

The strong spectral property for graphs

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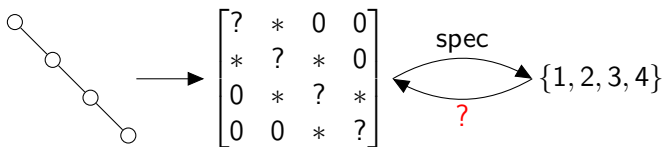
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Inverse eigenvalue problem of a graph (IEP- G)

Let G be a graph. Define $\mathcal{S}(G)$ as the family of all real symmetric matrices $A = [a_{ij}]$ such that

$$a_{ij} \begin{cases} \neq 0 & \text{if } ij \in E(G), i \neq j; \\ = 0 & \text{if } ij \notin E(G), i \neq j; \\ \in \mathbb{R} & \text{if } i = j. \end{cases}$$

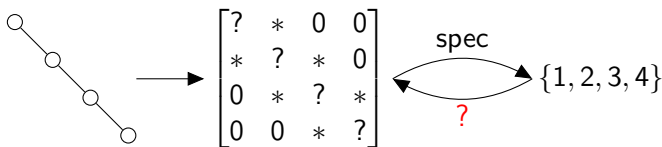


IEP- G : What are the possible spectra of a matrix in $\mathcal{S}(G)$?

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IEP- G : What are the possible spectra of a matrix in $\mathcal{S}(G)$?

Supergraph Lemma

Lemma (BFHHLS 2017)

Let H be a spanning subgraph of G . If $A \in \mathcal{S}(H)$ has the *strong spectral property (SSP)*, then there is a matrix $B \in \mathcal{S}(G)$ such that

- ▶ $\text{spec}(A) = \text{spec}(B)$,
- ▶ B has the SSP, and
- ▶ $\|B - A\|$ can be chosen arbitrarily small.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} \sim 1 & \epsilon & 0 & 0 \\ \epsilon & \sim 2 & \epsilon & 0 \\ 0 & \epsilon & \sim 3 & \epsilon \\ 0 & 0 & \epsilon & \sim 4 \end{bmatrix}$$

SSP will be defined later

Entrywise product \circ

$$A \circ X = O$$



$$(X)_{ij} \neq 0 \text{ only when } (A)_{ij} = 0$$

$$I \circ X = O$$



X is zero on the diagonal

Let $A \in \mathcal{S}(G)$. Then

$$A \circ X = O \text{ and } I \circ X = O$$



$(X)_{ij} \neq 0$ only when $ij \notin E(G)$

Strong spectral property (SSP)

Definition

A matrix A has the **strong spectral property (SSP)** if $X = O$ is the only real symmetric matrix that satisfies the following matrix equations:

- ▶ $A \circ X = O, I \circ X = O,$
- ▶ $AX - XA = O.$

Examples of matrices with the SSP:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Here we use the notation $[A, X]$ for $AX - XA$.

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Example of $A \in \mathcal{S}(P_4)$

Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} 0 & 0 & x & y \\ 0 & 0 & 0 & z \\ x & 0 & 0 & 0 \\ y & z & 0 & 0 \end{bmatrix}.$$

Then

$$[A, X] = \begin{bmatrix} 0 & -x & -y & -x+z \\ x & 0 & x-z & y \\ y & -x+z & 0 & z \\ x-z & -y & -z & 0 \end{bmatrix} = O.$$

$$\implies x=0, z=0, y=0 \implies X=O$$

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$A \in \mathcal{S}(P_4)$ always has the SSP

Let

$$A = \begin{bmatrix} d_1 & a_{12} & 0 & 0 \\ a_{12} & d_2 & a_{23} & 0 \\ 0 & a_{23} & d_3 & a_{34} \\ 0 & 0 & a_{34} & d_4 \end{bmatrix} \text{ and } X = \begin{bmatrix} 0 & 0 & x & y \\ 0 & 0 & 0 & z \\ x & 0 & 0 & 0 \\ y & z & 0 & 0 \end{bmatrix}.$$

Then $[A, X] =$

$$\begin{bmatrix} 0 & -a_{23}x & ?x - a_{34}y & ? \\ ? & 0 & ? & a_{12}y + ?z \\ ? & ? & 0 & a_{23}z \\ ? & ? & ? & 0 \end{bmatrix} = O.$$

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Example of $A \in \mathcal{S}(K_{1,3})$

Let

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \text{ and } X = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & x & y \\ 0 & x & 0 & z \\ 0 & y & z & 0 \end{bmatrix}.$$

Then $[A, X] =$

$$\begin{bmatrix} 0 & x+y & x+z & y+z \\ -x-y & 0 & 0 & 0 \\ -x-z & 0 & 0 & 0 \\ -y-z & 0 & 0 & 0 \end{bmatrix} = O \text{ implies } \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

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$$\text{Then } [A, X] = \begin{bmatrix} 0 & a_{13}x + a_{14}y & a_{12}x + a_{14}z & a_{12}y + a_{13}z \\ ? & 0 & ? & ? \\ ? & ? & 0 & ? \\ ? & ? & ? & 0 \end{bmatrix} = O$$

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$A \in \mathcal{S}(K_{1,3})$ always has the SSP

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Verification of the SSP

- ▶ Let $A \in \mathcal{S}(G)$.
- ▶ Let $E_{ij} = 0, 1$ -matrix with two ones on ij and ji .
- ▶ Define $X = \sum_{ij \in E(\bar{G})} x_{ij} E_{ij}$.

$$AX - XA = \sum_{ij \in E(\bar{G})} x_{ij} (AE_{ij} - E_{ij}A) = O$$

Verification:

A has the SSP $\iff \{AE_{ij} - E_{ij}A\}_{ij \in E(\bar{G})}$ is linearly independent

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Verification matrix

Let $\text{vec}_o(M)$ be the vector that records the off-diagonal entries of a skew-symmetric matrix M .

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ -1 & 0 & 4 & 5 \\ -2 & -4 & 0 & 6 \\ -3 & -5 & -6 & 0 \end{bmatrix} \xrightarrow{\text{vec}_o} [1 \ 2 \ 3 \ 4 \ 5 \ 6]$$

Definition

Let $A \in \mathcal{S}(G)$ and $p = |E(\overline{G})|$. The **SSP verification matrix** $\Psi_S(A)$ of A is a $p \times \binom{n}{2}$ matrix whose rows are composed of $\text{vec}_o(AE_{ij} - E_{ij}A)$ for $ij \in E(\overline{G})$.

A has the SSP $\iff \Psi_S(A)$ has full row-rank.

Verification matrix

Let $\text{vec}_o(M)$ be the vector that records the off-diagonal entries of a skew-symmetric matrix M .

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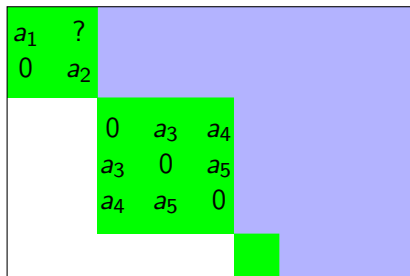
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Key idea

The verification matrix *always* has full row-rank if the green parts are always invertible and the white part is zero.



Forcing process: general setting

Let G be a graph.

- ▶ Each edge on G is considered as “black”.
- ▶ Each **non-edge** of G is initially white but can possibly be **blue** in the process.
- ▶ Color change rules will be defined later.

Theorem (L, Oblak, and Šmigoc 2020)

*If starting with all white and ending with all non-edge **blue**, then every $A \in S(G)$ has the SSP.*

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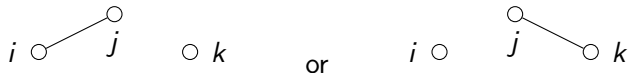
Theorem (L, Oblak, and Šmigoc 2020)

*If starting with all white and ending with all non-edge **blue**, then every $A \in \mathcal{S}(G)$ has the SSP.*

Forcing process: Rule 1

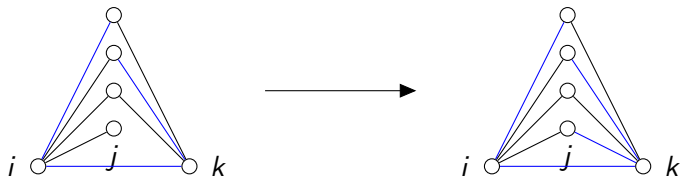
If

- ▶ ik is black or blue, and
- ▶ there is a unique black-white connection



between i and k (say the former case)

then jk turns blue.

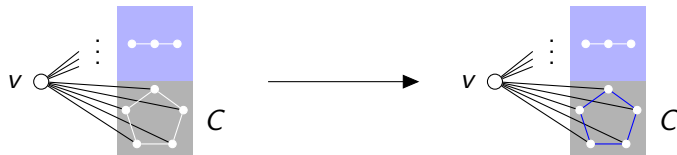


Forcing process: Rule 2

If

- ▶ $G[N(v)]$ contains a white odd cycle C as a component, and
- ▶ there are exactly two black-white connections between v and each vertex on C ,

then the edges in $E(C)$ turn blue.

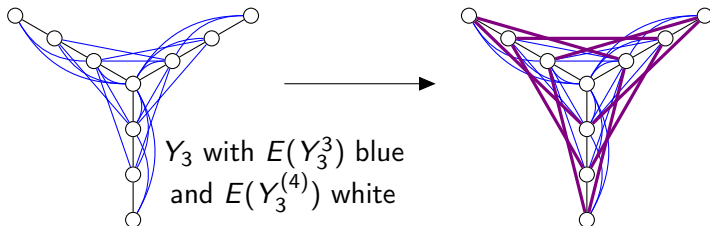


Forcing process: Rule 3

If

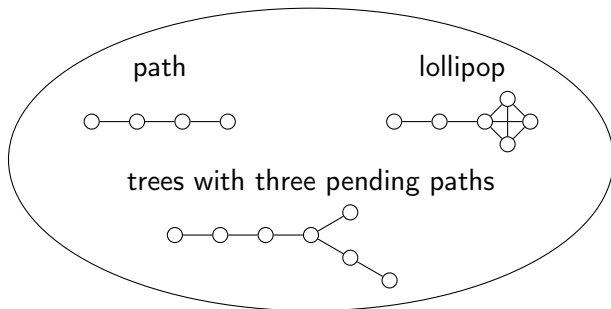
- ▶ G contains an induced subgraph Y_h ,
- ▶ edges in $E(Y_h^h)$ are blue, edges in $E(Y_h^{(h+1)})$ are white, and
- ▶ there are exactly two black-white connections between the two endpoints of each edge in $E(Y_h^{(h)})$,

then the edges in $E(Y_h^{(h+1)})$ turn blue.

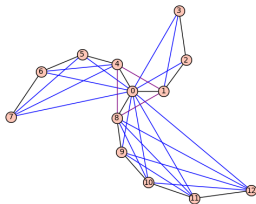
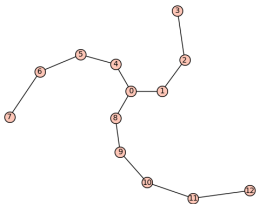


Graphs that guarantee the SSP

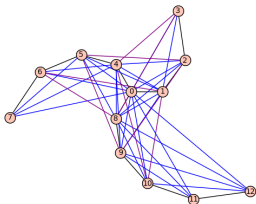
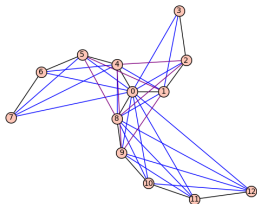
For the following graphs G , every $A \in \mathcal{S}(G)$ has the SSP.



This includes all graphs with $q(G) = n - 1$.

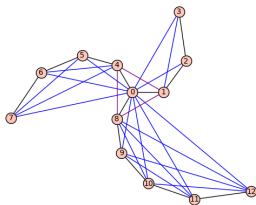
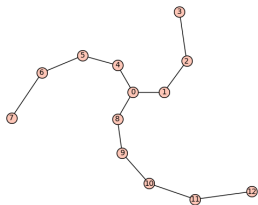


GIF version

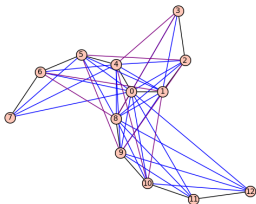
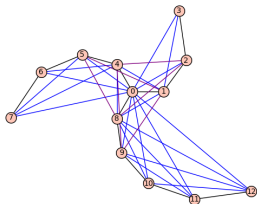


Thanks!







GIF version



Thanks!

References I

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-  J. C.-H. Lin, P. Oblak, and H. Šmigoc.
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