On the inverse eigenvalue problem for block graphs

Jephian C.-H. Lin 林晉宏

Department of Applied Mathematics, National Sun Yat-sen University

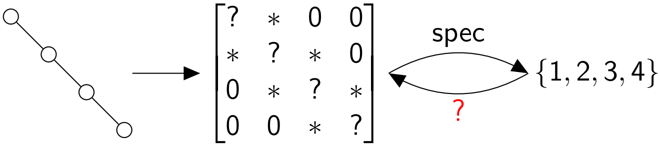
January 6 $\to$ April 7, 2022

2022 Joint Mathematics Meetings, Seattle, WA $\to$ virtual

Inverse eigenvalue problem of a graph (IEP-G)

Let G be a graph. Define $\mathcal{S}(G)$ as the family of all real symmetric matrices $A = [a_{ij}]$ such that

$$a_{ij} \begin{cases} \neq 0 & \text{if } ij \in E(G), i \neq j; \\ = 0 & \text{if } ij \notin E(G), i \neq j; \\ \in \mathbb{R} & \text{if } i = j. \end{cases}$$

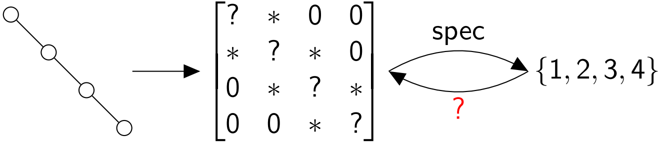


IEP-G: What are the possible spectra of a matrix in $\mathcal{S}(G)$?

Inverse eigenvalue problem of a graph (IEP-G)

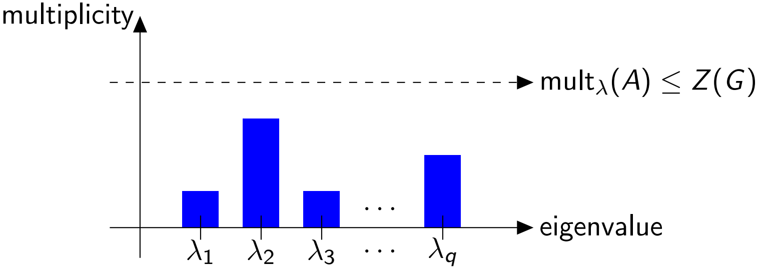
Let G be a graph. Define $\mathcal{S}(G)$ as the family of all real symmetric matrices $A = \left[a_{ij}\right]$ such that

$$a_{ij} \begin{cases} \neq 0 & \text{if } ij \in E(G), i \neq j; \\ = 0 & \text{if } ij \notin E(G), i \neq j; \\ \in \mathbb{R} & \text{if } i = j. \end{cases}$$



IEP-G: What are the possible spectra of a matrix in $\mathcal{S}(G)$?

Ordered multiplicity list



$$\mathrm{spec}(A) = \{\lambda_1^{(m_1)}, \ldots, \lambda_q^{(m_q)}\} \implies \begin{array}{l} \mathrm{m}(A) = (m_1, \ldots, m_q), \\ q(A) = q \end{array}$$

Supergraph Lemma

Lemma (BFHHLS 2017)

Let G and H' be two graphs with $V(G) = V(H')$ and $E(G) \subseteq E(H')$. If $A \in \mathcal{S}(G)$ has the SSP, then there is a matrix $A' \in \mathcal{S}(H')$ such that

- $\text{spec}(A') = \text{spec}(A)$,
- *A' has the SSP, and*
- *$\|A' - A\|$ can be chosen arbitrarily small.*

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} \sim 1 & \epsilon & 0 & 0 \\ \epsilon & \sim 2 & \epsilon & 0 \\ 0 & \epsilon & \sim 3 & \epsilon \\ 0 & 0 & \epsilon & \sim 4 \end{bmatrix}$$

SSP will be defined later

Matrix derivative

Definition

Let U and W be open subsets in vector spaces over $\mathbb{R}$ and $F : U \to W$ a function.
The derivative of F at a point $\mathrm{u}_0 \in U$ is

$$\dot{F} \cdot \mathrm{d} = \lim_{t \to 0} \frac{F(\mathrm{u}_0 + t\mathrm{d}) - F(\mathrm{u}_0)}{t},$$

which is a linear operator sending a direction to the directional derivative.

Example: $F(K) = e^K$

Define $F : \mathrm{Skew}_n(\mathbb{R}) \to \mathrm{Mat}_n(\mathbb{R})$ by $F(K) = e^K$.

Then $\dot{F}$ at O is $\dot{F} \cdot K = K$ since

$$\begin{aligned}
\dot{F} \cdot K &= \lim_{t \to 0} \frac{e^{O+Kt} - e^{O}}{t} \\
&= \lim_{t \to 0} \frac{1}{t} \left[\frac{(Kt)^0}{0!} + \frac{(Kt)^1}{1!} + \frac{(Kt)^2}{2!} + \frac{(Kt)^3}{3!} + \cdots - I \right] \\
&= \lim_{t \to 0} \left[\frac{K^1}{1!} + \frac{K^2 t^1}{2!} + \frac{K^3 t^2}{3!} + \cdots \right] = K.
\end{aligned}$$

Inverse function theorem

Theorem (Inverse function theorem)

Let $F : U \to W$ be a smooth function. If $\dot{F}$ at a point $\mathrm{u}_0 \in U$ is invertible, then F is locally invertible around u_0.

Theorem (FHLS 2021+)

Let $F : U \to W$ be a smooth function. If $\dot{F}$ at a point $\mathrm{u}_0 \in U$ is surjective, then F is locally surjective around u_0.

Sketch of the proof

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} \sim 1 & \epsilon & 0 & 0 \\ \epsilon & \sim 2 & \epsilon & 0 \\ 0 & \epsilon & \sim 3 & \epsilon \\ 0 & 0 & \epsilon & \sim 4 \end{bmatrix}$$

A $\qquad$ $A' = M - B'$

- $\mathcal{S}$: symmetric matrices that is nonzero only on the blue entries
- Define $F : \mathcal{S} \times \mathrm{Skew}_n(\mathbb{R}) \to \mathrm{Sym}_n(\mathbb{R})$ by $F(B, K) = e^{-K}Ae^{K} + B$.
- SSP $\iff$ $\dot{F}$ is surjective!
- For any M nearby A, there is B' and K' such that

$$e^{-K'}Ae^{K'} + B' = M.$$

- Choose proper M and let $A' = M - B'$.

Sketch of the proof

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} \sim 1 & \epsilon & 0 & 0 \\ \epsilon & \sim 2 & \epsilon & 0 \\ 0 & \epsilon & \sim 3 & \epsilon \\ 0 & 0 & \epsilon & \sim 4 \end{bmatrix} = A' = M - B'$$

- $\mathcal{S}$: symmetric matrices that is nonzero only on the blue entries
- Define $F : \mathcal{S} \times \mathrm{Skew}_n(\mathbb{R}) \to \mathrm{Sym}_n(\mathbb{R})$ by $F(B, K) = e^{-K}Ae^{K} + B$.
- SSP $\iff$ $\dot{F}$ is surjective!
- For any M nearby A, there is B' and K' such that

$$e^{-K'}Ae^{K'} + B' = M.$$

- Choose proper M and let $A' = M - B'$.

Sketch of the proof

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}}_{A} \longrightarrow \underbrace{\begin{bmatrix} \sim 1 & \epsilon & 0 & 0 \\ \epsilon & \sim 2 & \epsilon & 0 \\ 0 & \epsilon & \sim 3 & \epsilon \\ 0 & 0 & \epsilon & \sim 4 \end{bmatrix}}_{A' = M - B'}$$

- $\mathcal{S}$: symmetric matrices that is nonzero only on the blue entries
- Define $F : \mathcal{S} \times \mathrm{Skew}_n(\mathbb{R}) \to \mathrm{Sym}_n(\mathbb{R})$ by $F(B, K) = e^{-K} A e^{K} + B$.
- SSP $\iff$ $\dot{F}$ is surjective!
- For any M nearby A, there is B' and K' such that

$$e^{-K'} A e^{K'} + B' = M.$$

- Choose proper M and let $A' = M - B'$.

Sketch of the proof

$$\underset{A}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}} \longrightarrow \underset{A' = M - B'}{\begin{bmatrix} \sim 1 & \epsilon & 0 & 0 \\ \epsilon & \sim 2 & \epsilon & 0 \\ 0 & \epsilon & \sim 3 & \epsilon \\ 0 & 0 & \epsilon & \sim 4 \end{bmatrix}}$$

- $\mathcal{S}$: symmetric matrices that is nonzero only on the blue entries
- Define $F : \mathcal{S} \times \mathrm{Skew}_n(\mathbb{R}) \to \mathrm{Sym}_n(\mathbb{R})$ by $F(B, K) = e^{-K} A e^{K} + B$.
- SSP $\iff$ $\dot{F}$ is surjective!
- For any M nearby A, there is B' and K' such that

$$e^{-K'} A e^{K'} + B' = M.$$

- Choose proper M and let $A' = M - B'$.

Sketch of the proof

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}}_{A} \longrightarrow \underbrace{\begin{bmatrix} \sim 1 & \epsilon & 0 & 0 \\ \epsilon & \sim 2 & \epsilon & 0 \\ 0 & \epsilon & \sim 3 & \epsilon \\ 0 & 0 & \epsilon & \sim 4 \end{bmatrix}}_{A' = M - B'}$$

- $\mathcal{S}$: symmetric matrices that is nonzero only on the blue entries
- Define $F : \mathcal{S} \times \mathrm{Skew}_n(\mathbb{R}) \to \mathrm{Sym}_n(\mathbb{R})$ by $F(B, K) = e^{-K} A e^{K} + B$.
- SSP $\iff$ $\dot{F}$ is surjective!
- For any M nearby A, there is B' and K' such that

$$e^{-K'} A e^{K'} + B' = M.$$

- Choose proper M and let $A' = M - B'$.

The derivative of $F(B,K) = e^{-K}Ae^{K} + B$

At (O,O),

$$\dot{F} = K^\top A + AK + B$$

- $K \in \mathrm{Skew}_n(\mathbb{R})$
- $B \in \mathcal{S}^{\mathrm{cl}}(G)$, where $\mathcal{S}^{\mathrm{cl}}(G)$ is the topological closure of $\mathcal{S}(G)$. That is,

$$\mathcal{S}^{\mathrm{cl}}(G) = \{A = [a_{i,j}] \in \mathrm{Sym}_n(\mathbb{R}) : a_{i,j} = 0 \iff \{i,j\} \in E(\overline{G})\}.$$

$\dot{F}$ is surjective at (O,O) $\iff$
$\{K^\top A + AK : K \in \mathrm{Skew}_n(\mathbb{R})\} + \mathcal{S}^{\mathrm{cl}}(G) = \mathrm{Sym}_n(\mathbb{R})$.

The derivative of $F(B, K) = e^{-K} A e^{K} + B$

At (O, O),

$$\dot{F} = K^\top A + AK + B$$

- $K \in \mathrm{Skew}_n(\mathbb{R})$
- $B \in \mathcal{S}^{\mathrm{cl}}(G)$, where $\mathcal{S}^{\mathrm{cl}}(G)$ is the topological closure of $\mathcal{S}(G)$. That is,

$$\mathcal{S}^{\mathrm{cl}}(G) = \{A = [a_{i,j}] \in \mathrm{Sym}_n(\mathbb{R}) : a_{i,j} = 0 \iff \{i, j\} \in E(\overline{G})\}.$$

$$\dot{F} \text{ is surjective at } (O, O) \iff$$
$$\{K^\top A + AK : K \in \mathrm{Skew}_n(\mathbb{R})\} + \mathcal{S}^{\mathrm{cl}}(G) = \mathrm{Sym}_n(\mathbb{R}).$$

Strong spectral property (SSP)

Definition

A symmetric matrix A has the strong spectral property (SSP) if $X = O$ is the only real symmetric matrix that satisfies the following matrix equations:

- $A \circ X = O, I \circ X = O,$
- $AX - XA = O.$

Proposition (FHLS 2021+)

A symmetric matrix $A \in \mathcal{S}(G)$ has the SSP if and only if

$$\{K^\top A + AK : K \in \mathrm{Skew}_n(\mathbb{R})\} + \mathcal{S}^{\mathrm{cl}}(G) = \mathrm{Sym}_n(\mathbb{R}).$$

Extended SSP

Definition

Let G and H be two graphs such that $V(G) = V(H)$ and $E(G) \subseteq E(H)$. A matrix $A \in \mathcal{S}(G)$ has the SSP with respect to H if $X = O$ is the only real symmetric matrix that satisfies the following matrix equations:

- $X \in \mathcal{S}^{\mathrm{cl}}(\overline{H})$, $I \circ X = O$,
- $AX - XA = O$.

Proposition (FHLS 2021+)

A symmetric matrix $A \in \mathcal{S}(G)$ has the SSP with respect to H if and only if

$$\{K^\top A + AK : K \in \mathrm{Skew}_n(\mathbb{R})\} + \mathcal{S}^{\mathrm{cl}}(H) = \mathrm{Sym}_n(\mathbb{R}).$$

Extended supergraph lemma

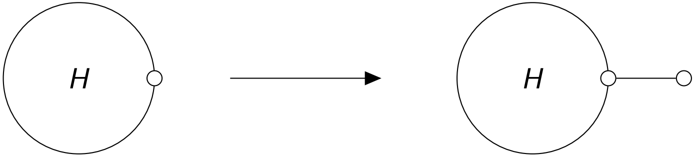
Lemma (L, Oblak, and Šmigoc 2021)

Let G, H, and H' be three graphs such that $V(G) = V(H) = V(H')$ and $E(G) \subseteq E(H) \subseteq E(H')$. If $A \in \mathcal{S}(G)$ has the SSP with respect to H, then there is a matrix $A' \in \mathcal{S}^{\mathrm{cl}}(H')$ such that

- $\operatorname{spec}(A) = \operatorname{spec}(A')$,
- *A' has the SSP, and*
- *$\|A' - A\|$ can be chosen arbitrarily small.*

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} \sim 1 & \sim 0 & \epsilon & 0 \\ \sim 0 & \sim 1 & 0 & \epsilon \\ \epsilon & 0 & \sim 2 & \sim 0 \\ 0 & \epsilon & \sim 0 & \sim 2 \end{bmatrix}$$

Appending a leaf



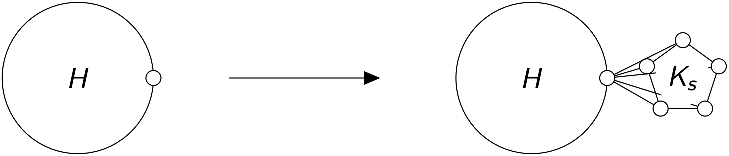
$A \in \mathcal{S}(H)$
SSP

$A' \in \mathcal{S}(H')$
SSP
$\operatorname{spec}(A') = \operatorname{spec}(A) \cup \{\lambda\}$

Theorem (BFHHLS 2017)

Let H be a graph and H' be obtained from H by appending a leaf. If $A \in \mathcal{S}(H)$ has the SSP and $\lambda \notin \operatorname{spec}(A)$, then there is a matrix $A' \in \mathcal{S}(H')$ such that $\operatorname{spec}(A') = \operatorname{spec}(A) \cup \{\lambda\}$.

Appending a clique

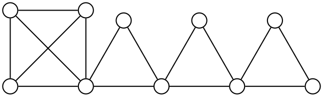


$A \in \mathcal{S}(H)$
SSP

$A' \in \mathcal{S}(H')$
SSP
$\mathrm{spec}(A') = \mathrm{spec}(A) \cup \{\lambda^{(s)}\}$

Theorem (L, Oblak, and Šmigoc 2021)

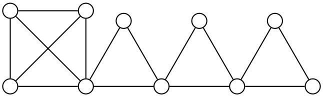
Let H be a graph and H' be obtained from H by appending a clique K_s. If $A \in \mathcal{S}(H)$ and $\lambda \notin \mathrm{spec}(A) \cup \mathrm{spec}(A(v))$ for all v, then there is a matrix $A' \in \mathcal{S}(H')$ such that $\mathrm{spec}(A') = \mathrm{spec}(A) \cup \{\lambda^{(s)}\}$.



allows ordered multiplicity list $(2, 2, 2, 2, 2)$

$$\left[\begin{array}{c|ccc} A & & O & \\ \hline & \lambda & & 0 \\ O & & \ddots & \\ & 0 & & \lambda \end{array}\right] \longrightarrow \left[\begin{array}{cc|ccc} \sim A & & & O & \\ & & \epsilon & \cdots & \epsilon \\ \hline & \epsilon & \sim \lambda & & \sim 0 \\ O & \vdots & & \ddots & \\ & \epsilon & \sim 0 & & \sim \lambda \end{array}\right]$$

Thanks!



allows ordered multiplicity list $(2, 2, 2, 2, 2)$

$$\left[\begin{array}{c|ccc} A & & O & \\ \hline & \lambda & & 0 \\ O & & \ddots & \\ & 0 & & \lambda \end{array}\right] \longrightarrow \left[\begin{array}{cc|ccc} \sim A & & & O & \\ & & \epsilon & \cdots & \epsilon \\ \hline & \epsilon & \sim\lambda & & \sim 0 \\ O & \vdots & & \ddots & \\ & \epsilon & \sim 0 & & \sim\lambda \end{array}\right]$$

Thanks!

References I

W. Barrett, S. M. Fallat, H. T. Hall, L. Hogben, J. C.-H. Lin, and B. Shader.
Generalizations of the Strong Arnold Property and the minimum number of distinct eigenvalues of a graph.
Electron. J. Combin., 24:#P2.40, 2017.

J. C.-H. Lin, P. Oblak, and H. Šmigoc.
On the inverse eigenvalue problem for block graphs.
Linear Algebra Appl., 631:379–397, 2021.