Applications of zero forcing number to the minimum rank problem

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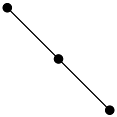
Abstract

- Introduction and some related properties
- Exhaustive zero forcing number and sieving process
- Summary and a counterexample to a problem on edge spread

Relation between Matrices and Graphs

$\mathcal{G}$:real symmetric matrices → graphs.

$$\begin{pmatrix} -3 & 3 & 0 \\ 3 & -5 & 2 \\ 0 & 2 & -2 \end{pmatrix} \xrightarrow{\mathcal{G}}$$



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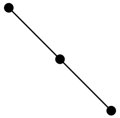

$$\mathcal{S}(G) = \{A \in M_{n\times n}(\mathbb{R}) : A = A^t, \mathcal{G}(A) = G\}.$$

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- $$\mathrm{mr}(G) + M(G) = |V(G)|.$$
- The minimum rank problem of a graph G is to determine the number $\mathrm{mr}(G)$ or $M(G)$.

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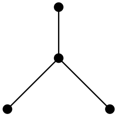
- The path cover number $P(G)$ of a graph G is the minimum number of vertex disjoint induced paths of G that cover $V(G)$.

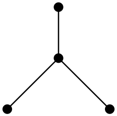
Example for Three Parameters

$$\begin{pmatrix} ? & * & * & * \\ * & ? & 0 & 0 \\ * & 0 & ? & 0 \\ * & 0 & 0 & ? \end{pmatrix} \xrightarrow{\mathcal{G}}$$



- rank ≥ 2.

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\mathcal{G}}$$



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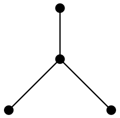

- $\text{rank} \geq 2$.
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- $\text{mr}(K_{1,3}) = 2$ and $M(K_{1,3}) = 4 - 2 = 2$.

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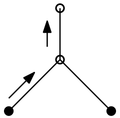

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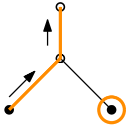

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- $Z(G) = 2$.
- $P(G) = 2$.

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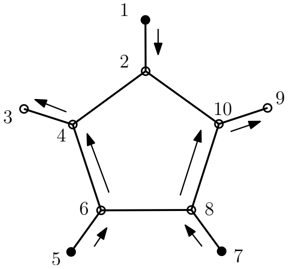
- $M(G)$ and $P(G)$ are not comparable in general.

Terminologies for $Z(G)$

- A chronological list record the order of forces.



chronological list

$1 \to 2$ $\quad 4 \to 3$
$5 \to 6$ $\quad 8 \to 10$
$7 \to 8$ $\quad 10 \to 9$
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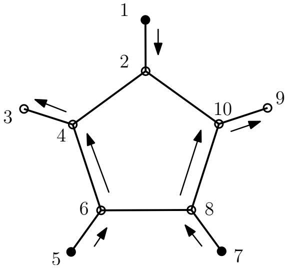

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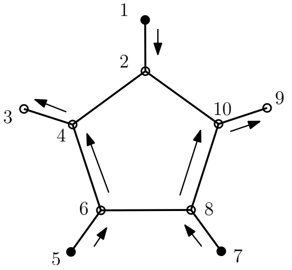

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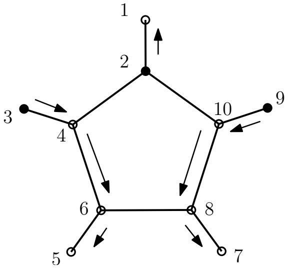
- The inverse chronological list gives another zero forcing set called reversal.



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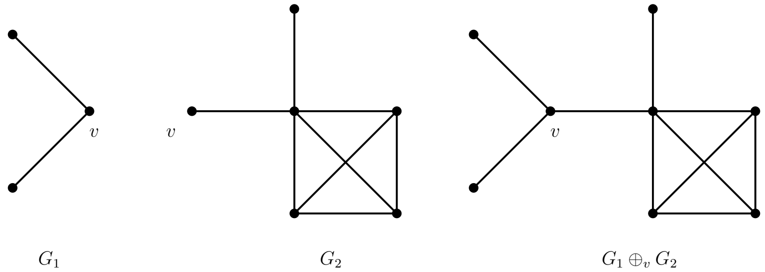

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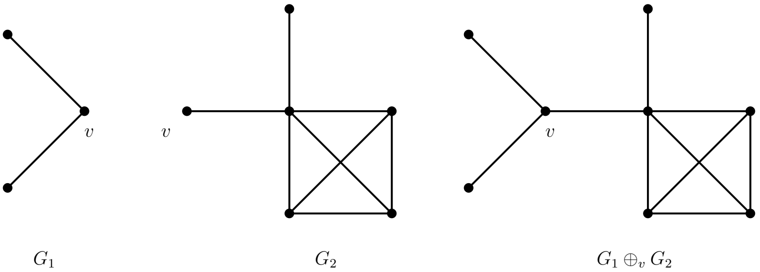
- If $G = G_1 \oplus_v G_2$, then

$$M(G) = \max\{M(G_1)+M(G_2)-1, M(G_1-v)+M(G_2-v)-1\}.[4]$$



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$$p_v(G) = \begin{cases} -1, & \text{if } v \text{ is simply terminal} \\ & \text{ of } G_1 \text{ and } G_2; \\ \min\{p_v(G_1), p_v(G_2)\}, & \text{otherwise.[5]} \end{cases}$$

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Sketch of Proof

- $$\begin{cases} -1 \leq z_v(G) \leq 1. \\ v \text{ is doubly terminal} \Leftrightarrow z_v = 0. \\ v \text{ is simply terminal} \Rightarrow z_v = 0. \end{cases}$$

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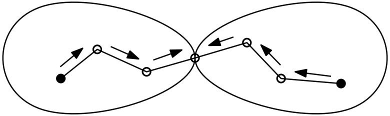

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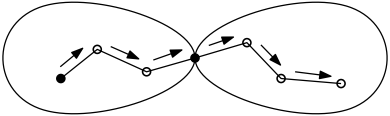

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- If $G = G_1 \oplus_v G_2$, then

$$Z(G) \leq Z(G_1) + Z(G_2 - v),\ Z(G) \leq Z(G_1 - v) + Z(G_2),$$

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- $z_v(G) = -1$, $z_v(G_1) = z_v(G_2) = 0$ is the only possibility. This implies v is simply terminal for G_1 and G_2.

Comparison of Reduction Formulae

- Denote $m_v(G) = M(G) - M(G - v)$, $p_v(G) = P(G) - P(G - v)$, and $z_v(G) = Z(G) - Z(G - v)$.

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- $-1 \leq m_v, p_v, r_v \leq 1$.
- If $G = G_1 \oplus_v G_2$, they have similar behavior.

$m_v(G_1 \backslash G_2)$	-1	0	1
-1	-1	-1	-1
0	-1	-1	0
1	-1	0	1

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$p_v, z_v(G_1 \backslash G_2)$	-1	0	1
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- Hard to apply on induction.

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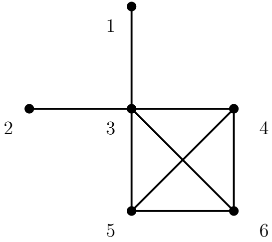

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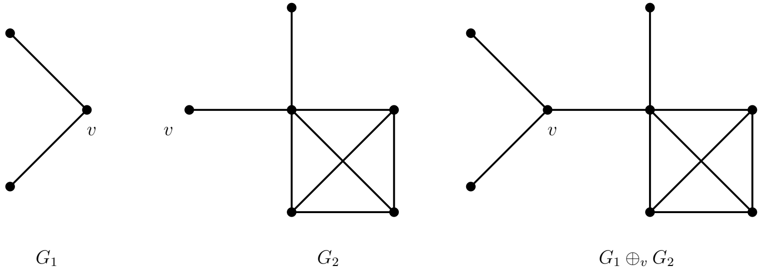
- A graph G satisfies the PZ condition iff $P(G) = Z(G)$.
- PZ condition is not hereditary.
- PZ condition does not preserve under vertex-sum operation.



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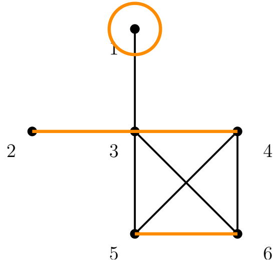

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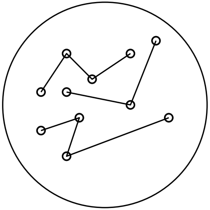

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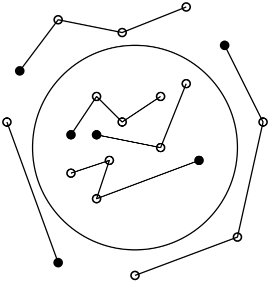

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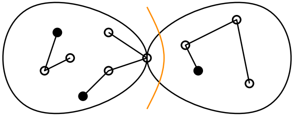

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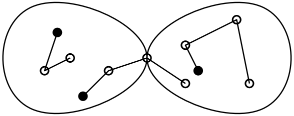

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Cactus graphs

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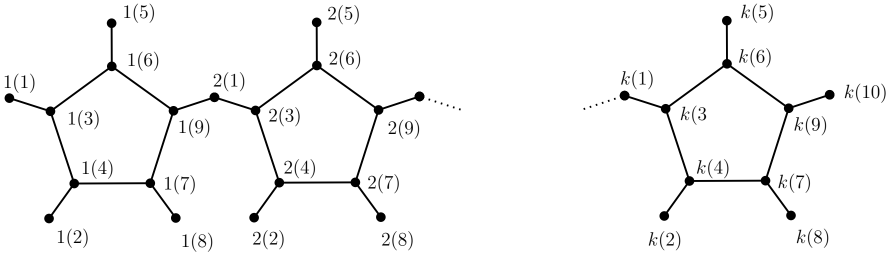
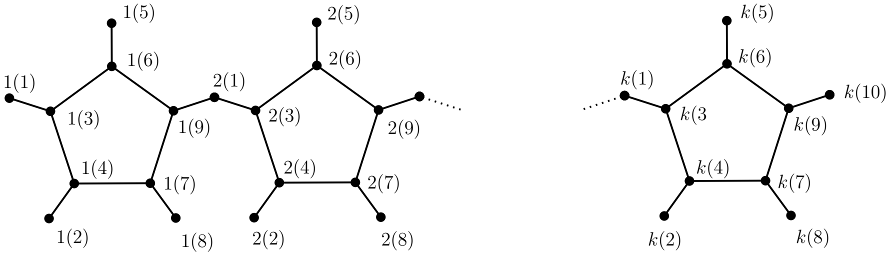
- A cactus G satisfies the strong PZ condition. Hence we have $P(G) = Z(G)$.

Large $Z(G) - M(G)$

- Let G_k be the k 5-sun sequence. Then $P(G_k) = Z(G_k) = 2k+1$ and $M(G_k) = k+1$.



- Let G_k be the k 5-sun sequence. Then $P(G_k) = Z(G_k) = 2k + 1$ and $M(G_k) = k + 1$.
- Actually, for all $1 \le p \le q \le 2p - 1$, there is a graph G such that $M(G) = p$ and $Z(G) = q$.



Large $Z(G) - M(G)$

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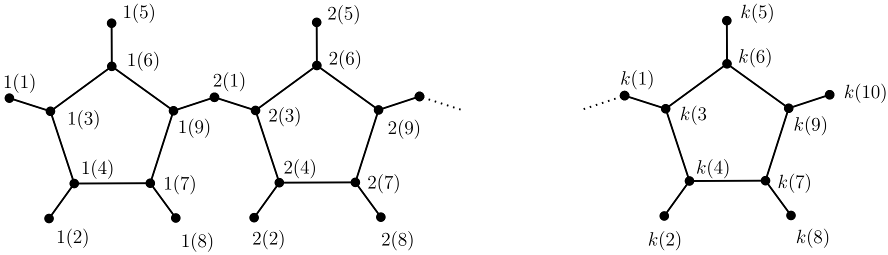
- Actually, for all $1 \le p \le q \le 2p - 1$, there is a graph G such that $M(G) = p$ and $Z(G) = q$.
- Q: Will the inequality $Z(G) \le 2M(G) - 1$ holds for all G?



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- The minimum rank of a pattern Q is

$$\mathrm{mr}(Q) = \mathsf{min}\{\mathrm{rank} A \colon A \cong Q\}.$$

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- The rank 2 is achievable. Hence $\mathrm{mr}(Q) = 2$.

Operation on S

- Define addition "+" and scalar multiplication "×" on S.

$$+ : S \times S \to S$$

+	0	$*$	u
0	0	$*$	u
$*$	$*$	u	u
u	u	u	u

$$\times : \{0, *\} \times S \to S$$

×	0	$*$	u
0	0	0	0
$*$	0	$*$	u

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- The rank of a pattern is the maximum number of independent row sign vectors.

Independence in different senses

Lemma

Suppose $V = \{v_1, v_2, \ldots, v_n\}$ *is a set of sign vectors, and* $W = \{w_1, w_2, \ldots, w_n\}$ *is a set of sign vectors such that* w_i *is obtained from* v_i *by replacing entries* u *by* 0 *or* $*$*. If* V *is linearly independent, then so is* W*.*

Suppose $R = \{r_1, r_2, \ldots, r_n\}$ *is a set of real vectors such that each entry in each vector matches the corresponding entry in elements of* W*. If* W *is linearly independent, then* R *is linearly independent as real vectors.*

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If Q *is a pattern matrix and* U *is the set of all pattern matrices obtained from* Q *by replacing* u *by* 0 *or* $*$, *then*

$$\operatorname{rank}(Q) \leq \min_{Q' \in U} \{\operatorname{rank}(Q')\} \leq \operatorname{mr}(Q).$$

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- When W and B is empty, $Z_W(G,B) = Z(G)$.

Natural Relation between Patterns and Bipartites

- Q is a given $m \times n$ pattern. $G = (X \cup Y, E)$ is the related bipartite defined by

$$X = \{a_1, a_2, \ldots, a_m\}, \ \ Y = \{b_1, b_2, \ldots, b_n\}, \ \ E = \{a_i b_j : Q_{ij} \neq 0\}.$$

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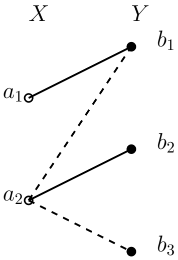

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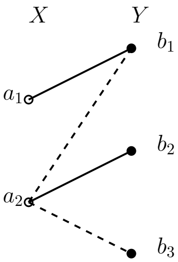

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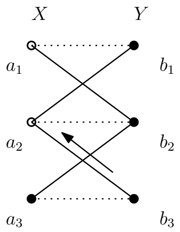

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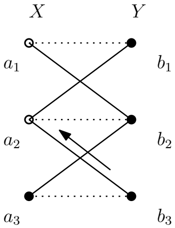

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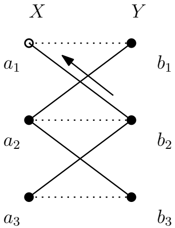

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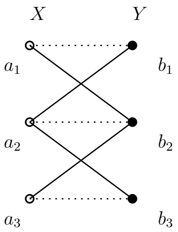

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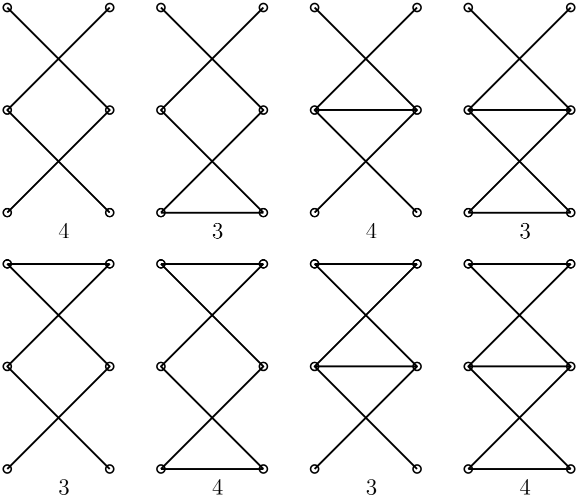
Bipartites related to P_3



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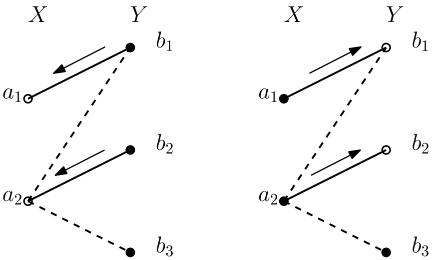

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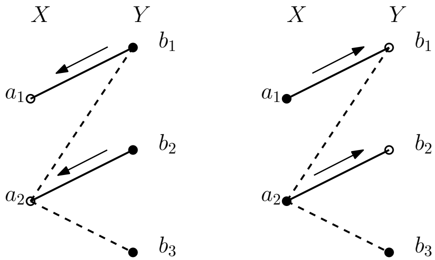

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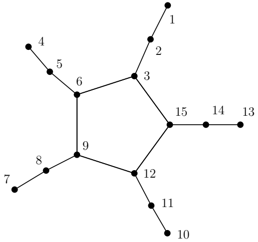
- The parameter $\widetilde{Z}(G)$ is still not sharp for some cactus.



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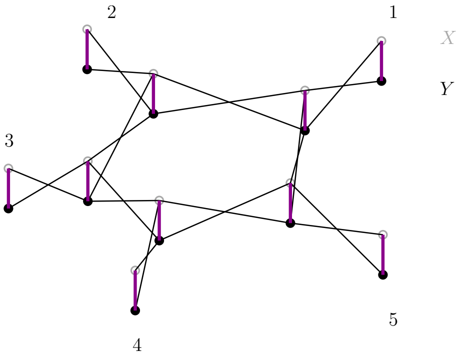

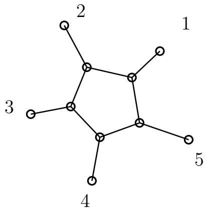

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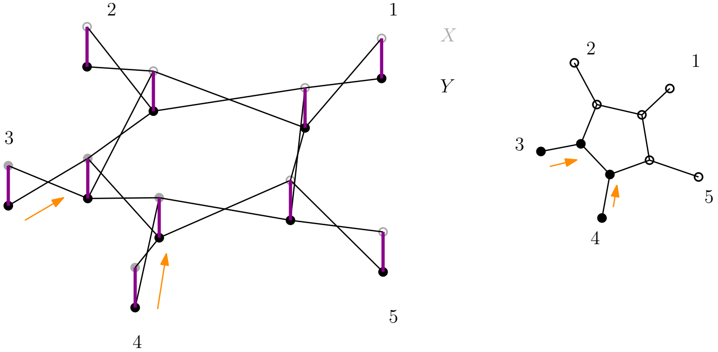

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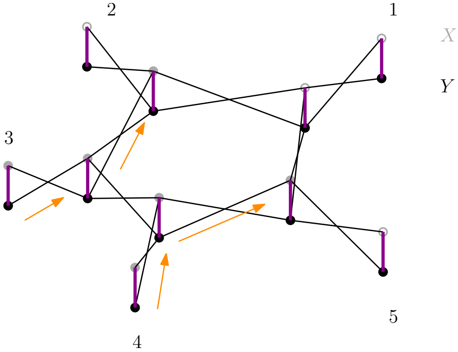

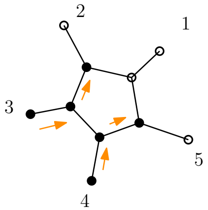

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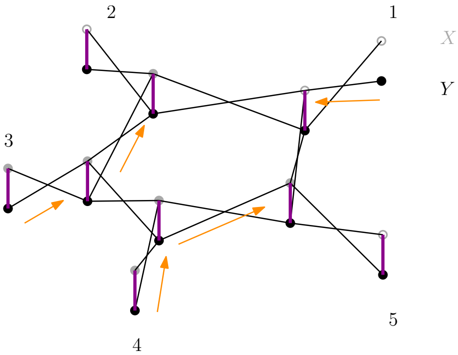

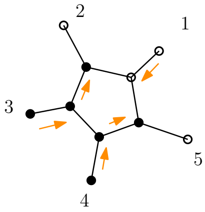

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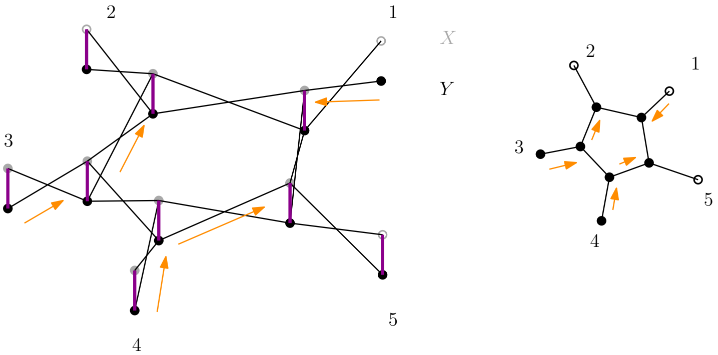

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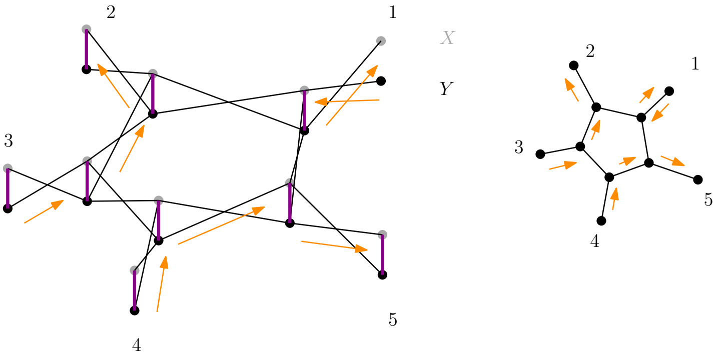

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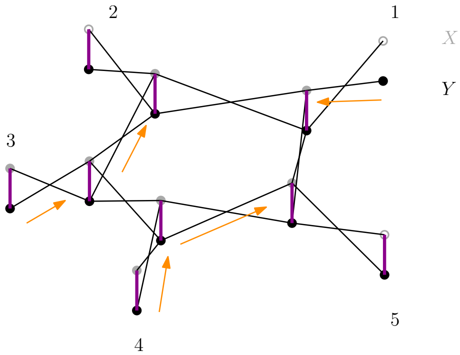

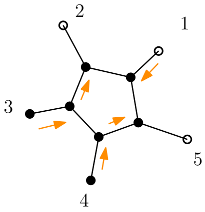

Example for Sieving Process

- If $Z(\widetilde{H_{5I}}) - 10 = 3$ for some I, then $1 \in I$ and $2 \notin I$, a contradiction.
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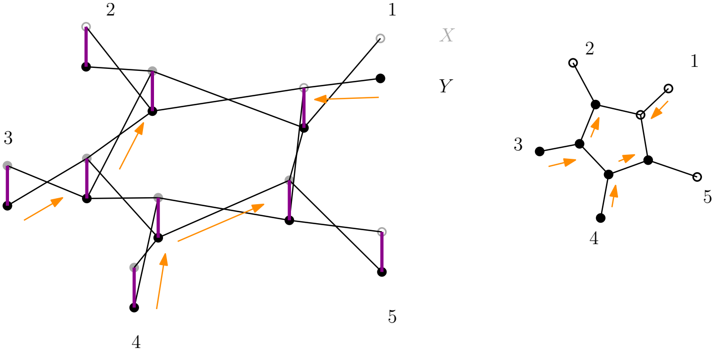

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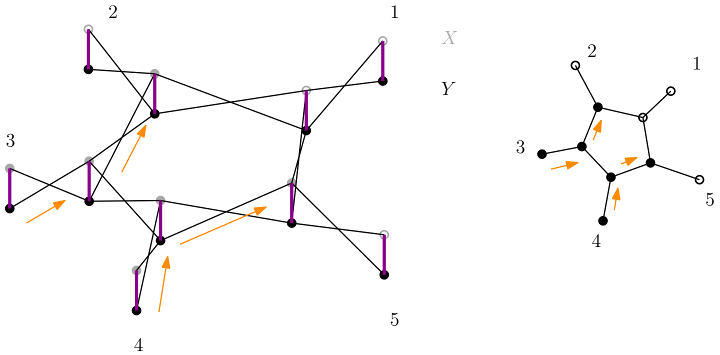

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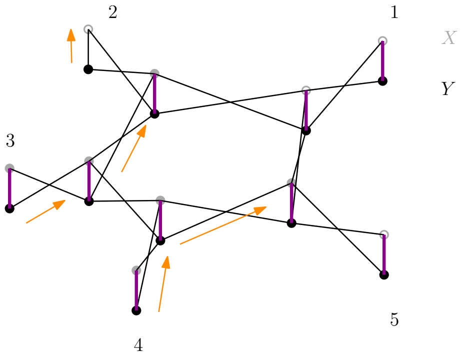

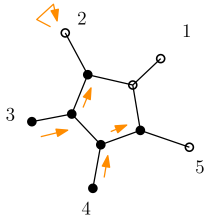

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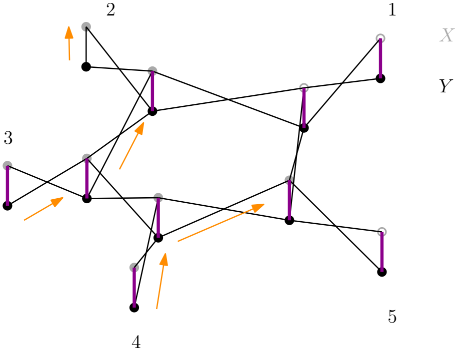

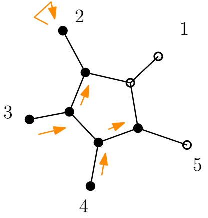

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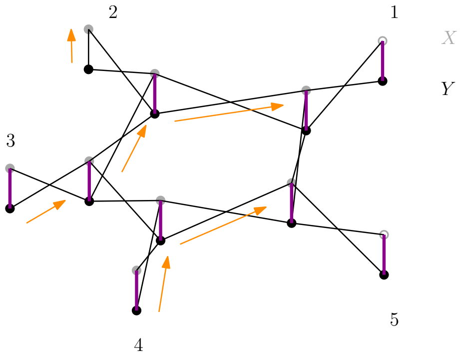

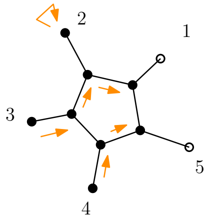

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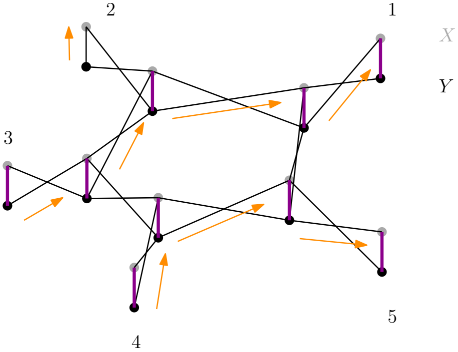

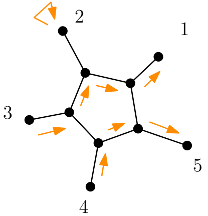

Edge vs Nonedge

- Edge: Increase number of neighbor; Increase possible route for passing.

Edge vs Nonedge

- Edge: Increase number of neighbor; Increase possible route for passing.
- Nonedge: Decrease number of neighbor; Decrease possible route for passing.

Edge vs Nonedge

- Edge: Increase number of neighbor; Increase possible route for passing.
- Nonedge: Decrease number of neighbor; Decrease possible route for passing.
- The BAD guy Banned Edge: Increase number of neighbor; Decrease possible route for passing.

Sieving Process

- Rewrite

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Sieving Process

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- $\widetilde{Z}(G) = \max\{k : \mathcal{I}_k \neq \varnothing\}$.
- Each $F \supseteq Y$ with size $n + k - 1$ is a sieve for $\mathcal{I}_k(G)$ to delete impossible index sets.

Nonzero-vertex and Zero-vertex

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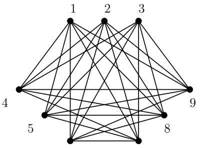
- For multi-partite G with more than one part and more than two vertices in each parts, each vertex is a zero-vertex in $\mathcal{I}_{n-2}(G)$, $n = |V(G)|$.
- We know $Z(G_k) = 2k + 1$ and $M(G_k) = k + 1$. By sieving process, $\widetilde{Z}(G_k) = k + 1$! Here G_k is the k 5-sun sequence.

Example for Stronger Upper Bound 1

- $M(G) \leq Z(G) = 7$. Each vertex is a zero-vertex in $\mathcal{I}_7$.



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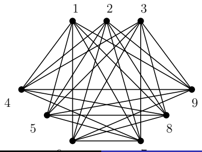
- If $A \in \mathcal{S}(G)$ has nullity 7, we may assume

$$A = \begin{pmatrix} O & J & J \\ J & O & B^t \\ J & B & O \end{pmatrix}.$$



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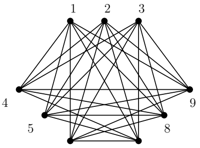

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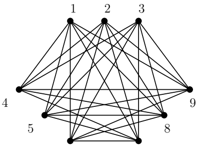

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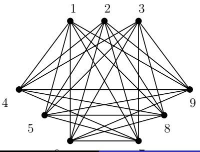
- $M(G) \leq 6$. And actually $M(G) = 6$.



Nonzero Elimination Lemma

Theorem

For a graph G, suppose i is a nonzero-vertex in $\mathcal{I}_k(G)$. And $\eta_i(G)$ denote the set of those graphs obtained from G by the following rules:

- *The vertex i should be deleted;*
- *For any neighbors x and y of i, the pair xy should be an edge if $xy \notin E(G)$ and could be an edge or a non-edge if $xy \in E(G)$.*

If the nullity k is achievable by some matrix in $\mathcal{S}(G)$, then

$$k \leq \max\{M(H) : H \in \eta_i(G)\}.$$

- If k is achievable by $A \in \mathcal{S}(G)$, assume

$$A = \begin{pmatrix} 1 & a^t & 0 \\ a & \widehat{A}_{11} & \widehat{A}_{12} \\ 0 & \widehat{A}_{21} & \widehat{A}_{22} \end{pmatrix}.$$

Sketch of Proof

- If k is achievable by $A \in \mathcal{S}(G)$, assume

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- The matrix Then the matrix

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has the same nullity, where $\widehat{B}_{11} = \widehat{A} - aa^t$.

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- The nullity of A should be less than the maximum nullity of each possible matrix P.

Zero Elimination Lemma

Theorem

For a graph G, suppose i is a zero-vertex in $\mathcal{I}_k(G)$ and j is a neighbor of i. Let

$$N_1 = \{v : iv \in E(G), v \neq j\}, \; N_2 = \{v : jv \in E(G), iv \notin E(G), v \neq i\}.$$

And $\eta_{i\to j}(G)$ denote the set of those graphs obtained from G by the following rules:

- *The vertex i and j should be deleted;*
- *For $x \in N_1$ and $y \in N_2$, the pair xy should be an edge if $xy \notin E(G)$ and could be an edge or a non-edge if $xy \in E(G)$;*
- *For x and y in N_1, the pair xy could be an edge or a non-edge.*

If the nullity k is achievable by some matrix in $\mathcal{S}(G)$, then

$$k \leq \max\{M(H) : H \in \eta_{i\to j}(G)\}.$$

Sketch of Proof

- If k is achievable by $A \in \mathcal{S}(G)$, assume

$$A = \begin{pmatrix} \alpha & a^t & O \\ a & \widehat{A}_{11} & \widehat{A}_{12} \\ O & \widehat{A}_{21} & \widehat{A}_{22} \end{pmatrix}.$$

Here α has the form $\begin{pmatrix} 0 & * \\ * & u \end{pmatrix}$ and α^{-1} has the form $\begin{pmatrix} u & * \\ * & 0 \end{pmatrix}$.

Sketch of Proof

- If k is achievable by $A \in \mathcal{S}(G)$, assume

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- The matrix Then the matrix

$$P = \begin{pmatrix} \alpha & O & O \\ O & \widehat{B}_{11} & \widehat{A}_{12} \\ O & \widehat{A}_{21} & \widehat{A}_{22} \end{pmatrix}$$

has the same nullity, where $\widehat{B}_{11} = \widehat{A} - a\alpha^{-1}a^t$.

Sketch of Proof

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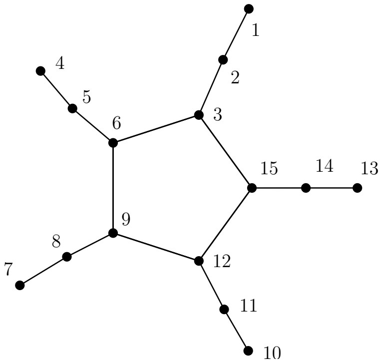
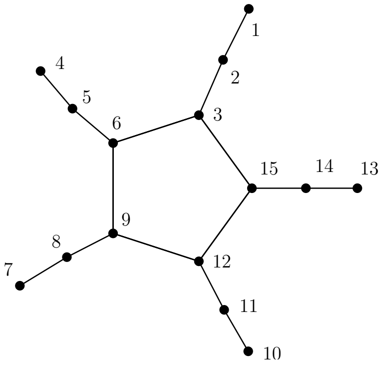
Example for Stronger Upper Bound 2

- $\widetilde{Z}(G) = Z(G) = P(G) = 3.$



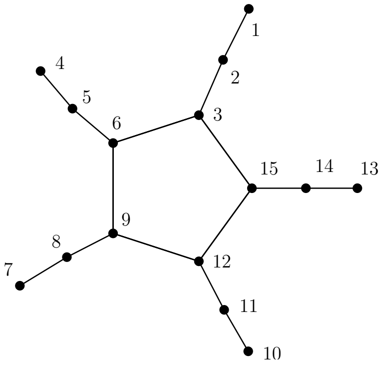
Example for Stronger Upper Bound 2

- $\widetilde{Z}(G) = Z(G) = P(G) = 3$.
- The vertex 1 is a nonzero-vertex in $\mathcal{I}_3$.



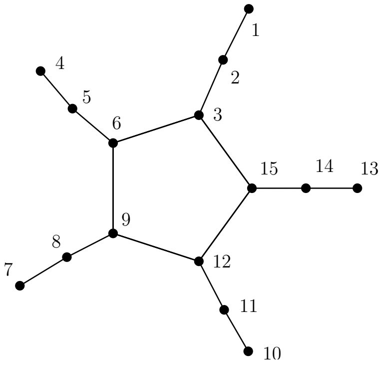
Example for Stronger Upper Bound 2

- $\widetilde{Z}(G) = Z(G) = P(G) = 3$.
- The vertex 1 is a nonzero-vertex in $\mathcal{I}_3$.
- $G - 1$ is the only graph in $\eta_1(G)$.

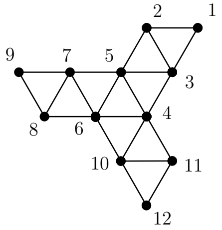


Example for Stronger Upper Bound 2

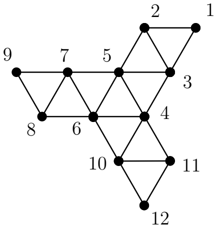
- $\widetilde{Z}(G) = Z(G) = P(G) = 3$.
- The vertex 1 is a nonzero-vertex in $\mathcal{I}_3$.
- $G - 1$ is the only graph in $\eta_1(G)$.
- If 3 is achievable, then $3 \leq M(G-1) \leq 2$, a contradiction. Hence $M(G) \leq 2$.



- $Z(G) = 4$ and $P(G) = 3$.



- $Z(G) = 4$ and $P(G) = 3$.
- The vertex 1 is a nonzero-vertex in $\mathcal{I}_4$.



- $Z(G) = 4$ and $P(G) = 3$.
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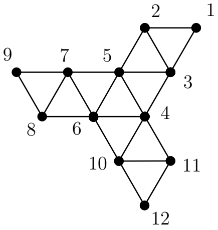
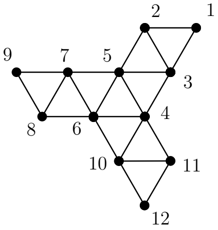
- Let $e = 23$. Then $G - 1$ and $G - 1 - e$ are the only two graphs in $\eta_1(G)$.

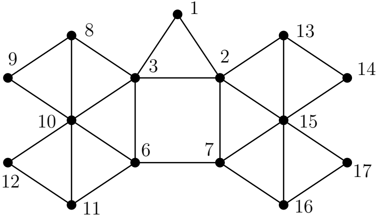


- $Z(G) = 4$ and $P(G) = 3$.
- The vertex 1 is a nonzero-vertex in $\mathcal{I}_4$.
- Let $e = 23$. Then $G - 1$ and $G - 1 - e$ are the only two graphs in $\eta_1(G)$.
- If 4 is achievable, then $4 \leq \max\{M(G-1), M(G-1-e)\} \leq 3$, a contradiction. Hence $M(G) \leq 3$.

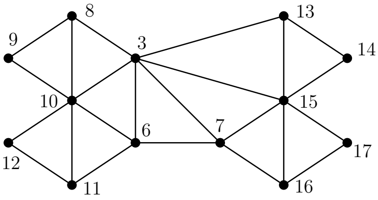


Example for Stronger Upper Bound 4

- $Z(G) = P(G) = 5$.



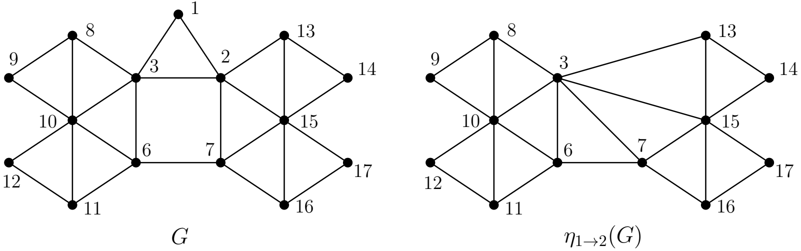
G



$\eta_{1\to2}(G)$

Example for Stronger Upper Bound 4

- $Z(G) = P(G) = 5$.
- The vertex 1 is a zero-vertex.



Example for Stronger Upper Bound 4

- $Z(G) = P(G) = 5$.
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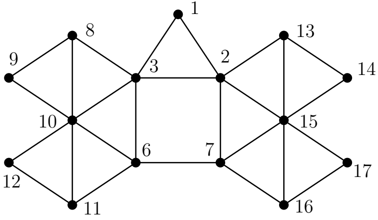
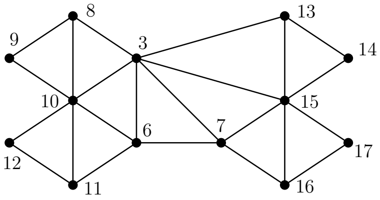
- $\eta_{1\to}(G)$ contains only one graph H.



G



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- $Z(G) = P(G) = 5$.
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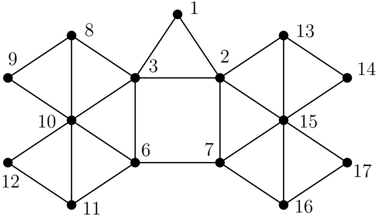
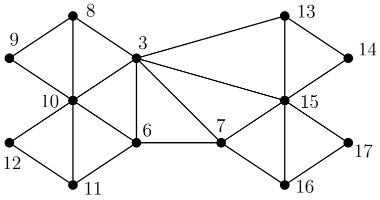
- $\eta_{1\to}(G)$ contains only one graph H.
- If 5 is achievable, then $5 \leq M(H) \leq 4$, a contradiction. Hence $M(G) \leq 4$.



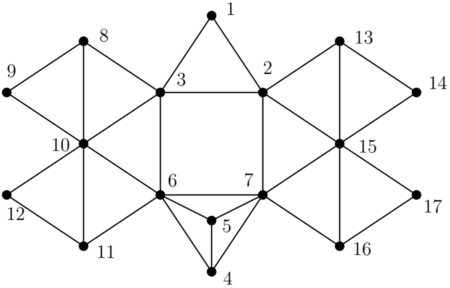
G



$\eta_{1\to 2}(G)$

Example for Stronger Upper Bound 5

- $Z(G) = P(G) = 6$.



Example for Stronger Upper Bound 5

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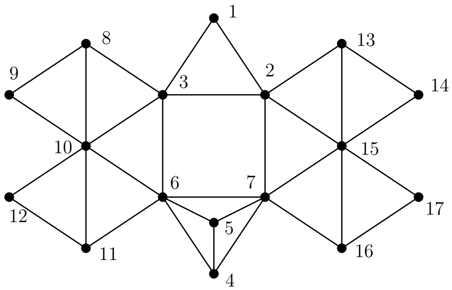
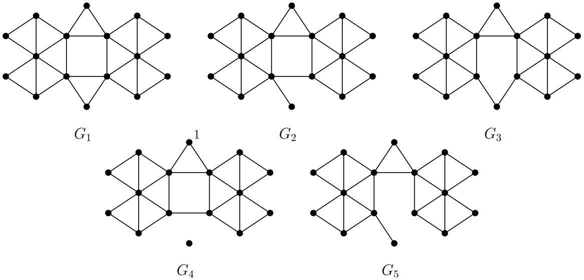
- The vertex 5 is a nonzero-vertex.



Example for Stronger Upper Bound 5

- $Z(G) = P(G) = 6$.
- The vertex 5 is a nonzero-vertex.
- List $\eta_1(G)$. $P(G_i) \leq 5$ for $i = 1, 2, 3, 4$. And they are outerplanar. $M(G_5) = 5$ by reduction formula. $M(G_4) \leq 5$ by doing nonzero elimination lemma again on 1.



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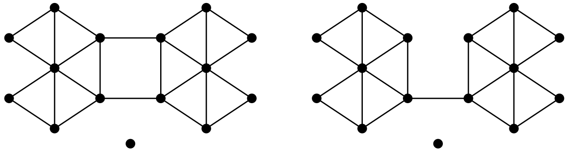

Corollary

If i is a vertex of a graph G and j is a neighbor of i, then

$$M(G) \le \max\{M(H) : H \in \eta_i(G) \cup \eta_{i \to j}(G)\}.$$

- A looped graph is a graph that allows loops. A vertex x is a neighbor of itself if and only if there is a loop on it.

Theorem

$\widetilde{Z}(G) = \widehat{Z}(G)$ *for all graph* G.

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Triangle Number on Pattern[3]

- A *t-triangle* of Q is a $t \times t$ subpattern that is permutation similar to a pattern that is upper triangular with all diagonal entries nonzero.

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- Question 2.22 in [7] ask whether the converse of Theorem 2.21 is true.

The Counterexample

- T is the turtle graph. $G = (X \cup Y, E)$ is construct from T by

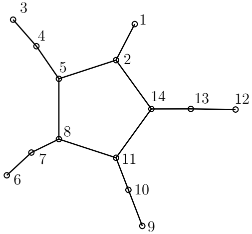
$$X = \{a_1, a_2, \ldots, a_{14}\}, \quad Y = \{b_1, b_2, \ldots, b_{14}\},$$

and

$$E(G) = E_1 \cup E_2,$$

where

$$E_1 = \{a_i a_j : i \neq j\} \cup \{b_i b_j : i \neq j\}, \quad E_2 = \{a_i b_j : ij \in E(T) \text{ or } i = j\}.$$



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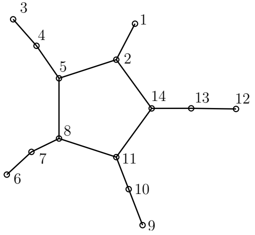

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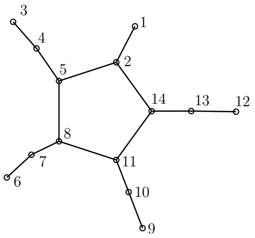

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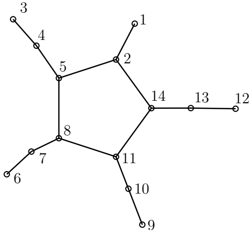

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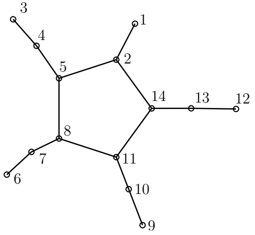
- The edge $e = a_1 b_1$ is used in each optimal zero forcing set. But $Z(G) = Z(G - e) = 16$ and so $z_e(G) = 0 \neq -1$.



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- The proof in [13] of $M(C_n) = 2$ could be generalized.
- $\mathrm{mr}(G) = \mathrm{mrs}(Q(G)) = \min\{\mathrm{mrs}(Q_I(G))\}$. So it is still valuable to consider zero-nonzero symmetric min rank problem.

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