Strong properties from a universal point of view

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Outline

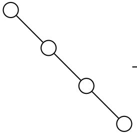
1. Introduction and motivation
2. Supergraph lemma and bifurcation lemma
3. Supergraph lemma and liberation lemma
4. Strong properties hidden in the history

Introduction and motivation

Inverse eigenvalue problem of a graph (IEP-G)

Let G be a graph. Define $\mathcal{S}(G)$ as the family of all real symmetric matrices $A = \left[a_{ij}\right]$ such that

$$a_{ij} \begin{cases} \neq 0 & \text{if } ij \in E(G), i \neq j; \\ = 0 & \text{if } ij \notin E(G), i \neq j; \\ \in \mathbb{R} & \text{if } i = j. \end{cases}$$



$$\longrightarrow \begin{bmatrix} ? & * & 0 & 0 \\ * & ? & * & 0 \\ 0 & * & ? & * \\ 0 & 0 & * & ? \end{bmatrix} \underset{?}{\overset{\text{spec}}{\rightleftarrows}} \{1, 2, 3, 4\}$$

IEP-G: What are the possible spectra of a matrix in $\mathcal{S}(G)$?

Inverse eigenvalue problem of a graph (IEP-G)

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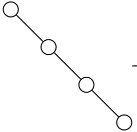

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IEP-G: What are the possible spectra of a matrix in $\mathcal{S}(G)$?

Supergraph lemma

Lemma (BFHHLS 2017)

Let G and H' be two graphs with $V(G) = V(H')$ and $E(G) \subseteq E(H')$. If $A \in \mathcal{S}(G)$ has the SSP, then there is a matrix $A' \in \mathcal{S}(H')$ such that

- $\operatorname{spec}(A') = \operatorname{spec}(A)$,
- *A' has the SSP, and*
- *$\|A' - A\|$ can be chosen arbitrarily small.*

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} \sim 1 & \epsilon & 0 & 0 \\ \epsilon & \sim 2 & \epsilon & 0 \\ 0 & \epsilon & \sim 3 & \epsilon \\ 0 & 0 & \epsilon & \sim 4 \end{bmatrix}$$

SSP will be defined later

BIRS Workshop in 2016

- Barrett, Fallat, Hall, Hogben, Lin, and Shader.
  Electron. J. Combin., 24:#P2.40, 2017.
- Barrett, Butler, Fallat, Hall, Hogben, Lin, Shader, and Young.
  J. Combin. Theory Ser. B, 142:276–306, 2020.

A long history ...

Strong Arnold Hypothesis

- Arnold 1971 studied matrices depending on parameters.
- Arnold 1972 introduced the transversality of deformation of operators.
- Colin de Verdière 1988 introduced the strong Arnold Hypothesis.
- Colin de Verdière 1990 introduced the parameter μ.

Inverse problems

- Nonnegative IEP: Laffey 1998.
- Sign pattern: DJOvdD 2000, Garnett and Shader 2013.
- IEP-G: Monfared and Shader 2013.

Yet more research works are ongoing!

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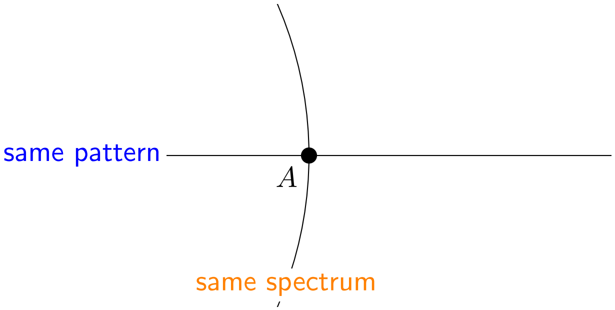
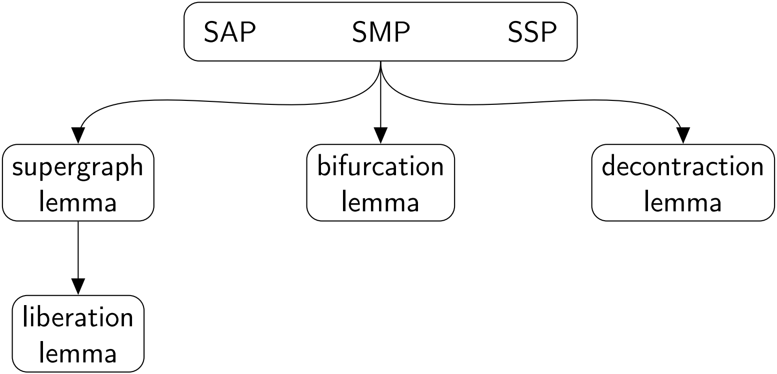
Big picture of the strong property



strong property $\sim$ transversal intersection $\sim$

a generic condition of A such that
the two coordinates controls the neighborhood locally

Big picture of the theory



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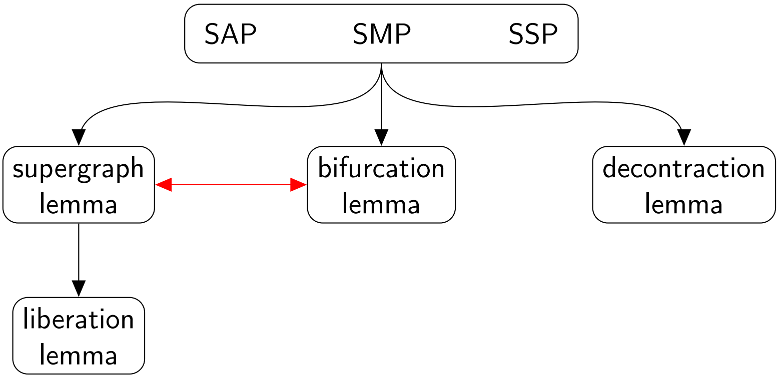

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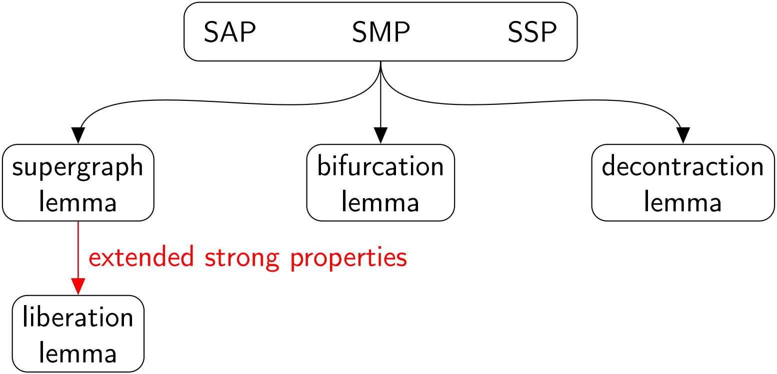

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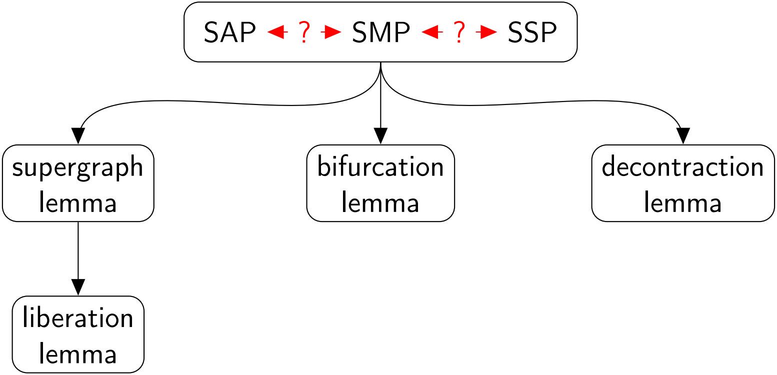
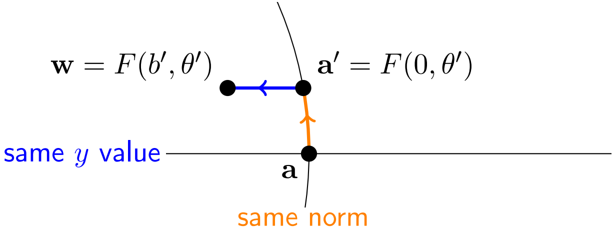


Supergraph lemma and bifurcation lemma

Inverse function theorem in $\mathbb{R}^2$

Fix a point $\mathbf{a} \in \mathbb{R}^2$. Combine two perturbations:

$$\begin{cases} \mathbf{a} + b\mathbf{e}_1 & \text{(same } y\text{)} \\ R_\theta \mathbf{a} & \text{(same norm)} \end{cases} \implies F(b, \theta) = R_\theta \mathbf{a} + b\mathbf{e}_1$$



$\frac{dF}{db,\theta}$ invertible $\implies$ any nearby $\mathbf{w}$ can be written as $\mathbf{w} = F(b', \theta')$

For whatever y value nearby, there is $\mathbf{a}'$ with $\|\mathbf{a}'\| = \|\mathbf{a}\|$.

Inverse function theorem

Theorem (Inverse function theorem)

Let $F : U \to W$ be a smooth function. If $\dot{F}$ at a point $\mathbf{u}_0 \in U$ is invertible, then F is locally invertible around $\mathbf{u}_0$.

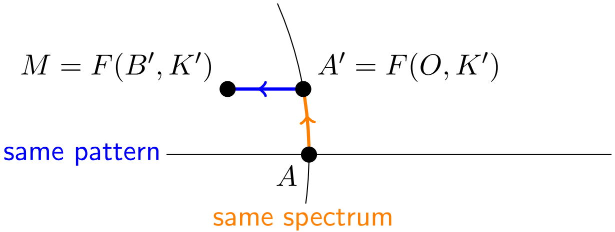
Theorem (FHLS 2022)

Let $F : U \to W$ be a smooth function. If $\dot{F}$ at a point $\mathbf{u}_0 \in U$ is surjective, then F is locally surjective around $\mathbf{u}_0$.

Inverse function theorem in $\mathrm{Sym}_n(\mathbb{R})$

Fix a point $A \in \mathcal{S}(G)$. Combine two perturbations:

$$\begin{cases} A + B & \text{(same pattern)} \\ e^{-K}Ae^{K} & \text{(same spectrum)} \end{cases} \implies F(B,K) = e^{-K}Ae^{K} + B$$



$\dot{F}$ surjective $\implies$ any nearby M can be written as $M = F(B', K')$

For whatever pattern nearby, there is A' with $\mathrm{spec}(A') = \mathrm{spec}(A)$.

Pattern perturbation

$$F(B, K) = e^{-K} A e^{K} + B$$

Define $\mathcal{S}^{\mathrm{cl}}(G)$ as the topological closure of $\mathcal{S}(G)$:

$$\mathcal{S}^{\mathrm{cl}}(G) = \{A = \left[a_{i,j}\right] \in \mathrm{Sym}_n(\mathbb{R}) : a_{i,j} = 0 \iff \{i, j\} \in E(\overline{G})\}.$$

Let $A \in \mathcal{S}(G)$. Then $A + B \in \mathcal{S}(G)$ when $\|B\|$ is small enough.

The tangent space of $F(B, K)$ at (O, O) with respect to B is $\mathcal{S}^{\mathrm{cl}}(G)$.

Isospectral perturbation

$$F(B,K) = e^{-K}Ae^{K} + B$$

The function e^K is a bijection between

$$\{\text{skew-symmetric matrices nearby } O\} \to \{\text{orthogonal matrices nearby } I\}$$

for real matrices.

The tangent space of $F(B,Q)$ at (O,O) with respect to Q is

$$\{-KA + AK : K \in \mathrm{Skew}_n(\mathbb{R})\}.$$

Strong spectral property, transversality, and surjectivity

Definition

Let A be a real symmetric matrix. Then A has the strong spectral property (SSP) if $X = O$ is the only real symmetric matrix that satisfies

$$A \circ X = I \circ X = [A, X] = O.$$

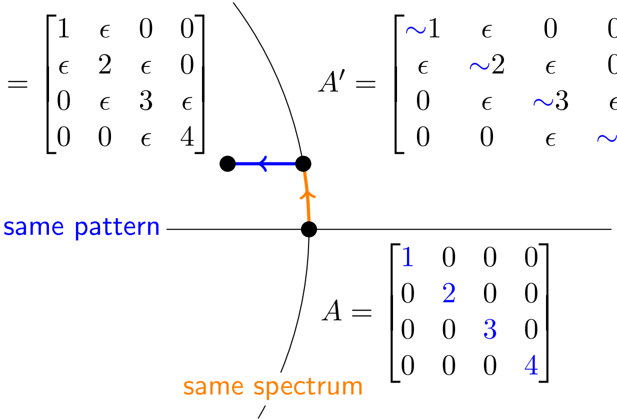
Let $F(B, K) = e^{-K} A e^{K} + B$. Then the following are equivalent:

1. A has the SSP.
2. $\mathcal{S}^{\mathrm{cl}}(G) + \{-KA + AK : K \in \mathrm{Skew}_n(\mathbb{R})\} = \mathrm{Sym}_n(\mathbb{R})$.
3. The derivative $\dot{F}$ is surjective.

Illustration of the supergraph lemma

$$F(B,K) = e^{-K}Ae^{K} + B$$

$$M = \begin{bmatrix} 1 & \epsilon & 0 & 0 \\ \epsilon & 2 & \epsilon & 0 \\ 0 & \epsilon & 3 & \epsilon \\ 0 & 0 & \epsilon & 4 \end{bmatrix} \qquad A' = \begin{bmatrix} \sim 1 & \epsilon & 0 & 0 \\ \epsilon & \sim 2 & \epsilon & 0 \\ 0 & \epsilon & \sim 3 & \epsilon \\ 0 & 0 & \epsilon & \sim 4 \end{bmatrix}$$



$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

For whatever pattern nearby, there is A' with $\operatorname{spec}(A') = \operatorname{spec}(A)$.

They must be true, right?

Let $A \in \mathcal{S}(G)$ with the SSP. People *believed* that ...

- For any set of real numbers Λ' nearby $\operatorname{spec}(A)$, there is a matrix $A' \in \mathcal{S}(G)$ with $\operatorname{spec}(A') = \Lambda'$.
- For any refinement $\mathbf{m}'$ of $\mathbf{m}(A)$, there is a matrix $A' \in \mathcal{S}(G)$ with $\mathbf{m}(A') = \mathbf{m}'$.
- For any $k > q(A)$, there is a matrix $A' \in \mathcal{S}(G)$ with $q(A') = k$.

Let $A \in \mathcal{Q}(P)$ be a nilpotent matrix with the nSSP. People *knew* that ...

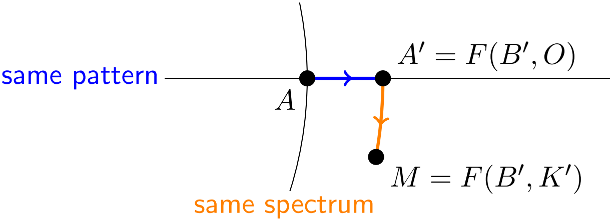
- For any set of complex numbers Λ' (invariant under conjugation) nearby $\{0, \ldots, 0\}$, there is a matrix $A' \in \mathcal{Q}(P)$ with $\operatorname{spec}(A') = \Lambda'$.

nSSP = the condition of the nilpotent-centralizer method

Bifurcation lemma

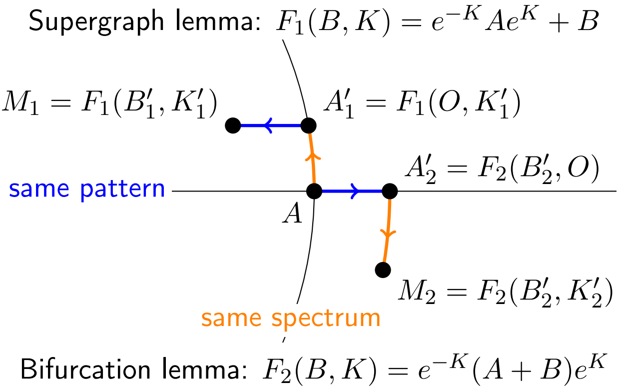
Lemma (FHLS 2022)

Let $A \in \mathcal{S}(G)$ with the SSP. Then for any set of real numbers Λ' nearby $\operatorname{spec}(A)$, there is a matrix A' with $\operatorname{spec}(A') = \Lambda'$.



$$F(B, K) = e^{-K}(A + B)e^{K}$$

Supergraph lemma and bifurcation lemma



The argument also applies to ...

The universal space $\mathrm{Sym}_n(\mathbb{R})$ can also be $\mathrm{Mat}_n(\mathbb{R})$, or matrices over other fields.

The spectrum can be replaced by:

- nullity (strong Arnold property),
- ordered multiplicity list (strong multiplicity property),
- orthogonality (strong inner product property),
- nullity and nullity of some principal submatrix (strong nullity interlacing property) ...

The pattern can be replaced by:

- hollow matrices,
- discrete Schrödinger operators (Colin de Verdière parameter)
- weighted normalized Laplacian matrices (Colin de Verdière 1988),
- sign patterns (nilpotent-centralizer method, non-symmetric strong spectral property, similarity-transversality property) ...

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Supergraph lemma and liberation lemma

Supergraph lemma and liberation lemma

The supergraph lemma allows us to add arbitrary edges.

Lemma (BFHHLS 2017)

Let G be a graph, $A \in \mathcal{S}(G)$ with the SSP, and $\beta \subseteq E(\overline{G})$. Then there exists a matrix $A' \in \mathcal{S}(G+\beta)$ with the SSP and $\operatorname{spec}(A') = \operatorname{spec}(A)$.

When A does not have the SSP, the liberation lemma allows us to add some specific set of edges.

Lemma (BBFHHLSY 2020, L, Oblak, and Šmigoc 2023)

Let G be a graph, $A \in \mathcal{S}(G)$, and β an SSP liberation set of A. Then there exists a matrix $A' \in \mathcal{S}(G+\beta)$ with the SSP and $\operatorname{spec}(A') = \operatorname{spec}(A)$.

Strong spectral property

Let G be a graph on n vertices and A, X be $n \times n$ real symmetric matrices.

- $X \circ G = O$ means X is zero on those entries corresponding to $E(G)$ and on the diagonal.
- $[A, X] = AX - XA$.

Definition

A matrix $A \in \mathcal{S}(G)$ has the strong spectral property (SSP) if $X \circ G = O$ and $[A, X] = O$ implies $X = O$.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad X = \begin{bmatrix} 0 & 0 & x & y \\ 0 & 0 & 0 & z \\ x & 0 & 0 & 0 \\ y & z & 0 & 0 \end{bmatrix}$$

Example of no SSP

Let $A \in \mathcal{S}(K_2 \dot\cup K_2)$.

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \qquad X = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

Then $[A, X] = O$.

Extended strong spectral property

Definition

Let G be a spanning subgraph of H. A matrix $A \in \mathcal{S}(G)$ has the strong spectral property with respect to H if $X \circ H = O$ and $[A, X] = O$ implies $X = O$.

Consider $G = 2K_2$ and $H = P_4$. Then $A \in \mathcal{S}(G)$ has the SSP with respect to H.

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \qquad X = \begin{bmatrix} 0 & 0 & x & y \\ 0 & 0 & 0 & z \\ x & 0 & 0 & 0 \\ y & z & 0 & 0 \end{bmatrix}$$

This property treat A as a matrix in $\mathcal{S}^{\mathrm{cl}}(H)$, where entries in $E(H)$ can move freely.

The logic

Let $A \in \mathcal{S}(G)$.

- A has the SSP $=$ A has the SSP with respect to G.
- A has the SSP with respect to $H \implies$
  A has the SSP with respect to H'
  if H is a spanning subgraph of H'
- Any $n \times n$ matrix A has the SSP with respect to K_n.

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Extended supergraph lemma

Supergraph lemma: Free entries in $\mathcal{S}(G)$ changes correspondingly for any perturbation in $E(\overline{G})$.

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} \sim 0 & \sim 1 & \epsilon & 0 \\ \sim 1 & \sim 0 & \sim 1 & 0 \\ \epsilon & \sim 1 & \sim 0 & \sim 1 \\ 0 & 0 & \sim 1 & \sim 0 \end{bmatrix}$$

Extended supergraph lemma: Free entries in $\mathcal{S}(H)$ changes correspondingly for any perturbation in $E(\overline{H})$.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} \sim 1 & \sim 1 & \epsilon & 0 \\ \sim 1 & \sim 1 & \sim 0 & 0 \\ \epsilon & \sim 0 & \sim -1 & \sim 1 \\ 0 & 0 & \sim 1 & \sim -1 \end{bmatrix}$$

Liberation set

Definition

Let G be a graph and $A \in \mathcal{S}(G)$. A nonempty set of edges $\beta \subseteq E(\overline{G})$ is called an SSP liberation set of A if A has the SSP with respect to $G + \beta'$ for all $\beta' \subset \beta$ with $|\beta| = |\beta| - 1$.

For example, $\beta = \{\{1,3\},\{2,3\}\}$ is an SSP liberation set of

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

Illustration of the liberation lemma

A has the SSP with respect to $G + \beta'$ for all $\beta' \subset \beta$ with $|\beta| = |\beta| - 1$, so it can be perturbed in many different ways.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{1} \qquad \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{2}$$

Key: Make new nonzero entries one by one.

$$A \xrightarrow{1} \begin{bmatrix} \sim 1 & \sim 1 & \epsilon_1 & 0 \\ \sim 1 & \sim 1 & \sim 0 & 0 \\ \epsilon_1 & \sim 0 & \sim -1 & \sim 1 \\ 0 & 0 & \sim 1 & \sim -1 \end{bmatrix} \xrightarrow{2} \begin{bmatrix} \sim 1 & \sim 1 & \sim \epsilon_1 & 0 \\ \sim 1 & \sim 1 & \sim \epsilon_2 & 0 \\ \sim \epsilon_1 & \sim \epsilon_2 & \sim -1 & \sim 1 \\ 0 & 0 & \sim 1 & \sim -1 \end{bmatrix}$$

Strong properties hidden in the history

They all talked about different things ...

- Arnold 1971 studied matrices depending on parameters.
- Arnold 1972 introduced the transversality of deformation of operators.
- Colin de Verdière 1988 introduced the strong Arnold Hypothesis.
- Colin de Verdière 1990 introduced the parameter μ.

	property	domain	notes
[1]	$\sim$SSP	discrete	$\mathrm{Mat}_n(\mathbb{C})$, bifurcation
[2]	$\sim$SMP	continuous discrete	"*The hypothesis of transversality must be accepted without proof.*"
[3]	SAP $\to$ SSP	continuous discrete	weak Arnold hypothesis weighted normalized Laplacian matrices $D^{-\frac{1}{2}}LD^{-\frac{1}{2}}$
[4]	SAP	discrete	$\mu(G) \leq 3 + \mathrm{cr}(G)$ $\mu(G) \leq m(X)$ if G can be embedded into manifold X

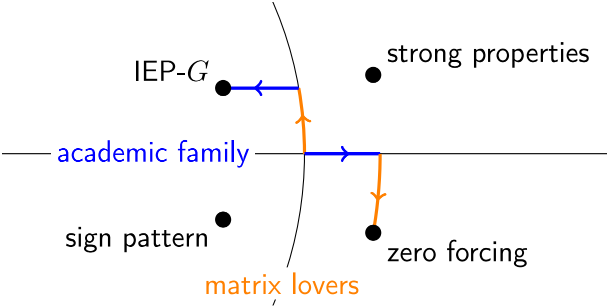
Yet more to be explored ...

Colin de Verdière.
Sur une hypothèse de transversalité d'Arnold.
Comment. Math. Helv., 63:184–193, 1988.

Tous ces résultats sont utilisables dans la construction de métriques riemanniennes ou de domaines euclidiens de $\mathbf{R}^n$ dont une partie finie du spectre est prescrit ([C–C], [CV3]). L'ensemble de ces résultats est annoncé dans [CV2].

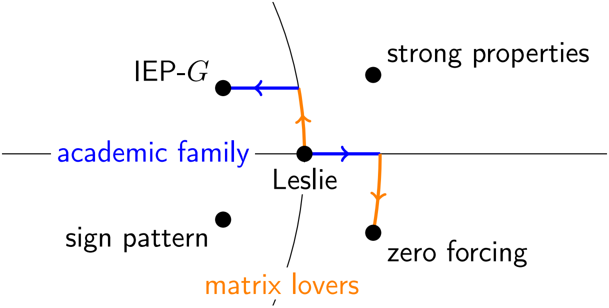
All these results are applicable to the construction of the Riemannian metrics or the Euclidean domain of $\mathbb{R}^n$ where part of spectrum is prescribed ([C-C], [CV3]). All of these results is announced in [CV2].

Strong advisor property



all nearby problems can be resolved through collaboration between students and colleagues

Strong advisor property



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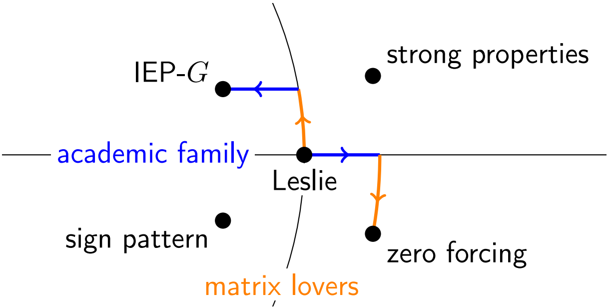
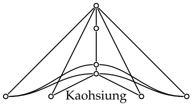


all nearby problems can be resolved through collaboration between students and colleagues

Thanks!

ILAS2025 in Taiwan



https://ilas2025.tw/

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