

Inverse eigenvalue problem of a graph

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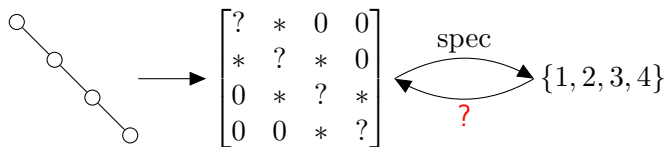
November 8, 2028

2024 Japan-Taiwan Joint Workshop
on Inverse Problems and Related Topics

Inverse eigenvalue problem of a graph (IEP- G)

Let G be a graph. Define $\mathcal{S}(G)$ as the family of all real symmetric matrices $A = [a_{ij}]$ such that

$$a_{ij} \begin{cases} \neq 0 & \text{if } ij \in E(G), i \neq j; \\ = 0 & \text{if } ij \notin E(G), i \neq j; \\ \in \mathbb{R} & \text{if } i = j. \end{cases}$$

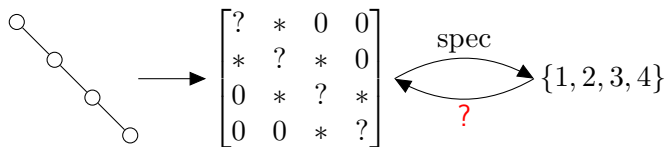


IEP- G : What are the possible spectra of a matrix in $\mathcal{S}(G)$?

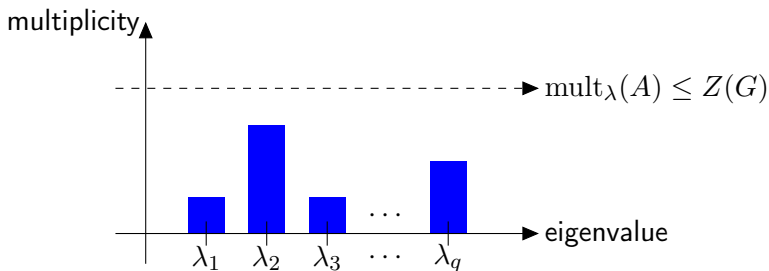
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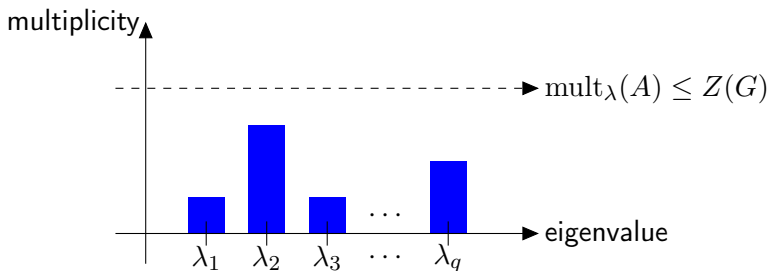
$$a_{ij} \begin{cases} \neq 0 & \text{if } ij \in E(G), i \neq j; \\ = 0 & \text{if } ij \notin E(G), i \neq j; \\ \in \mathbb{R} & \text{if } i = j. \end{cases}$$



IEP- G : What are the possible spectra of a matrix in $\mathcal{S}(G)$?



$$\text{spec}(A) = \{\lambda_1^{(m_1)}, \dots, \lambda_q^{(m_q)}\} \implies \begin{aligned} \mathbf{m}(A) &= (m_1, \dots, m_q), \\ q(A) &= q \end{aligned}$$



Questions

What are possible $\mathbf{m}(A)$ and what are

$$M(G) = \max\{\text{mult}_\lambda(A) : A \in \mathcal{S}(G), \lambda \in \text{spec}(A)\},$$

$$q(G) = \min\{q(A) : A \in \mathcal{S}(G)\}?$$

$$\begin{bmatrix} ? & * & & & \\ * & ? & * & & \\ & * & \ddots & \ddots & \\ & & \ddots & & * \\ & & & * & ? \end{bmatrix} \in \mathcal{S}(P_n) \quad \lambda_1 < \cdots < \lambda_n$$

- $\{\text{Rows } 2 \sim n\}$ and $\{\text{Rows } 1 \sim n-1\}$ are always independent.
- $\text{mult}(\lambda) = \text{null}(A - \lambda I) \leq 1$ for any $A \in \mathcal{S}(P_n)$ and $\lambda \in \mathbb{R}$.

$$\begin{bmatrix} ? & * & & & \\ * & ? & * & & \\ & * & \ddots & \ddots & \\ & & \ddots & & * \\ & & & * & ? \end{bmatrix} \in \mathcal{S}(P_n) \quad \lambda_1 < \cdots < \lambda_n$$

Theorem (Gray and Wilson 1976; and Hald 1976)

For any set $\Lambda = \{\lambda_1, \dots, \lambda_n\}$ of n *distinct* real numbers, there is a matrix $A \in \mathcal{S}(P_n)$ such that $\text{spec}(A) = \Lambda$.

$$\begin{bmatrix} ? & * & & & * \\ * & ? & * & & \\ & * & \ddots & \ddots & \\ & & \ddots & & * \\ * & & & * & ? \end{bmatrix} \in \mathcal{S}(C_n)$$

$$\lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 < \dots, \text{ or}$$

$$\lambda_1 < \lambda_2 \leq \lambda_3 < \lambda_4 \leq \lambda_5 < \dots$$

For example, $(2, 1, 2)$ is not possible for C_5 .

- $\{\text{Rows } 2 \sim n-1\}$ is always independent.
- $\text{mult}(\lambda) = \text{null}(A - \lambda I) \leq 2$ for any $A \in \mathcal{S}(C_n)$ and $\lambda \in \mathbb{R}$.

$$\begin{bmatrix} ? & * & & & * \\ * & ? & * & & \\ & * & \ddots & \ddots & \\ & & \ddots & & * \\ * & & & * & ? \end{bmatrix} \in \mathcal{S}(C_n)$$

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For example, $(2, 1, 2)$ is not possible for C_5 .

Theorem (Ferguson 1980)

For any set $\Lambda = \{\lambda_1, \dots, \lambda_n\}$ of n real numbers satisfying one of the conditions above, there is a matrix $A \in \mathcal{S}(C_n)$ such that $\text{spec}(A) = \Lambda$.

Signature similarity

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 3 & -4 & 0 \\ 0 & -4 & 5 & -6 \\ 0 & 0 & -6 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Definition

A **signature matrix** is a matrix whose diagonal entries are 1 or -1 . Two matrices A and B are **signature similar** if $B = DAD$ for some signature matrix.

Observation

Every matrix $A \in \mathcal{S}(P_n)$ is signature similar to a matrix $A' \in \mathcal{S}(P_n)$ whose off-diagonal entries are nonnegative.

What do we know about the eigenvectors?

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 3 & 4 & 0 \\ 0 & 4 & 5 & 6 \\ 0 & 0 & 6 & 7 \end{bmatrix} = QDQ^T \text{ with } Q = \begin{bmatrix} 0.33 & 0.86 & 0.38 & 0.05 \\ -0.58 & -0.13 & 0.75 & 0.28 \\ 0.63 & -0.36 & 0.17 & 0.67 \\ -0.4 & 0.34 & -0.5 & 0.69 \end{bmatrix}$$

Theorem (Ferguson 1980)

Let $A \in \mathcal{S}(P_n)$ with nonnegative off-diagonal entries. Suppose $\lambda_1 < \dots < \lambda_n$ are the eigenvalues of A and $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ the corresponding orthonormal eigenbasis. Then $(\mathbf{v}_i)_1(\mathbf{v}_i)_n$ is sign alternating for $i = 1, \dots, n$.

Adjugate

Let A be an $n \times n$ matrix.

- $A(i, j)$ is the submatrix of A by removing the i -th row and the j -th column.
- The i, j -cofactor of A is

$$c_{i,j}(A) = (-1)^{i+j} \det(A(i, j)).$$

- The **adjugate** of A is $A^{\text{adj}} = [c_{i,j}]^T$.
- It is known that $AA^{\text{adj}} = A^{\text{adj}}A = \det(A)I$, so

$$A^{\text{adj}} = \begin{cases} \det(A)A^{-1} & \text{if } \text{rank}(A) = n, \\ O & \text{if } \text{rank}(A) \leq n - 2. \end{cases}$$

Eigenvector-eigenvalue identity

Theorem (Eigenvector-eigenvalue identity)

Let A be an $n \times n$ real symmetric matrix. Suppose $\lambda_1 \leq \dots \leq \lambda_n$ are the eigenvalues of A and $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ the corresponding orthonormal eigenbasis. If $\text{mult}(\lambda_i) = 1$, then

$$(A - \lambda_i I)^{\text{adj}} = \left(\prod_{j \neq i} (\lambda_j - \lambda_i) \right) \mathbf{v}_i \mathbf{v}_i^{\top}.$$

Corollary

When $A \in \mathcal{S}(P_n)$ is a matrix with nonnegative off-diagonal entries,

$$\text{sgn}((\mathbf{v}_i)_1 (\mathbf{v}_i)_n) = (-1)^{1+n} \det((A - \lambda_i)(n, 1)) \prod_{j \neq i} (\lambda_j - \lambda_i).$$

Spectrum of $A \in \mathcal{S}(C_n)$

Theorem (Ferguson 1980)

Let $A \in \mathcal{S}(C_n)$ with eigenvalues $\lambda_1 \leq \dots \leq \lambda_n$. If $\lambda_i = \lambda_{i+1}$ and $\lambda_j = \lambda_{j+1}$ for some $i < j$, then $j - i$ is even.

Sketch of the proof

By signature similarity, we may assume

$$A = \begin{bmatrix} A^{(n)} & \mathbf{b} \\ \mathbf{b}^\top & c \end{bmatrix} \text{ with } \mathbf{b} = \begin{bmatrix} x \\ \mathbf{0} \\ y \end{bmatrix}$$

such that $A^{(n)} \in \mathcal{S}(P_{n-1})$ with **nonnegative off-diagonal** entries and eigenvalues $\mu_1 < \dots < \mu_{n-1}$.

(Let $\{\mathbf{v}_1, \dots, \mathbf{v}_{n-1}\}$ be the corresponding orthonormal eigenbasis of $A^{(n)}$.)

Spectrum of $A \in \mathcal{S}(C_n)$

Theorem (Ferguson 1980)

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Sketch of the proof

By the Cauchy interlacing theorem,

$$\lambda_i \leq \mu_i \leq \lambda_{i+1} = \lambda_i$$

implies $\lambda_i = \mu_i = \lambda_{i+1}$ and similarly $\lambda_j = \mu_j = \lambda_{j+1}$.

Spectrum of $A \in \mathcal{S}(C_n)$

Theorem (Ferguson 1980)

Let $A \in \mathcal{S}(C_n)$ with eigenvalues $\lambda_1 \leq \dots \leq \lambda_n$. If $\lambda_i = \lambda_{i+1}$ and $\lambda_j = \lambda_{j+1}$ for some $i < j$, then $j - i$ is even.

Sketch of the proof

$$A = \begin{bmatrix} A(n) & \mathbf{b} \\ \mathbf{b}^\top & c \end{bmatrix} \text{ with } \mathbf{b} = \begin{bmatrix} x \\ \mathbf{0} \\ y \end{bmatrix}$$

- Since $\text{mult}_A(\lambda_i) = 2$ and $\text{mult}_{A(n)}(\lambda_i) = 1$, $\mathbf{b} \in \text{Col}(A(n) - \mu_i I)$.
- $\mathbf{b} \perp \mathbf{v}_i \implies \text{sgn}(xy) = -\text{sgn}((\mathbf{v}_i)_1)(\mathbf{v}_i)_{n-1}$ (same for j)
- Since $\text{sgn}(xy)$ is fixed and $\text{sgn}((\mathbf{v}_j)_1)(\mathbf{v}_j)_{n-1})$ is **sign alternating**, $j - i$ is even.

For the IEP- G , we need . . .

- Combinatorial tools: zero forcing, unique shortest path, variants of zero forcing . . .
- Theory of symmetric matrices: Cauchy interlacing theorem, eigenvector-eigenvalue identity, Rayleigh quotient, Parter–Wiener theorem, Godsil’s lemma, . . .
- Analytic tools: implicit function theorem, inverse function theorem, . . .

An introductory article

S. M. Fallat, L. Hogben, J. C.-H. Lin, and B. Shader.

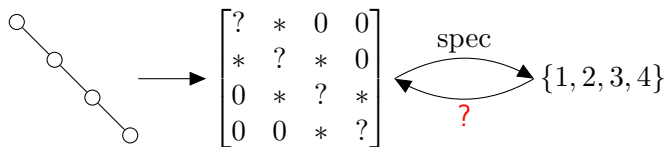
The inverse eigenvalue problem of a graph,
zero forcing, and related parameters.

Notices Amer. Math. Soc., 67:257–261, February, 2020.

Inverse eigenvalue problem of a graph (IEP- G)

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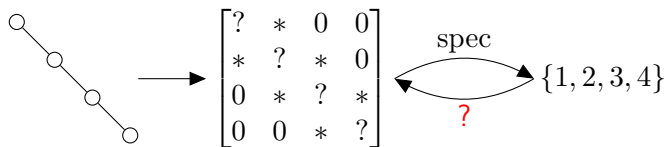


IEP- G : What are the possible spectra of a matrix in $\mathcal{S}(G)$?

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IEP- G : What are the possible spectra of a matrix in $\mathcal{S}(G)$?

Supergraph lemma

Lemma (BFHHLS 2017)

Let G and H' be two graphs with $V(G) = V(H')$ and $E(G) \subseteq E(H')$. If $A \in \mathcal{S}(G)$ has the **SSP**, then there is a matrix $A' \in \mathcal{S}(H')$ such that

- $\text{spec}(A') = \text{spec}(A)$,
- A' has the **SSP**, and
- $\|A' - A\|$ can be chosen arbitrarily small.

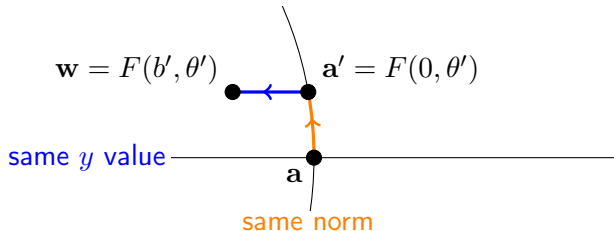
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} \sim 1 & \epsilon & 0 & 0 \\ \epsilon & \sim 2 & \epsilon & 0 \\ 0 & \epsilon & \sim 3 & \epsilon \\ 0 & 0 & \epsilon & \sim 4 \end{bmatrix}$$

SSP will be defined later

Inverse function theorem in \mathbb{R}^2

Fix a point $\mathbf{a} \in \mathbb{R}^2$. Combine two perturbations:

$$\begin{cases} \mathbf{a} + b\mathbf{e}_1 & \text{(same } y) \\ R_\theta \mathbf{a} & \text{(same norm)} \end{cases} \implies F(b, \theta) = R_\theta \mathbf{a} + b\mathbf{e}_1$$



$\frac{dF}{db, \theta}$ invertible \implies any nearby \mathbf{w} can be written as $\mathbf{w} = F(b', \theta')$

For whatever y value nearby, there is \mathbf{a}' with $\|\mathbf{a}'\| = \|\mathbf{a}\|$.

Inverse function theorem

Theorem (Inverse function theorem)

Let $F : U \rightarrow W$ be a smooth function. If \dot{F} at a point $\mathbf{u}_0 \in U$ is invertible, then F is locally invertible around \mathbf{u}_0 .

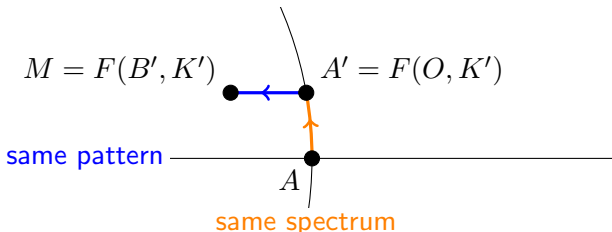
Theorem (FHLS 2022)

Let $F : U \rightarrow W$ be a smooth function. If \dot{F} at a point $\mathbf{u}_0 \in U$ is surjective, then F is locally surjective around \mathbf{u}_0 .

Inverse function theorem in $\text{Sym}_n(\mathbb{R})$

Fix a point $A \in \mathcal{S}(G)$. Combine two perturbations:

$$\begin{cases} A + B & \text{(same pattern)} \\ e^{-K} A e^K & \text{(same spectrum)} \end{cases} \implies F(B, K) = e^{-K} A e^K + B$$



\dot{F} surjective \implies any nearby M can be written as $M = F(B', K')$

For whatever pattern nearby, there is A' with $\text{spec}(A') = \text{spec}(A)$.

Pattern perturbation

$$F(B, K) = e^{-K} A e^K + B$$

Define $\mathcal{S}^{\text{cl}}(G)$ as the topological closure of $\mathcal{S}(G)$:

$$\mathcal{S}^{\text{cl}}(G) = \{A = [a_{i,j}] \in \text{Sym}_n(\mathbb{R}) : a_{i,j} = 0 \iff \{i, j\} \in E(\overline{G})\}.$$

Let $A \in \mathcal{S}(G)$. Then $A + B \in \mathcal{S}(G)$ when $\|B\|$ is small enough.

The tangent space of $F(B, K)$ at (O, O) with respect to B is $\mathcal{S}^{\text{cl}}(G)$.

Isospectral perturbation

$$F(B, K) = e^{-K} A e^K + B$$

The function e^K is a bijection between

$\{\text{skew-symmetric matrices nearby } O\} \rightarrow \{\text{orthogonal matrices nearby } I\}$

for real matrices.

The tangent space of $F(B, Q)$ at (O, O) with respect to Q is

$$\{-KA + AK : K \in \text{Skew}_n(\mathbb{R})\}.$$

Strong spectral property, transversality, and surjectivity

Definition

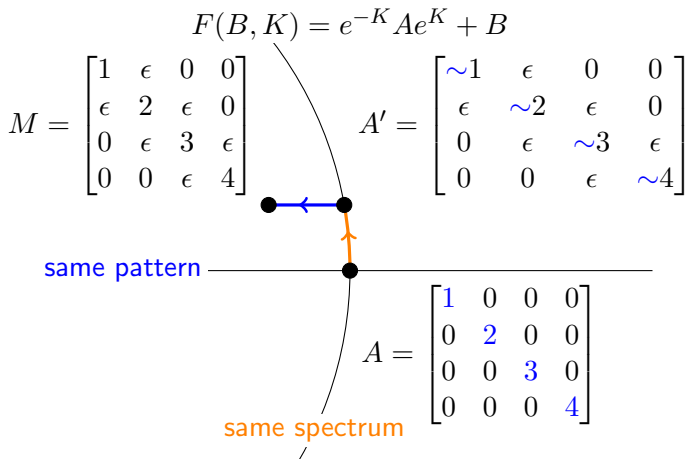
Let A be a real symmetric matrix. Then A has the **strong spectral property (SSP)** if $X = O$ is the only real symmetric matrix that satisfies

$$A \circ X = I \circ X = [A, X] = O.$$

Let $F(B, K) = e^{-K} A e^K + B$. Then the following are equivalent:

- 1 A has the SSP.
- 2 $\mathcal{S}^{\text{cl}}(G) + \{-KA + AK : K \in \text{Skew}_n(\mathbb{R})\} = \text{Sym}_n(\mathbb{R})$.
- 3 The derivative \dot{F} is surjective.

Illustration of the supergraph lemma



For whatever pattern nearby, there is A' with $\text{spec}(A') = \text{spec}(A)$.

They must be true, right?

Let $A \in \mathcal{S}(G)$ with the **SSP**. People *believed* that ...

- For any set of real numbers Λ' **nearby** $\text{spec}(A)$, there is a matrix $A' \in \mathcal{S}(G)$ with $\text{spec}(A') = \Lambda'$.
- For any **refinement** \mathbf{m}' of $\mathbf{m}(A)$, there is a matrix $A' \in \mathcal{S}(G)$ with $\mathbf{m}(A') = \mathbf{m}'$.
- For any $k > q(A)$, there is a matrix $A' \in \mathcal{S}(G)$ with $q(A') = k$.

Let $A \in \mathcal{Q}(P)$ be a nilpotent matrix with the **nSSP**. People *knew* that ...

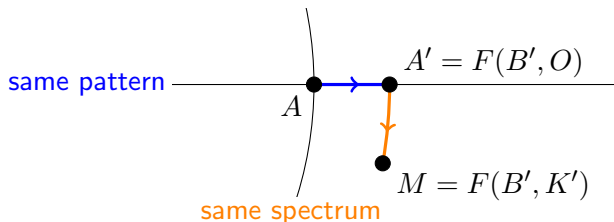
- For any set of complex numbers Λ' (**invariant under conjugation**) **nearby** $\{0, \dots, 0\}$, there is a matrix $A' \in \mathcal{Q}(P)$ with $\text{spec}(A') = \Lambda'$.

nSSP = the condition of the nilpotent-centralizer method

Bifurcation lemma

Theorem (FHLS 2022)

Let $A \in \mathcal{S}(G)$ with the SSP. Then for any set of real numbers Λ' nearby $\text{spec}(A)$, there is a matrix A' with $\text{spec}(A') = \Lambda'$.



$$F(B, K) = e^{-K}(A + B)e^K$$

The nSSP

Definition

Let A be a real matrix. Then A has the **non-symmetric strong spectral property (nSSP)** if $X = O$ is the only real matrix that satisfies

$$A \circ X = [A, X^T] = O.$$

Let $Q^v(P)$ be the set of matrices with the same zero entries as P .

Let $F(B, Q) = Q^{-1}(A + B)Q$, where $B \in Q^v(P)$. Then the following are equivalent:

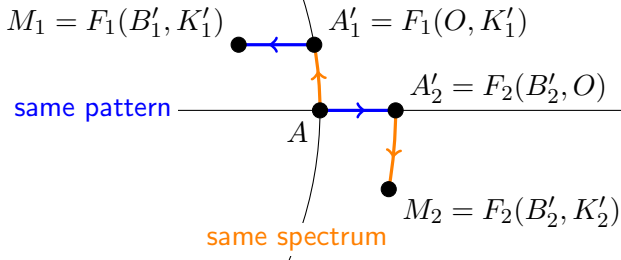
- 1 A has the nSSP.
- 2 $Q^v(P) + \{-LA + AL : L \in \text{Mat}_n(\mathbb{R})\} = \text{Mat}_n(\mathbb{R})$.
- 3 The derivative \dot{F} is surjective.

Bifurcation lemma for non-symmetric matrices

Theorem (FHLS 2022)

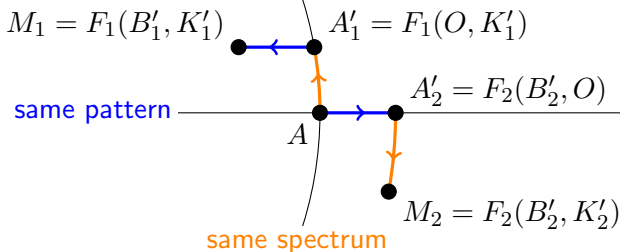
Let $A \in Q(P)$ with the n SSP for some sign pattern P . Then for any set of complex numbers Λ' (invariant under conjugation) nearby $\text{spec}(A)$, there is a matrix $A' \in Q(P)$ with $\text{spec}(A') = \Lambda'$.

Supergraph lemma: $F_1(B, K) = e^{-K} A e^K + B$



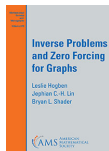
Bifurcation lemma: $F_2(B, K) = e^{-K} (A + B) e^K$

Supergraph lemma: $F_1(B, K) = e^{-K} A e^K + B$



Bifurcation lemma: $F_2(B, K) = e^{-K} (A + B) e^K$

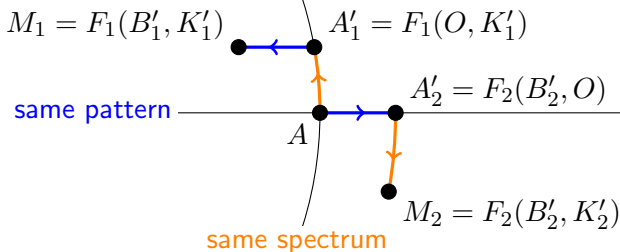
For more IEP-G...



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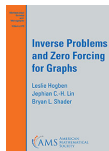


Supergraph lemma: $F_1(B, K) = e^{-K} A e^K + B$



Bifurcation lemma: $F_2(B, K) = e^{-K} (A + B) e^K$

For more IEP-G...



Thanks!

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References I



W. Barrett, S. M. Fallat, H. T. Hall, L. Hogben, J. C.-H. Lin, and B. Shader.

Generalizations of the Strong Arnold Property and the minimum number of distinct eigenvalues of a graph.

Electron. J. Combin., 24:#P2.40, 2017.



S. M. Fallat, L. Hogben, J. C.-H. Lin, and B. Shader.

The inverse eigenvalue problem of a graph, zero forcing, and related parameters.

Notices Amer. Math. Soc., 67:257–261, February, 2020.



W. E. Ferguson, Jr.

The construction of Jacobi and periodic Jacobi matrices with prescribed spectra.

Math. Comp., 35:1203–1220, 1980.

References II



L. J. Gray and D. G. Wilson.

Construction of a Jacobi matrix from spectral data.

Linear Algebra Appl., 14:131–134, 1976.



O. H. Hald.

Inverse eigenvalue problems for Jacobi matrices.

Linear Algebra Appl., 14:63–85, 1976.



L. Hogben, J. C.-H. Lin, and B. Shader.

Inverse Problems and Zero Forcing for Graphs.

American Mathematical Society, Providence, 2022.