The inverse eigenvalue problem of a graph: Multiplicities and minors

Jephian C.-H. Lin

Department of Mathematics and Statistics, University of Victoria

Jan 13, 2018

Joint Mathematics Meetings, San Diego, CA

Joint work with Wayne Barrett, Steve Butler, Shaun M. Fallat, H. Tracy Hall, Leslie Hogben, Bryan L. Shader and Michael Young.

Inverse eigenvalue problem of a graph

Let G be a simple graph on n vertices. The family $\mathcal{S}(G)$ consists of all $n \times n$ real symmetric matrix $M = \left[M_{i,j}\right]$ with

$$\begin{cases} M_{i,j} = 0 & \text{if } i \neq j \text{ and } \{i,j\} \text{ is not an edge,} \\ M_{i,j} \neq 0 & \text{if } i \neq j \text{ and } \{i,j\} \text{ is an edge,} \\ M_{i,j} \in \mathbb{R} & \text{if } i = j. \end{cases}$$

$$\mathcal{S}(\circ\!-\!\circ\!-\!\circ) \ni \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0.1 & 0 \\ 0.1 & 1 & \pi \\ 0 & \pi & 0 \end{bmatrix}, \cdots$$

The inverse eigenvalue problem of a graph (IEPG) asks what are all spectra appeared in $\mathcal{S}(G)$ for a given graph G.

Theorem (Monfared and Shader 2013)

Let G be a graph on n vertices. For any n distinct real numbers $\{\lambda_1, \ldots, \lambda_n\}$, there is a matrix $A \in \mathcal{S}(G)$ with

$$\text{spec}(A) = \{\lambda_1, \ldots, \lambda_n\}.$$

Key idea: Use *Implicit Function Theorem* to perturb the diagonal matrix.

$$\begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & \lambda_n \end{bmatrix} \longrightarrow \begin{bmatrix} \sim \lambda_1 & \epsilon & 0 & \epsilon & 0 \\ \epsilon & \sim \lambda_2 & \epsilon & 0 & \epsilon \\ 0 & \epsilon & \sim \lambda_3 & \epsilon & 0 \\ \epsilon & 0 & \epsilon & \ddots & \epsilon \\ 0 & \epsilon & 0 & \epsilon & \sim \lambda_n \end{bmatrix}$$

Theorem (Monfared and Shader 2013)

Let G be a graph on n vertices. For any n distinct real numbers $\{\lambda_1, \ldots, \lambda_n\}$, there is a matrix $A \in \mathcal{S}(G)$ with

$$\operatorname{spec}(A) = \{\lambda_1, \ldots, \lambda_n\}.$$

Key idea: Use *Implicit Function Theorem* to perturb the diagonal matrix.

$$\begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & \lambda_n \end{bmatrix} \longrightarrow \begin{bmatrix} \sim \lambda_1 & \epsilon & 0 & \epsilon & 0 \\ \epsilon & \sim \lambda_2 & \epsilon & 0 & \epsilon \\ 0 & \epsilon & \sim \lambda_3 & \epsilon & 0 \\ \epsilon & 0 & \epsilon & \ddots & \epsilon \\ 0 & \epsilon & 0 & \epsilon & \sim \lambda_n \end{bmatrix}$$

Example on P_3

$$A(x_1, x_2, x_3, y) = \begin{bmatrix} x_1 & y & 0 \\ y & x_2 & y \\ 0 & y & x_3 \end{bmatrix}$$

Goal: Given λ_i's, find x_i's and $y \neq 0$ such that

$$\text{spec}(A(x_1, x_2, x_3, y)) = \{\lambda_i\}_{i=1}^{3}.$$

Note:

$$\text{spec}(A(\lambda_1, \lambda_2, \lambda_3, 0)) = \{\lambda_i\}_{i=1}^{3},$$

but $y = 0$.

Example on P_3

$$A(x_1, x_2, x_3, y) = \begin{bmatrix} x_1 & y & 0 \\ y & x_2 & y \\ 0 & y & x_3 \end{bmatrix}$$

Consider the function f

$$(x_1, x_2, x_3, y) \mapsto (\mathrm{tr}(A), \frac{1}{2}\mathrm{tr}(A^2), \frac{1}{3}\mathrm{tr}(A^3)).$$

The right hand side controls the spectrum. When $x_i = \lambda_i$ and $y = 0$, it has the desired spectrum.

Example on P_3

$$A(x_1, x_2, x_3, y) = \begin{bmatrix} x_1 & y & 0 \\ y & x_2 & y \\ 0 & y & x_3 \end{bmatrix}$$

$$(\overbrace{x_1, x_2, x_3}^{\text{independent}}, \underbrace{y}_{\text{independent}}) \mapsto \underbrace{(\operatorname{tr}(A), \frac{1}{2}\operatorname{tr}(A^2), \frac{1}{3}\operatorname{tr}(A^3))}_{\text{dependent}}.$$

Example on P_3

$$A(x_1, x_2, x_3, y) = \begin{bmatrix} x_1 & y & 0 \\ y & x_2 & y \\ 0 & y & x_3 \end{bmatrix}$$

$$(\overbrace{x_1, x_2, x_3}^{\text{dependent}}, \underbrace{y}_{\text{independent}}) \mapsto \underbrace{(\operatorname{tr}(A), \frac{1}{2}\operatorname{tr}(A^2), \frac{1}{3}\operatorname{tr}(A^3))}_{\text{fixed}}.$$

Example on P_3

$$A(x_1, x_2, x_3, y) = \begin{bmatrix} x_1 & y & 0 \\ y & x_2 & y \\ 0 & y & x_3 \end{bmatrix}$$

$$(\overbrace{x_1, x_2, x_3}^{\text{dependent}}, \underbrace{y}_{\text{independent}}) \mapsto \underbrace{(\operatorname{tr}(A), \frac{1}{2}\operatorname{tr}(A^2), \frac{1}{3}\operatorname{tr}(A^3))}_{\text{fixed}}.$$

$$\begin{aligned} \operatorname{tr}(A) &= x_1 + x_2 + x_3 \\ \operatorname{tr}(A^2) &= x_1^2 + x_2^2 + x_3^2 + y(???) \\ \operatorname{tr}(A^3) &= x_1^3 + x_2^3 + x_3^3 + y(???) \end{aligned} \implies \operatorname{Jac}\Big|_{\substack{x_i=\lambda_i \\ y=0}} = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix}$$

Example on P_3

$$A(x_1, x_2, x_3, y) = \begin{bmatrix} x_1 & y & 0 \\ y & x_2 & y \\ 0 & y & x_3 \end{bmatrix}$$

$$(\overbrace{x_1, x_2, x_3}^{\text{dependent}}, \underbrace{y}_{\text{independent}}) \mapsto \underbrace{(\operatorname{tr}(A), \frac{1}{2}\operatorname{tr}(A^2), \frac{1}{3}\operatorname{tr}(A^3))}_{\text{fixed}}.$$

When λ_i's are all distinct, the Jacobian matrix is invertible. We may perturb y to $\epsilon \neq 0$ and $x_i \sim \lambda_i$, while preserving the same spectrum.

(This proof follows from [Monfared and Khanmohammadi 2018].)

Another point of view of the theorem

Theorem (Monfared and Shader 2013)

Let $\overline{K_n}$ be a spanning subgraph of G. If $A \in \mathcal{S}(\overline{K_n})$ has some nice property, then there is $B \in \mathcal{S}(G)$ with

$$\text{spec}(B) = \text{spec}(A).$$

Theorem (BFHHLS 2017)

Let H be a spanning subgraph of G. If $A \in \mathcal{S}(H)$ has the Strong Spectral Property, then there is $B \in \mathcal{S}(G)$ with

$$\text{spec}(B) = \text{spec}(A).$$

Another point of view of the theorem

Theorem (Monfared and Shader 2013)

Let $\overline{K_n}$ be a spanning subgraph of G. If $A \in \mathcal{S}(\overline{K_n})$ has some nice property, then there is $B \in \mathcal{S}(G)$ with

$$\mathrm{spec}(B) = \mathrm{spec}(A).$$

Theorem (BFHHLS 2017)

Let H be a spanning subgraph of G. If $A \in \mathcal{S}(H)$ has the Strong Spectral Property, then there is $B \in \mathcal{S}(G)$ with

$$\mathrm{spec}(B) = \mathrm{spec}(A).$$

Isospectral manifolds and pattern manifolds

Let $A \in \mathcal{S}(H)$. The isospectral manifold is

$$\mathcal{E}_A = \{Q^\top A Q : Q \text{ orthogonal}\}.$$

The pattern manifold is

$$\mathcal{S}^{cl}(H) = \{M : M_{i,j} = 0 \text{ if } \{i,j\} \in E(\overline{H})\}.$$

Also define

$$\mathcal{S}^{cl}_y(H,G) = \{M \in \mathcal{S}^{cl}(G) : M_{i,j} = y \text{ if } \{i,j\} \in E(G) \setminus E(H)\}$$

such that

$$\mathcal{S}^{cl}(H) = \mathcal{S}^{cl}_0(H,G) \parallel \mathcal{S}^{cl}_y(H,G) \subset \mathcal{S}^{cl}(G).$$

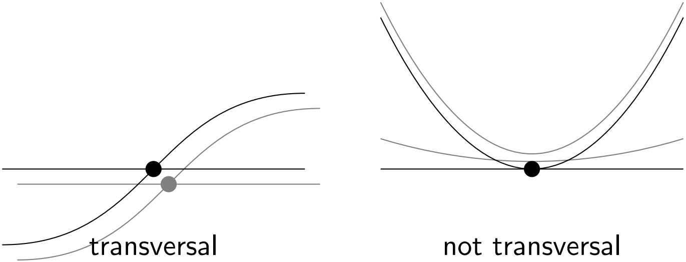
Transversality and Strong Arnold Property

Two manifolds intersect transversally at a point A if their normal spaces only have trivial intersection.

$$\mathcal{M}_1 \overset{☺}{\cap} \mathcal{M}_2 \iff \mathrm{Nor}_{\mathcal{M}_1.A} \cap \mathrm{Nor}_{\mathcal{M}_2.A} = \{\mathbf{0}\}$$

Let $A \in \mathcal{S}(H)$. Then A has the Strong Spectral Property if

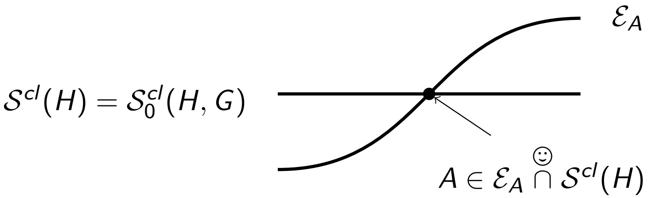
$$\mathcal{E}_A \overset{☺}{\cap} \mathcal{S}^{cl}(H).$$



Theorem (BFHHLS 2017)

Let H be a spanning subgraph of G. If $A \in \mathcal{S}(H)$ has the Strong Spectral Property, then there is $B \in \mathcal{S}(G)$ with

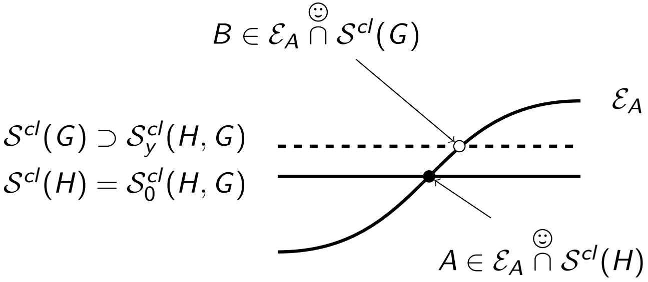
$$\text{spec}(B) = \text{spec}(A).$$



Theorem (BFHHLS 2017)

Let H be a spanning subgraph of G. If $A \in \mathcal{S}(H)$ has the Strong Spectral Property, then there is $B \in \mathcal{S}(G)$ with

$$\mathrm{spec}(B) = \mathrm{spec}(A).$$



Tangent spaces

Let $A \in \mathcal{S}(H)$. Then the tangent spaces are

$$\begin{aligned} \mathrm{Tan}_{\mathcal{E}_A.A} &= \{K^\top A + AK : K \text{ skew-symmetric}\} \\ \mathrm{Tan}_{\mathcal{S}^{cl}(H).A} &= \mathcal{S}^{cl}(H). \end{aligned}$$

Let $Q(t)$ be an orthogonal matrix with $Q(0) = I$. Then

$$\frac{d}{dt}[Q(t)^\top A Q(t)]\Big|_{t=0} = \dot{Q}(0)^\top A A\dot{Q}(0).$$

Tangent spaces

Let $A \in \mathcal{S}(H)$. Then the tangent spaces are

$$\begin{aligned} \mathrm{Tan}_{\mathcal{E}_A.A} &= \{K^\top A + AK : K \text{ skew-symmetric}\} \\ \mathrm{Tan}_{\mathcal{S}^{cl}(H).A} &= \mathcal{S}^{cl}(H). \end{aligned}$$

Let $Q(t)$ be an orthogonal matrix with $Q(0) = I$. Then

$$\frac{d}{dt}[Q(t)^\top A Q(t)]\Big|_{t=0} = \dot{Q}(0)^\top A Q(0) + Q(0)^\top A \dot{Q}(0).$$

Tangent spaces

Let $A \in \mathcal{S}(H)$. Then the tangent spaces are

$$\begin{aligned} \mathrm{Tan}_{\mathcal{E}_A.A} &= \{K^\top A + AK : K \text{ skew-symmetric}\} \\ \mathrm{Tan}_{\mathcal{S}^{cl}(H).A} &= \mathcal{S}^{cl}(H). \end{aligned}$$

Let $Q(t)$ be an orthogonal matrix with $Q(0) = I$. Then

$$\frac{d}{dt}[Q(t)^\top A Q(t)]\Big|_{t=0} = \dot{Q}(0)^\top A Q(0) + Q(0)^\top A \dot{Q}(0).$$

Tangent spaces

Let $A \in \mathcal{S}(H)$. Then the tangent spaces are

$$\begin{aligned} \mathrm{Tan}_{\mathcal{E}_A.A} &= \{K^\top A + AK : K \text{ skew-symmetric}\} \\ \mathrm{Tan}_{\mathcal{S}^{cl}(H).A} &= \mathcal{S}^{cl}(H). \end{aligned}$$

Let $Q(t)$ be an orthogonal matrix with $Q(0) = I$. Then

$$\frac{d}{dt}[Q(t)^\top A Q(t)]\Big|_{t=0} = \dot{Q}(0)^\top A + A\dot{Q}(0).$$

Tangent spaces

Let $A \in \mathcal{S}(H)$. Then the tangent spaces are

$$\begin{aligned} \mathrm{Tan}_{\mathcal{E}_A . A} &= \{K^\top A + AK : K \text{ skew-symmetric}\} \\ \mathrm{Tan}_{\mathcal{S}^{cl}(H) . A} &= \mathcal{S}^{cl}(H). \end{aligned}$$

Let $Q(t)$ be an orthogonal matrix with $Q(0) = I$. Then

$$\frac{d}{dt}[Q(t)^\top A Q(t)]\Big|_{t=0} = \dot{Q}(0)^\top A + A\dot{Q}(0).$$

$$Q(t)^\top Q(t) = I \implies \dot{Q}(0)^\top Q(0) + Q(0)^\top \dot{Q}(0) = 0,$$

so $\dot{Q}(0)$ is skew-symmetric.

Tangent spaces

Let $A \in \mathcal{S}(H)$. Then the tangent spaces are

$$\begin{aligned} \mathrm{Tan}_{\mathcal{E}_A.A} &= \{K^\top A + AK : K \text{ skew-symmetric}\} \\ \mathrm{Tan}_{\mathcal{S}^{cl}(H).A} &= \mathcal{S}^{cl}(H). \end{aligned}$$

Let $Q(t)$ be an orthogonal matrix with $Q(0) = I$. Then

$$\frac{d}{dt}[Q(t)^\top A Q(t)]\Big|_{t=0} = \dot{Q}(0)^\top A + A\dot{Q}(0).$$

$$Q(t)^\top Q(t) = I \implies \dot{Q}(0)^\top Q(0) + Q(0)^\top \dot{Q}(0) = 0,$$

so $\dot{Q}(0)$ is skew-symmetric.

Tangent spaces

Let $A \in \mathcal{S}(H)$. Then the tangent spaces are

$$\operatorname{Tan}_{\mathcal{E}_A.A} = \{K^\top A + AK : K \text{ skew-symmetric}\}$$
$$\operatorname{Tan}_{\mathcal{S}^{cl}(H).A} = \mathcal{S}^{cl}(H).$$

Let $Q(t)$ be an orthogonal matrix with $Q(0) = I$. Then

$$\frac{d}{dt}[Q(t)^\top A Q(t)]\Big|_{t=0} = \dot{Q}(0)^\top A + A\dot{Q}(0).$$

$$Q(t)^\top Q(t) = I \implies \dot{Q}(0)^\top + \dot{Q}(0) = 0,$$

so $\dot{Q}(0)$ is skew-symmetric.

What if no Strong Spectral Property?

- Strong Spectral Property $\implies$ can add any edge.
- ??? ??? Property $\implies$ can add some specific edges.

Theorem (Matrix Liberation Lemma, BBFHHLSY 2018)

Let H be a spanning subgraph of G. If $A \in \mathcal{S}(H)$ has the property that

- $\mathcal{E}_A \overset{\smile}{\cap} \mathcal{S}^{cl}(G)$ *and*
- *there is* $Y \in \mathrm{Tan}_{\mathcal{E}_A.A} \cap \mathrm{Tan}_{\mathcal{S}^{cl}(G)}$ *with* $\mathrm{supp}(Y) \supseteq E(G) \setminus E(H)$,

then there is $B \in \mathcal{S}(G)$ with

$$\mathrm{spec}(B) = \mathrm{spec}(A).$$

What if no Strong Spectral Property?

- Strong Spectral Property $\implies$ can add any edge.
- ??? ??? Property $\implies$ can add some specific edges.

Theorem (Matrix Liberation Lemma, BBFHHLSY 2018)

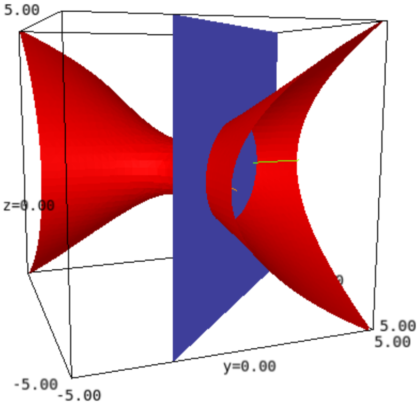
Let H be a spanning subgraph of G. If $A \in \mathcal{S}(H)$ has the property that

- $\mathcal{E}_A \overset{☺}{\cap} \mathcal{S}^{cl}(G)$ *and*
- *there is* $Y \in \mathrm{Tan}_{\mathcal{E}_A.A} \cap \mathrm{Tan}_{\mathcal{S}^{cl}(G)}$ *with* $\mathrm{supp}(Y) \supseteq E(G) \setminus E(H)$,

then there is $B \in \mathcal{S}(G)$ with

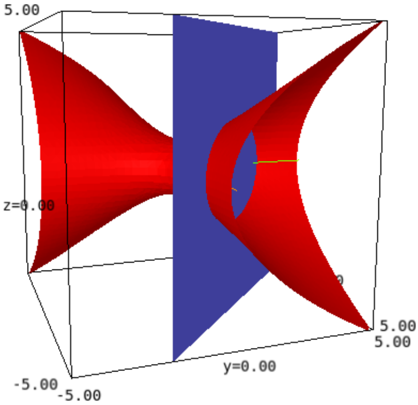
$$\mathrm{spec}(B) = \mathrm{spec}(A).$$

The isospectral manifold $\left\{ \begin{bmatrix} x & z \\ z & y \end{bmatrix} : \mathrm{tr} = 3, \det = 1 \right\}$.



Click here to
play with the interactive figure.

The isospectral manifold $\left\{ \begin{bmatrix} x & z \\ z & y \end{bmatrix} : \mathrm{tr} = 3, \det = 1 \right\}$.



Click here to
play with the interactive figure.

Thank you!

References I

W. Barrett, S. Butler, S. M. Fallat, H. T. Hall, L. Hogben, J. C.-H. Lin, B. Shader, and M. Young.
The inverse eigenvalue problem of a graph: Multiplicities and minors.
https://arxiv.org/abs/1708.00064.
(under review).

W. Barrett, S. M. Fallat, H. T. Hall, L. Hogben, J. C.-H. Lin, and B. Shader.
Generalizations of the Strong Arnold Property and the minimum number of distinct eigenvalues of a graph.
Electron. J. Combin., 24:P2.40, 2017.

References II

K. H. Monfared and E. Khanmohammadi.
A structured inverse spectrum problem for infinite graphs.
Linear Algebra Appl., 539:28–43, 2018.

K. H. Monfared and B. Shader.
Construction of matrices with a given graph and prescribed interlaced spectral data.
Linear Algebra Appl., 438:4348–4358, 2013.