The liberation set in the inverse eigenvalue problem of a graph

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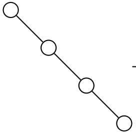
Inverse eigenvalue problem of a graph (IEP-G)

Let G be a graph. Define $\mathcal{S}(G)$ as the family of all real symmetric matrices $A = [a_{ij}]$ such that

$$a_{ij} \begin{cases} \neq 0 & \text{if } ij \in E(G), i \neq j; \\ = 0 & \text{if } ij \notin E(G), i \neq j; \\ \in \mathbb{R} & \text{if } i = j. \end{cases}$$



$$\longrightarrow \begin{bmatrix} ? & * & 0 & 0 \\ * & ? & * & 0 \\ 0 & * & ? & * \\ 0 & 0 & * & ? \end{bmatrix} \underset{?}{\overset{\text{spec}}{\rightleftarrows}} \{1, 2, 3, 4\}$$

IEP-G: What are the possible spectra of a matrix in $\mathcal{S}(G)$?

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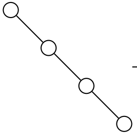

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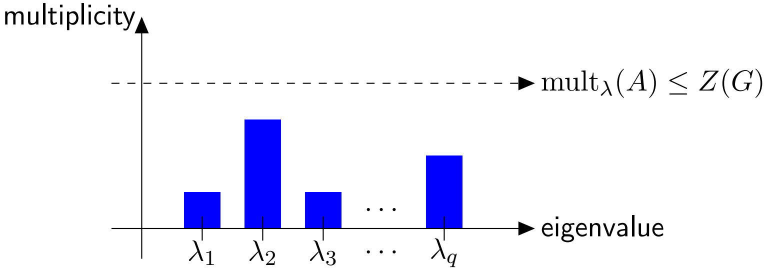
IEP-G: What are the possible spectra of a matrix in $\mathcal{S}(G)$?

Ordered multiplicity list



$$\mathrm{spec}(A) = \{\lambda_1^{(m_1)}, \ldots, \lambda_q^{(m_q)}\} \implies \begin{array}{l} \mathbf{m}(A) = (m_1, \ldots, m_q), \\ q(A) = q \end{array}$$

Strong spectral property

Let G be a graph on n vertices and A, X be $n \times n$ real symmetric matrices.

- $X \circ G = O$ means X is zero on those entries corresponding to $E(G)$ and on the diagoal.
- $[A, X] = AX - XA$.

Definition

A matrix $A \in \mathcal{S}(G)$ has the strong spectral property (SSP) if $X \circ G = O$ and $[A, X] = O$ implies $X = O$.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad X = \begin{bmatrix} 0 & 0 & x & y \\ 0 & 0 & 0 & z \\ x & 0 & 0 & 0 \\ y & z & 0 & 0 \end{bmatrix}$$

Supergraph lemma

Lemma (BFHHLS 2017)

Let G be a spanning subgraph of H'. If $A \in \mathcal{S}(G)$ has the SSP, then there is a matrix $A' \in \mathcal{S}(H')$ such that

- $\operatorname{spec}(A') = \operatorname{spec}(A)$,
- *A' has the SSP, and*
- *$\|A' - A\|$ can be chosen arbitrarily small.*

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} \sim 1 & \epsilon & 0 & 0 \\ \epsilon & \sim 2 & \epsilon & 0 \\ 0 & \epsilon & \sim 3 & \epsilon \\ 0 & 0 & \epsilon & \sim 4 \end{bmatrix}$$

Example of no SSP

Let $A \in \mathcal{S}(K_2 \dot{\cup} K_2)$.

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \qquad X = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

Then $[A, X] = O$.

Matrix liberation lemma — vector version

Theorem (BBFHHLSY 2020)

Let G be a graph, $A \in \mathcal{S}(G)$, and $\Psi = \Psi_{\mathrm{SSP}}(A)$. Suppose there is a vector $\mathbf{x} \in \mathrm{Col}(\Psi)$ such that the rows of Ψ corresponding to zeros of $\mathbf{x}$ form a linearly independent set. Then there exists a matrix $A' \in \mathcal{S}(G + \mathrm{supp}(\mathbf{x}))$ with the SSP and $\mathrm{spec}(A') = \mathrm{spec}(A)$.

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad \Psi = \begin{bmatrix} 0 & 2 & -1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & -1 & 2 & 0 \end{bmatrix}$$

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- Ψ is an $|E(\overline{G})| \times \binom{n}{2}$ matrix—usually HUGE.
- It is not clear about how to find such $\mathbf{x}$.
- The outcome only depends on $\mathrm{supp}(\mathbf{x})$ but not the precise values on $\mathbf{x}$.

Extended strong spectral property

Definition

Let G be a spanning subgraph of H. A matrix $A \in \mathcal{S}(G)$ has the strong spectral property with respect to H if $X \circ H = O$ and $[A, X] = O$ implies $X = O$.

Consider $G = 2K_2$ and $H = P_4$. Then $A \in \mathcal{S}(G)$ has the SSP with respect to H.

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \qquad X = \begin{bmatrix} 0 & 0 & x & y \\ 0 & 0 & 0 & z \\ x & 0 & 0 & 0 \\ y & z & 0 & 0 \end{bmatrix}$$

Liberation set

Definition

Let G be a graph and $A \in \mathcal{S}(G)$. A nonempty set of edges $\beta \subseteq E(\overline{G})$ is called an SSP liberation set of A if A has the SSP with respect to $G + \beta'$ for all $\beta' \subset \beta$ with $|\beta| = |\beta| - 1$.

For example, $\beta = \{\{1,3\},\{2,3\}\}$ is an SSP liberation set of

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

Matrix liberation lemma — set version

Theorem (L, Oblak, and Šmigoc 2023)

Let G be a graph, $A \in \mathcal{S}(G)$, and β an SSP liberation set of A. Then there exists a matrix $A' \in \mathcal{S}(G+\beta)$ with the SSP and $\operatorname{spec}(A') = \operatorname{spec}(A)$.

- No need to find the HUGE matrix Ψ.
- No need to find $\mathbf{x}$.
- Behind the scene, $\beta = \operatorname{supp}(\mathbf{x})$ for some desired $\mathbf{x}$.

SSP for direct sum of matrices

Proposition (BBFHHLSY 2020)

Let A and B be matrices with the SSP. Then $A \oplus B$ has the SSP if and only if $AY = YB$ implies $Y = O$.

$$A \oplus B = \begin{bmatrix} A & \\ & B \end{bmatrix} \qquad X = \begin{bmatrix} X_A & Y \\ Y^\top & X_B \end{bmatrix}.$$

$$[A \oplus B, X] = \begin{bmatrix} AX_A - X_A A & AY - YB \\ BY - YA & BX_B - X_B B \end{bmatrix}$$

Extended SSP for direct sum of matrices

Proposition (L, Oblak, and Šmigoc 2023)

Let $A \in \mathcal{S}(G)$ and $B \in \mathcal{S}(H)$ be matrices with the SSP. Then $A \oplus B$ has the SSP with respect to $G \,\dot\cup\, H + \beta'$ if and only if $AY = YB$ and $Y|_{\beta'} = O$ implies $Y = O$.

Theorem (L, Oblak, and Šmigoc 2023)

Let G and H graphs, and β' a zero forcing set of $G \Box H$. Then $A \oplus B$ has the SSP with respect to $G \,\dot\cup\, H + \beta'$ for any $A \in \mathcal{S}(G)$ and $B \in \mathcal{S}(H)$ both with the SSP.

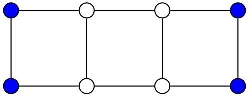
Zero forcing cover

Definition

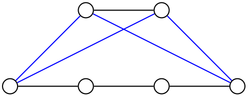
Let G be a graph. A set $F \subseteq V(G)$ is a zero forcing cover of G if F' is a zero forcing set of G for any $F' \subset F$ with $|F'| = |F| - 1$.

Theorem (L, Oblak, and Šmigoc 2023)

Let G and H be graphs. Suppose $A \in \mathcal{S}(G)$ and $B \in \mathcal{S}(H)$ have the SSP. If β is a zero forcing cover of $G \square H$, then β is an SSP liberation set of $A \oplus B$.

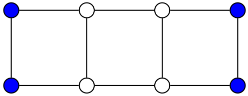


$P_2 \square P_4$ and a zero forcing cover β



$P_2 \,\dot\cup\, P_4 + \beta$

Let σ, τ be realizable spectra in $\mathcal{S}(P_2)$ and $\mathcal{S}(P_4)$. Then $\sigma \,\dot\cup\, \tau$ is realizable in $\mathcal{S}(P_2 \,\dot\cup\, P_4 + \beta)$.



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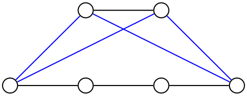

$P_2 \dot\cup P_4 + \beta$

Thanks!

Kaohsiung

$\lambda_2^L = 1.6$

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