity of this pattern is 0.

Zero forcing I

Thinking the matrix as a linear system, if a variable is known as zero, then color it blue.

$$\begin{array}{cc} & \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & * & * & 0 \\ * & * & * & 0 \\ * & * & * & * \\ 0 & 0 & * & 0 \end{bmatrix} \end{array}$$

The only vector in the right kernel is $(0, 0, 0, 0, 0)$, so the maximum nullity is 0.

Zero forcing I

Thinking the matrix as a linear system, if a variable is known as zero, then color it blue.

$$\begin{array}{cc} & \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & * & * & 0 \\ * & * & * & 0 \\ * & * & * & * \\ 0 & 0 & * & 0 \end{bmatrix} \end{array}$$

The only vector in the right kernel is $(0, 0, 0, 0, 0)$, so the maximum nullity is 0.

Zero forcing I

Thinking the matrix as a linear system, if a variable is known as zero, then color it blue.

$$\begin{array}{c c} & \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & * & * & 0 \\ * & * & * & 0 \\ * & * & * & * \\ 0 & 0 & * & 0 \end{bmatrix} \end{array}$$

The only vector in the right kernel is $(0, 0, 0, 0, 0)$, so the maximum nullity is 0.

Zero forcing I

Thinking the matrix as a linear system, if a variable is known as zero, then color it blue.

$$
\begin{array}{c c}
 & \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & * & * & 0 \\ * & * & * & 0 \\ * & * & * & * \\ 0 & 0 & * & 0 \end{bmatrix}
\end{array}
$$

The only vector in the right kernel is $(0,0,0,0,0)$, so the maximum nullity is 0.

Zero forcing I

Thinking the matrix as a linear system, if a variable is known as zero, then color it blue.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 0 & * & * & 0 \\ * & * & * & 0 \\ * & * & * & * \\ 0 & 0 & * & 0 \end{bmatrix} \end{array}$$

The only vector in the right kernel is $(0, 0, 0, 0, 0)$, so the maximum nullity is 0.

Zero forcing I

Thinking the matrix as a linear system, if a variable is known as zero, then color it blue.

$$\begin{array}{cc} & \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & * & * & 0 \\ * & * & * & 0 \\ * & * & * & * \\ 0 & 0 & * & 0 \end{bmatrix} \end{array}$$

The only vector in the right kernel is $(0, 0, 0, 0, 0)$, so the maximum nullity is 0.

Zero forcing I

Thinking the matrix as a linear system, if a variable is known as zero, then color it blue.

$$\begin{array}{c c} & \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & * & * & 0 \\ * & * & * & 0 \\ * & * & * & * \\ 0 & 0 & * & 0 \end{bmatrix} \end{array}$$

The only vector in the right kernel is $(0, 0, 0, 0, 0)$, so the maximum nullity is 0.

Zero forcing I

Thinking the matrix as a linear system, if a variable is known as zero, then color it blue.

$$
\begin{array}{c c}
 & \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & * & * & 0 \\ * & * & * & 0 \\ * & * & * & * \\ 0 & 0 & * & 0 \end{bmatrix}
\end{array}
$$

The only vector in the right kernel is $(0, 0, 0, 0, 0)$, so the maximum nullity is 0.

Zero forcing I

Thinking the matrix as a linear system, if a variable is known as zero, then color it blue.

$$\begin{array}{c|cccc} & x_1 & x_2 & x_3 & x_4 \\ \hline 1 & 0 & * & * & 0 \\ 2 & * & * & * & 0 \\ 3 & * & * & * & * \\ 4 & 0 & 0 & * & 0 \end{array}$$

The only vector in the right kernel is $(0, 0, 0, 0, 0)$, so the maximum nullity is 0.

Zero forcing II

Color x_4 in advance. The remaining process is the same.

$$\begin{array}{c c} & \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & * & * & 0 \\ * & * & * & 0 \\ * & * & * & * \\ 0 & 0 & * & * \end{bmatrix} \end{array}$$

The first three columns are always independent, so the the maximum nullity is at most 1.

$$\text{maximum nullity } \leq \# \text{ initial blue variables}$$

Zero forcing II

Color x_4 in advance. The remaining process is the same.

$$\begin{array}{c c} & \begin{array}{cccc} x_1 & x_2 & x_3 & \boxed{x_4} \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & * & * & 0 \\ * & * & * & 0 \\ * & * & * & * \\ 0 & 0 & * & * \end{bmatrix} \end{array}$$

The first three columns are always independent, so the the maximum nullity is at most 1.

$$\text{maximum nullity} \leq \#\ \text{initial blue variables}$$

Zero forcing II

Color x_4 in advance. The remaining process is the same.

$$
\begin{array}{c|cccc}
 & x_1 & x_2 & x_3 & \boxed{x_4} \\
\hline
1 & 0 & * & * & 0 \\
2 & * & * & * & 0 \\
3 & * & * & * & * \\
\rightarrow 4 & 0 & 0 & * & *
\end{array}
$$

The first three columns are always independent, so the the maximum nullity is at most 1.

$$\text{maximum nullity } \leq \# \text{ initial blue variables}$$

Zero forcing II

Color x_4 in advance. The remaining process is the same.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 0 & * & * & 0 \\ * & * & * & 0 \\ * & * & * & * \\ 0 & 0 & * & * \end{bmatrix} \end{array}$$

The first three columns are always independent, so the the maximum nullity is at most 1.

$$\text{maximum nullity } \leq \# \text{ initial blue variables}$$

Zero forcing II

Color x_4 in advance. The remaining process is the same.

$$\begin{array}{c c} & \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{array}{c} \to 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & * & * & 0 \\ * & * & * & 0 \\ * & * & * & * \\ 0 & 0 & * & * \end{bmatrix} \end{array}$$

The first three columns are always independent, so the the maximum nullity is at most 1.

$$\text{maximum nullity} \le \#\ \text{initial blue variables}$$

Zero forcing II

Color x_4 in advance. The remaining process is the same.

$$\begin{array}{c|cccc} & x_1 & x_2 & x_3 & x_4 \\ \hline 1 & 0 & * & * & 0 \\ 2 & * & * & * & 0 \\ 3 & * & * & * & * \\ 4 & 0 & 0 & * & * \end{array}$$

The first three columns are always independent, so the the maximum nullity is at most 1.

maximum nullity $\leq \#$ initial blue variables

Zero forcing II

Color x_4 in advance. The remaining process is the same.

$$\begin{array}{cc} & \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & * & * & 0 \\ * & * & * & 0 \\ * & * & * & * \\ 0 & 0 & * & * \end{bmatrix} \end{array}$$

The first three columns are always independent, so the the maximum nullity is at most 1.

maximum nullity $\leq \#$ initial blue variables

Zero forcing II

Color x_4 in advance. The remaining process is the same.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \left[\begin{array}{cccc} 0 & * & * & 0 \\ * & * & * & 0 \\ * & * & * & * \\ 0 & 0 & * & * \end{array}\right] \end{array}$$

The first three columns are always independent, so the the maximum nullity is at most 1.

$$\text{maximum nullity} \leq \# \text{ initial blue variables}$$

Zero forcing III

$$\begin{array}{cc} & \begin{array}{cccc} x_3 & x_2 & x_1 & x_4 \end{array} \\ \begin{array}{c} 4 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} * & 0 & 0 & * \\ * & * & 0 & 0 \\ * & * & * & 0 \\ * & * & * & * \end{bmatrix} \end{array}$$

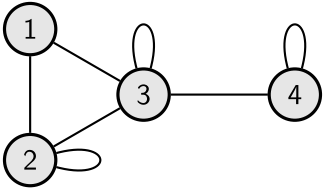
The zero forcing is actually finding the largest lower triangular pattern.

$$\text{maximum nullity } \leq \# \text{ initial blue variables}$$

The minimum rank of loop graphs

The maximum nullity $M(\mathfrak{G})$ of a loop graph $\mathfrak{G}$ is the maximum nullity over real symmetric matrices following its zero-nonzero pattern.

$$\begin{bmatrix} 0 & * & * & 0 \\ * & * & * & 0 \\ * & * & * & * \\ 0 & 0 & * & * \end{bmatrix}$$



The zero forcing number $Z(\mathfrak{G})$ is the minimum number of initial blue vertices required to make all vertices blue by the color-change rule:

For a vertex x, if y is the only white neighbor of x, then y turns blue.

$M(\mathfrak{G})$ and $Z(\mathfrak{G})$

Theorem (Hogben '10)

For any loop graph $\mathfrak{G}$, $M(\mathfrak{G}) \le Z(\mathfrak{G})$.

In general, $Z(\mathfrak{G})$ gives a nice bound; however, for loopless odd cycles $\mathfrak{C}^0_{2k+1}$, $0 = M(\mathfrak{G}) < Z(\mathfrak{G}) = 1$.

$$\det \begin{bmatrix} 0 & a & 0 & 0 & f \\ a & 0 & b & 0 & 0 \\ 0 & b & 0 & c & 0 \\ 0 & 0 & c & 0 & d \\ f & 0 & 0 & d & 0 \end{bmatrix} = 2abcdf \neq 0, \text{ if } a, b, c, d, f \neq 0.$$

Main idea: eliminate the odd cycles

The odd cycle zero forcing number $Z_{oc}(\mathfrak{G})$ of a loop graph $\mathfrak{G}$ is the minimum number of initial blue vertices required to make all vertices blue by:

- If y is the only white neighbor of x, then y turns blue.
- If the subgraph induced by the white vertices contains a component, which is a loopless odd cycle, then all vertices in this component turn blue.

Theorem (JL '16)

For any loop graph $\mathfrak{G}$, $M(\mathfrak{G}) \leq Z_{oc}(\mathfrak{G}) \leq Z(\mathfrak{G})$.

Main idea: eliminate the odd cycles

The odd cycle zero forcing number $Z_{oc}(\mathfrak{G})$ of a loop graph $\mathfrak{G}$ is the minimum number of initial blue vertices required to make all vertices blue by:

- If y is the only white neighbor of x, then y turns blue.
- If the subgraph induced by the white vertices contains a component, which is a loopless odd cycle, then all vertices in this component turn blue.

Theorem (JL '16)

For any loop graph $\mathfrak{G}$, $M(\mathfrak{G}) \leq Z_{oc}(\mathfrak{G}) \leq Z(\mathfrak{G})$.

Thank you

Odd cycle zero forcing

$$\begin{array}{c c} & \begin{array}{ccccc} x_1 & x_3 & x_4 & x_5 & x_1 \end{array} \\ \begin{array}{c} 2 \\ 3 \\ 4 \\ 5 \\ 1 \end{array} & \begin{bmatrix} * & 0 & 0 & 0 & 0 \\ * & 0 & a & b & 0 \\ 0 & a & 0 & c & 0 \\ 0 & b & c & 0 & 0 \\ * & * & 0 & 0 & * \end{bmatrix} \end{array}$$

References I

L. Hogben.
Minimum rank problems.
Linear Algebra Appl., 432:1961–1974, 2010.

J. C.-H. Lin.
Odd cycle zero forcing parameters and the minimum rank of graph blowups.
Electron. J. Linear Algebra, 31:42–59, 2016.