Variants of zero forcing and their applications to the minimum rank problem

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Final Defense

Outline

1. **Overview**: Zero forcing vs. Minimum rank
2. **New upper bound**: odd cycle zero forcing Z_{oc}
3. **Sufficient condition for the Strong Arnold Property**: SAP zero forcing Z_{SAP}
4. **Conclusion**

The minimum rank problem

- The minimum rank problem refers to finding the minimum rank or the maximum nullity of matrices under certain restrictions.
- The restrictions can be the zero-nonzero pattern, conditions on the inertia, or other properties of a matrix.
- The minimum rank problem is motivated by
  - inverse eigenvalue problem — Matrix theory, Engineering
  - Colin de Verdière parameter, orthogonal representation — Graph theory

Example of the maximum nullity

$*$ =nonzero

$$\begin{bmatrix} 0 & * & * & 0 \\ * & * & * & 0 \\ * & * & * & * \\ 0 & 0 & * & 0 \end{bmatrix}$$

Any matrix following this pattern is always nonsingular, meaning the maximum nullity of this pattern is 0.

Zero forcing I

Thinking the matrix as a linear system, if a variable is known as zero, then color it blue.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 0 & * & * & 0 \\ * & * & * & 0 \\ * & * & * & * \\ 0 & 0 & * & 0 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The only vector in the right kernel is $(0, 0, 0, 0)$, so the maximum nullity is 0.

Zero forcing I

Thinking the matrix as a linear system, if a variable is known as zero, then color it blue.

$$
\begin{array}{c}
 \\ 1 \\ 2 \\ 3 \\ \to 4
\end{array}
\begin{array}{c}
\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\
\begin{bmatrix}
0 & * & * & 0 \\
* & * & * & 0 \\
* & * & * & * \\
0 & 0 & * & 0
\end{bmatrix}
\end{array}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}
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$$

$$
\begin{bmatrix} 0 & * & * & 0 \\ * & * & * & 0 \\ * & * & * & * \\ 0 & 0 & * & 0 \end{bmatrix}
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Zero forcing II

Color x_4 in advance. The remaining process is the same.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 0 & * & * & 0 \\ * & * & * & 0 \\ * & * & * & * \\ 0 & 0 & * & * \end{bmatrix} \end{array}$$

The first three columns are always independent, so the the maximum nullity is at most 1.

$$\text{maximum nullity} \leq \#\text{ initial blue variables}$$

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$$\begin{array}{c|cccc} & x_1 & x_2 & x_3 & x_4 \\ \hline 1 & 0 & * & * & 0 \\ 2 & * & * & * & 0 \\ 3 & * & * & * & * \\ 4 & 0 & 0 & * & * \end{array}$$

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$$\text{maximum nullity} \leq \#\ \text{initial blue variables}$$

Zero forcing II

Color x_4 in advance. The remaining process is the same.

	x_1	x_2	x_3	x_4
1	0	$*$	$*$	0
2	$*$	$*$	$*$	0
3	$*$	$*$	$*$	$*$
4	0	0	$*$	$*$

The first three columns are always independent, so the the maximum nullity is at most 1.

maximum nullity $\leq$ # initial blue variables

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$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 0 & * & * & 0 \\ * & * & * & 0 \\ * & * & * & * \\ 0 & 0 & * & * \end{bmatrix} \end{array}$$

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Zero forcing III

$$
\begin{array}{c} \\ 4 \\ 1 \\ 2 \\ 3 \end{array}
\begin{array}{c}
\begin{array}{cccc} x_3 & x_2 & x_1 & x_4 \end{array} \\
\left[\begin{array}{cccc}
* & 0 & 0 & * \\
* & * & 0 & 0 \\
* & * & * & 0 \\
* & * & * & *
\end{array}\right]
\end{array}
$$

Zero forcing is a process of finding the largest lower triangular pattern.

$$\text{maximum nullity} \leq \# \text{ initial blue variables}$$

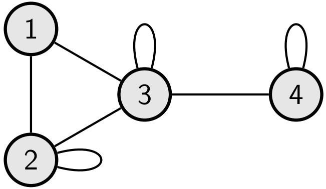
New upper bound

odd cycle zero forcing Z_{oc}

The minimum rank of loop graphs

The maximum nullity $M(\mathfrak{G})$ of a loop graph $\mathfrak{G}$ is the maximum nullity over real symmetric matrices following its zero-nonzero pattern.

$$\begin{bmatrix} 0 & * & * & 0 \\ * & * & * & 0 \\ * & * & * & * \\ 0 & 0 & * & * \end{bmatrix}$$



The zero forcing number $Z(\mathfrak{G})$ is the minimum number of initial blue vertices required to make all vertices blue through the color-change rule:

For a vertex x, if y is its only white neighbor, then y turns blue.

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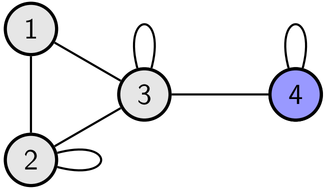

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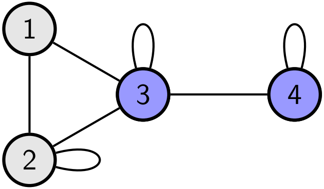

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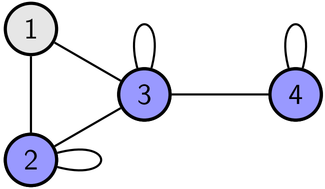

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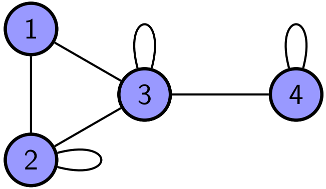


The zero forcing number $Z(\mathfrak{G})$ is the minimum number of initial blue vertices required to make all vertices blue through the color-change rule:

For a vertex x, if y is its only white neighbor, then y turns blue.

$M(\mathfrak{G})$ and $Z(\mathfrak{G})$

Theorem (Hogben '10)

For any loop graph $\mathfrak{G}$, $M(\mathfrak{G}) \leq Z(\mathfrak{G})$.

In general, $Z(\mathfrak{G})$ gives a nice bound; however, for loopless odd cycles $\mathfrak{C}^0_{2k+1}$, $0 = M(\mathfrak{G}) < Z(\mathfrak{G}) = 1$.

$$\det \begin{bmatrix} 0 & a & 0 & 0 & f \\ a & 0 & b & 0 & 0 \\ 0 & b & 0 & c & 0 \\ 0 & 0 & c & 0 & d \\ f & 0 & 0 & d & 0 \end{bmatrix} = 2abcdf \neq 0, \text{ if } a, b, c, d, f \neq 0.$$

Main idea: eliminate the odd cycles

The odd cycle zero forcing number $Z_{oc}(\mathfrak{G})$ of a loop graph $\mathfrak{G}$ is the minimum number of initial blue vertices required to make all vertices blue by:

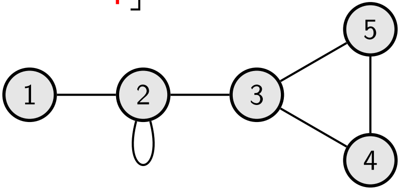
- For a vertex x, if y is its only white neighbor, then y turns blue.
- If the subgraph induced by the white vertices contains a component, which is a loopless odd cycle, then all vertices in this component turn blue.

Theorem (L '16)

For any loop graph $\mathfrak{G}$, $M(\mathfrak{G}) \leq Z_{oc}(\mathfrak{G}) \leq Z(\mathfrak{G})$.

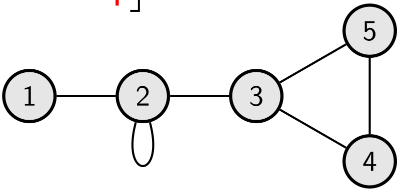
Odd cycle zero forcing

$$\begin{array}{c} \\ 1 \\ 3 \\ 4 \\ 5 \\ 2 \end{array}
\begin{array}{c} \begin{array}{ccccc} x_2 & x_3 & x_4 & x_5 & x_1 \end{array} \\
\left[\begin{array}{ccccc}
* & 0 & 0 & 0 & 0 \\
* & 0 & a & b & 0 \\
0 & a & 0 & c & 0 \\
0 & b & c & 0 & 0 \\
* & * & 0 & 0 & *
\end{array}\right] \end{array}$$



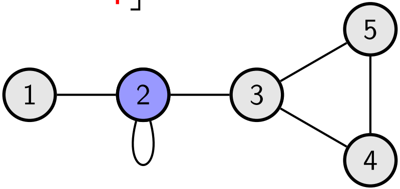
Odd cycle zero forcing

	x_2	x_3	x_4	x_5	x_1
1	$*$	0	0	0	0
3	$*$	0	a	b	0
4	0	a	0	c	0
5	0	b	c	0	0
2	$*$	$*$	0	0	$*$



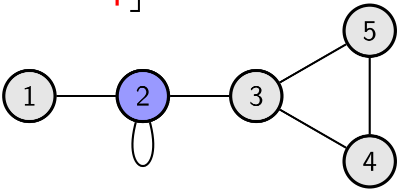
Odd cycle zero forcing

$$
\begin{array}{c}
 \\ 1 \\ 3 \\ 4 \\ 5 \\ 2
\end{array}
\begin{array}{c}
\begin{array}{ccccc} x_2 & x_3 & x_4 & x_5 & x_1 \end{array} \\
\begin{bmatrix}
* & 0 & 0 & 0 & 0 \\
* & 0 & a & b & 0 \\
0 & a & 0 & c & 0 \\
0 & b & c & 0 & 0 \\
* & * & 0 & 0 & *
\end{bmatrix}
\end{array}
$$



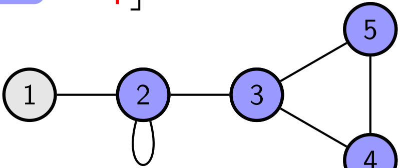
Odd cycle zero forcing

	x_2	x_3	x_4	x_5	x_1
1	$*$	0	0	0	0
→ 3	$*$	0	a	b	0
→ 4	0	a	0	c	0
→ 5	0	b	c	0	0
2	$*$	$*$	0	0	$*$



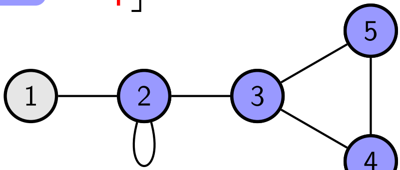
Odd cycle zero forcing

	x_2	x_3	x_4	x_5	x_1
1	$*$	0	0	0	0
3	$*$	0	a	b	0
4	0	a	0	c	0
5	0	b	c	0	0
2	$*$	$*$	0	0	$*$



Odd cycle zero forcing

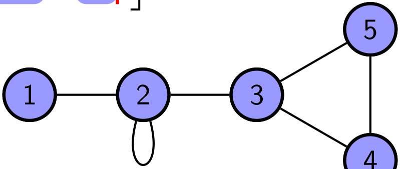
	x_2	x_3	x_4	x_5	x_1
1	$*$	0	0	0	0
3	$*$	0	a	b	0
4	0	a	0	c	0
5	0	b	c	0	0
→ 2	$*$	$*$	0	0	$*$



Odd cycle zero forcing

$$
\begin{array}{c} \\ 1 \\ 3 \\ 4 \\ 5 \\ 2 \end{array}
\begin{array}{ccccc} x_2 & x_3 & x_4 & x_5 & x_1 \\ \end{array}
$$

$$
\begin{bmatrix}
* & 0 & 0 & 0 & 0 \\
* & 0 & a & b & 0 \\
0 & a & 0 & c & 0 \\
0 & b & c & 0 & 0 \\
* & * & 0 & 0 & *
\end{bmatrix}
$$

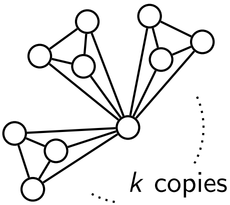


Remarks on Z_{oc}

Corollary (L '16)

For any loop configuration $\mathfrak{G}$ *of a complete graph or a cycle,* $M(\mathfrak{G}) = Z_{oc}(\mathfrak{G})$.

- $Z_{oc}(\mathfrak{G})$ fills in the gaps for many loop graphs that contains loopless odd cycles as induced subgraphs.
- $Z(\mathfrak{G}) - Z_{oc}(\mathfrak{G})$ can be arbitrarily large.



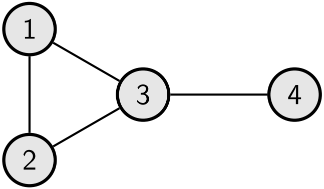
$Z(\mathfrak{G}) = k + 1$

$Z_{oc}(\mathfrak{G}) = 1$

The minimum rank of simple graphs

The maximum nullity $M(G)$ of a simple graph G is the maximum nullity over real symmetric matrices following its zero-nonzero pattern, where diagonal entries are free.

$$\begin{bmatrix} ? & * & * & 0 \\ * & ? & * & 0 \\ * & * & ? & * \\ 0 & 0 & * & ? \end{bmatrix}$$



The zero forcing number $Z(G)$ is the minimum number of initial blue vertices required to make all vertices blue through the color-change rule:

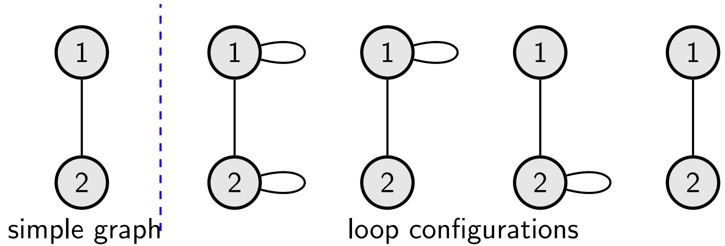
For a blue vertex x, if y is its only white neighbor, then y turns blue.

Inverse eigenvalue problem

- Let $S(G)$ be the family of real symmetric matrices that follow the zero-nonzero pattern of G. (Diagonal entries are free.)
- The inverse eigenvalue problem of a graph (IEPG) asks what are the possible spectra of matrices in $S(G)$.
- The maximum nullity $M(G)$ is an upper bound for all multiplicity.
- $M(G) \leq Z(G)$ [AIM '08]
- E.g., for path graphs P_n, $M(P_n) = 1$, so all matrices in $S(G)$ have only simple eigenvalues.

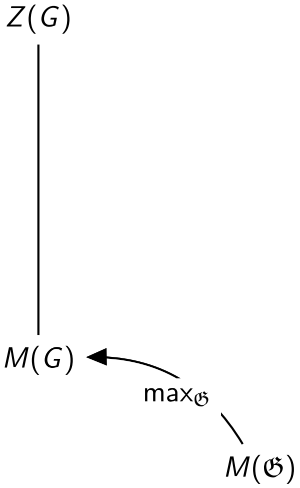
Loop configurations

A loop configuration of a simple graph G is a loop graph $\mathfrak{G}$ obtained from G by designating each vertex as having or not having a loop.

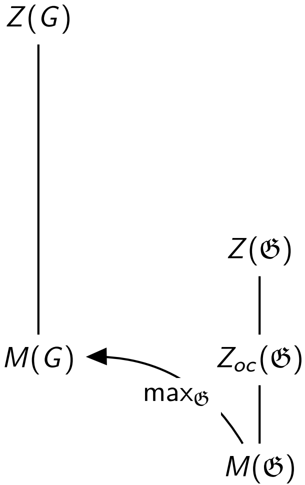


$M(G) = \max_{\mathfrak{G}} M(\mathfrak{G})$, taking maximum over all loop configurations $\mathfrak{G}$.

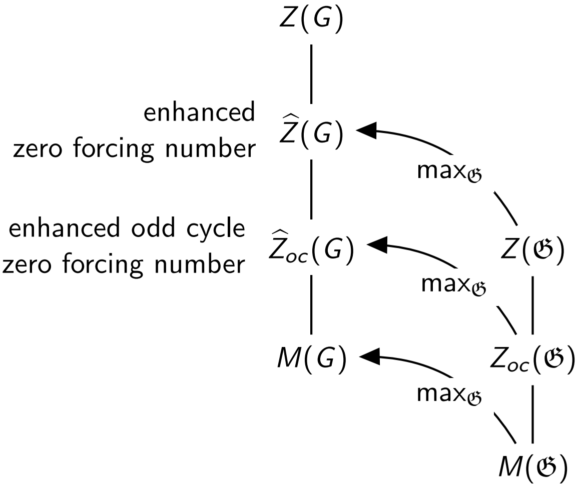
Relations of all mentioned parameters



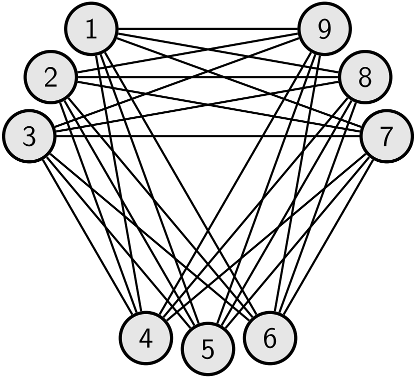
Relations of all mentioned parameters



Relations of all mentioned parameters



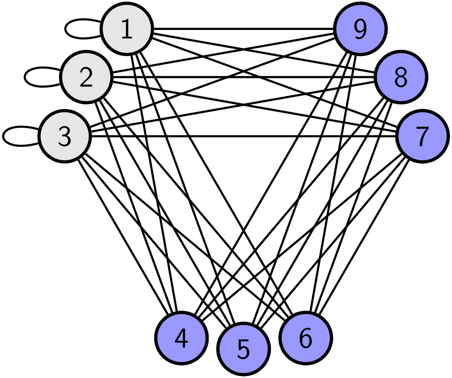
Example: $K_{3,3,3}$



$$\widehat{Z}_{oc}(K_{3,3,3}) = 6 < \widehat{Z}(K_{3,3,3}) = 7$$

Example: $K_{3,3,3}$

1,2,3 have loops
others are unknown



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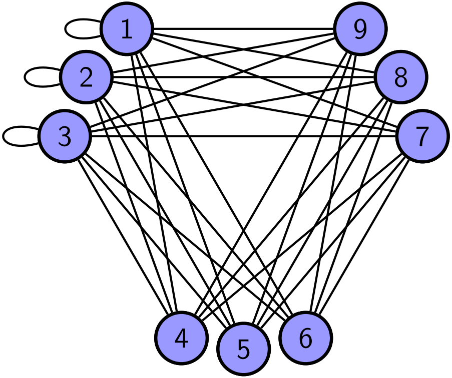
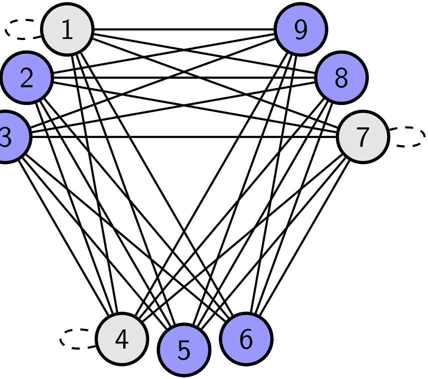


$$\widehat{Z}_{oc}(K_{3,3,3}) = 6 < \widehat{Z}(K_{3,3,3}) = 7$$

Example: $K_{3,3,3}$

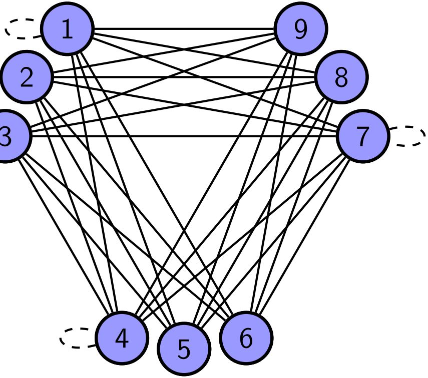
1,4,7 have no loops
others are unknown



$$\widehat{Z}_{oc}(K_{3,3,3}) = 6 < \widehat{Z}(K_{3,3,3}) = 7$$

Example: $K_{3,3,3}$

1,4,7 have no loops
others are unknown



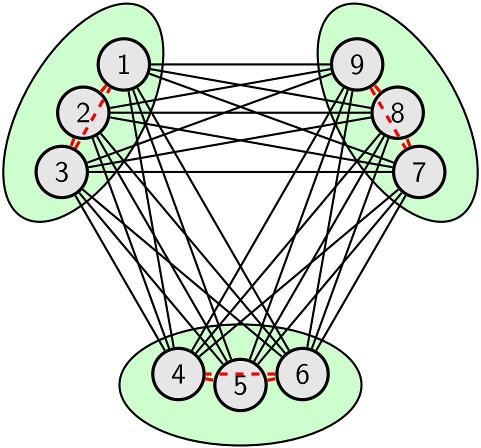
$$\widehat{Z}_{oc}(K_{3,3,3}) = 6 < \widehat{Z}(K_{3,3,3}) = 7$$

Remarks on new parameters

$$M(G) \leq \widehat{Z}_{oc}(G) \leq \widehat{Z}(G) \leq Z(G)$$

- The enhanced odd cycle zero forcing number $\widehat{Z}_{oc}(G)$ inserts a new parameter between $M(G)$ and $\widehat{Z}(G)$.
- $M(K_{3,3,3}) = 6 = \widehat{Z}_{oc}(K_{3,3,3}) < \widehat{Z}(K_{3,3,3}) = 7$
- $M(\mathfrak{C}^0_{2k+1}) = 0 = Z_{oc}(\mathfrak{C}^0_{2k+1}) < Z(\mathfrak{C}^0_{2k+1}) = 1$

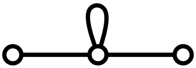
$\mathfrak{C}_3^0$ vs $K_{3,3,3}$



$$\widehat{Z}_{oc}(K_{3,3,3}) = 6 < \widehat{Z}(K_{3,3,3}) = 7$$

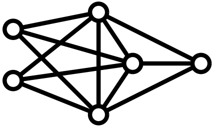
Graph & Matrix blowups

loop graph $\mathfrak{G}$



(2, 3, 1)-blowup

simple graph H



$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 4 & 7 \\ 0 & 7 & 0 \end{bmatrix} \xrightarrow{(2,3,1)\text{-blowup}} \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 4 & 4 & 4 & 7 \\ 1 & 1 & 4 & 4 & 4 & 7 \\ 1 & 1 & 4 & 4 & 4 & 7 \\ 0 & 0 & 7 & 7 & 7 & 0 \end{bmatrix}$$

$A \in S(\mathfrak{G})$

$A' \in S(H)$

Simple graph $\longleftarrow$ loop graph

- The notation $H \xleftarrow{(t_1,\ldots,t_n)} \mathfrak{G}$ means H is the simple graph obtained from the loop graph $\mathfrak{G}$ by $(t_1,\ldots,t_n)$-blowup.
- E.g., $K_{3,3,3} \xleftarrow{(3,3,3)} \mathfrak{C}_3^0$.

Theorem (L '16)

Suppose $H \xleftarrow{(t_1,\ldots,t_n)} \mathfrak{G}$ with $t_i \geq 3$ and $M(\mathfrak{G}) = Z_{oc}(\mathfrak{G})$. Then $M(H) = \widehat{Z}_{oc}(H) = Z_{oc}(\mathfrak{G}) + \ell$, where $\ell = \sum_{i=1}^{n}(t_i - 1)$.

E.g., since $M(\mathfrak{C}_3^0) = Z_{oc}(\mathfrak{C}_3^0) = 0$,

$$M(K_{3,3,3}) = \widehat{Z}_{oc}(K_{3,3,3}) = 0 + (2+2+2) = 6.$$

Sufficient condition for the Strong Arnold Property

SAP zero forcing Z_{SAP}

The Strong Arnold Property

- A real symmetric matrix A is said to have the Strong Arnold Property (SAP) if the only real symmetric matrix X that satisfies

$$\begin{cases} A \circ X = O \\ I \circ X = O \\ AX = O \end{cases}$$

  is $X = O$. Here $\circ$ is the Hadamard (entrywise) product.
- If A is nonsingular, then A has the SAP.
- If $A \in \mathcal{S}(K_n)$, then A has the SAP.

Example of not having the SAP

Let

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \end{bmatrix}.$$

Then $A \circ X = I \circ X = O$ and $AX = O$, so A does not have the SAP.

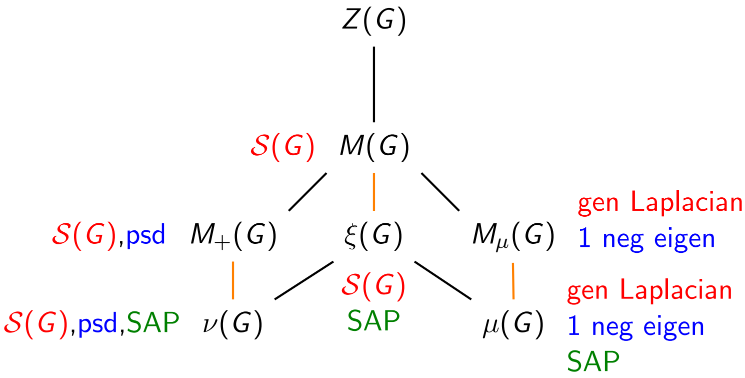
Motivation: Colin de Verdière parameter $\mu(G)$

- For a simple graph G, the Colin de Verdière parameter $\mu(G)$ [Colin de Verdière '90] is the maximum nullity over matrices A such that
  - $A \in \mathcal{S}(G)$ and all off-diagonal entries are zero or negative. (Called generalized Laplacian.)
  - A has exactly one negative eigenvalue (counting multiplicity).
  - A has the SAP.
- Characterizations:
  - $\mu(G) \leq 1$ iff G is a disjoint union of paths. (No K_3 minor)
  - $\mu(G) \leq 2$ iff G is outer planar. (No $K_4, K_{2,3}$ minor)
  - $\mu(G) \leq 3$ iff G is planar. (No $K_5, K_{3,3}$ minor)
- It is conjectured that $\mu(G) + 1 \geq \chi(G)$.

Other Colin de Verdière type parameters

- $\xi(G) = \max\{\text{null}(A) : A \in \mathcal{S}(G), A \text{ has the SAP}\}$
- $\nu(G) = \max\{\text{null}(A) : A \in \mathcal{S}(G), A \text{ is PSD}, A \text{ has the SAP}\}$
- For Colin de Verdière type parameters $\beta \in \{\mu, \nu, \xi\}$, they are all minor monotone. That is, $\beta(H) \leq \beta(G)$ if H is a minor of G. [C '90, C '98, BFH '05]
- By graph minor theorem, $\beta(G) \leq k$ if and only if G does not contain a family of finite graphs as minors. (Called forbidden minors.)

Colin de Verdière type parameters



The meaning of Strong Arnold Property

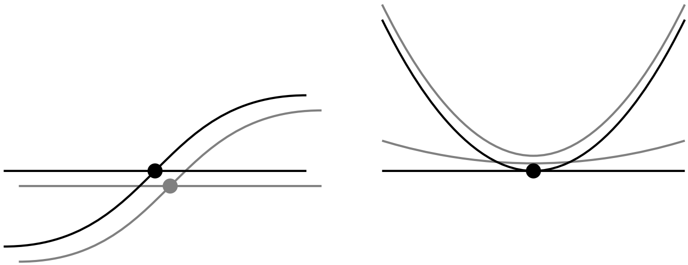
A real symmetric matrix A is said to have the Strong Arnold Property if $X = O$ is the only symmetric matrix that satisfies

$$\underbrace{A \circ X = I \circ X = O}_{\substack{\text{normal space of} \\ \text{the pattern manifold}}} \text{ and } \underbrace{AX = O}_{\substack{\text{normal space of} \\ \text{the rank manifold}}}.$$

- Pattern manifold: symmetric matrices with the same zero-nonzero pattern as A.
- Rank manifold: symmetric matrices with the same rank as A.

Two manifolds intersect transversally if the intersection of their normal spaces is $\{\mathbf{0}\}$. Equivalently, A has the SAP means the pattern manifold and the rank manifold of A intersect transverally.

Transversality: perturbation allowed



transversal

not transversal

If a matrix A has the SAP, then A can be perturbed slightly yet maintain the same rank.

How to verify the SAP?

- Let G be a graph and $A \in \mathcal{S}(G)$ with $\mathbf{v}_j$ its j-th column vector. Let $\overline{m} = |E(\overline{G})|$.
- The SAP matrix Ψ of A is an $n^2 \times \overline{m}$ matrix with
  - row indexed by (i, j) with $i, j \in \{1, \ldots, n\}$
  - column indexed by $\{i, j\} \in E(\overline{G})$
  - the $\{i, j\}$-th column of Ψ is

$$(\mathbf{0}, \ldots, \mathbf{0}, \underbrace{\mathbf{v}_j}_{i\text{-th block}}, \mathbf{0}, \ldots, \mathbf{0}, \underbrace{\mathbf{v}_i}_{j\text{-th block}}, \mathbf{0}, \ldots, \mathbf{0})^\top$$

- A has the SAP if and only if Ψ is full-rank.

Example of the SAP matrix: forcing triples

- Recall the SAP: $A \circ X = I \circ X = AX = O \implies X = O$.
- Let $G = P_4$ and $A \in \mathcal{S}(G)$ with $\mathbf{v}_j$ its j-th column vector.

$$AX = \begin{bmatrix} d_1 & a_1 & 0 & 0 \\ a_1 & d_2 & a_2 & 0 \\ 0 & a_2 & d_3 & a_3 \\ 0 & 0 & a_3 & d_4 \end{bmatrix} \begin{bmatrix} 0 & 0 & x_{\{1,3\}} & x_{\{1,4\}} \\ 0 & 0 & 0 & x_{\{2,4\}} \\ x_{\{1,3\}} & 0 & 0 & 0 \\ x_{\{1,4\}} & x_{\{2,4\}} & 0 & 0 \end{bmatrix} = O.$$

- This is equivalent to

$$\begin{bmatrix} \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{v}_4 \\ \mathbf{v}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} \begin{bmatrix} x_{\{1,3\}} \\ x_{\{1,4\}} \\ x_{\{2,4\}} \end{bmatrix} = \Psi \begin{bmatrix} x_{\{1,3\}} \\ x_{\{1,4\}} \\ x_{\{2,4\}} \end{bmatrix} = \mathbf{0}.$$

Zero forcing $\Longrightarrow$ full-rank

$$
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{c}
\begin{array}{ccc} x_{\{1,3\}} & x_{\{1,4\}} & x_{\{2,4\}} \end{array} \\
\begin{bmatrix} \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{v}_4 \\ \mathbf{v}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}
\end{array}
$$

Idea: If the zero forcing number is zero, then every matrix has the SAP.

Zero forcing $\implies$ full-rank

$$
\begin{array}{c} \\ 1 \\ \rightarrow 2 \\ 3 \\ 4 \end{array}
\begin{array}{c} \begin{matrix} x_{\{1,3\}} & x_{\{1,4\}} & x_{\{2,4\}} \end{matrix} \\ \begin{bmatrix} \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{v}_4 \\ \mathbf{v}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} \end{array}
$$

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Zero forcing $\Longrightarrow$ full-rank

$$
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{ccc} x_{\{1,3\}} & x_{\{1,4\}} & x_{\{2,4\}} \\ \end{array}
$$

$$
\begin{array}{c|ccc} & x_{\{1,3\}} & x_{\{1,4\}} & x_{\{2,4\}} \\ \hline 1 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{0} \\ 2 & \mathbf{0} & \mathbf{0} & \mathbf{v}_4 \\ 3 & \mathbf{v}_1 & \mathbf{0} & \mathbf{0} \\ 4 & \mathbf{0} & \mathbf{v}_1 & \mathbf{v}_2 \end{array}
$$

Idea: If the zero forcing number is zero, then every matrix has the SAP.

Zero forcing $\implies$ full-rank

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} x_{\{1,3\}} \quad x_{\{1,4\}} \quad x_{\{2,4\}} \\ \begin{bmatrix} \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{v}_4 \\ \mathbf{v}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} \end{array}$$

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Idea: If the zero forcing number is zero, then every matrix has the SAP.

Zero forcing $\implies$ full-rank

	$x_{\{1,3\}}$	$x_{\{1,4\}}$	$x_{\{2,4\}}$
1	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{0}$
2	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{v}_4$
3	$\mathbf{v}_1$	$\mathbf{0}$	$\mathbf{0}$
4	$\mathbf{0}$	$\mathbf{v}_1$	$\mathbf{v}_2$

Idea: If the zero forcing number is zero, then every matrix has the SAP.

Zero forcing $\implies$ full-rank

$$
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{ccc} x_{\{1,3\}} & x_{\{1,4\}} & x_{\{2,4\}} \\ \left[\begin{array}{c} \mathbf{v}_3 \\ \mathbf{0} \\ \mathbf{v}_1 \\ \mathbf{0} \end{array}\right. & \begin{array}{c} \mathbf{v}_4 \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{v}_1 \end{array} & \left.\begin{array}{c} \mathbf{0} \\ \mathbf{v}_4 \\ \mathbf{0} \\ \mathbf{v}_2 \end{array}\right] \end{array}
$$

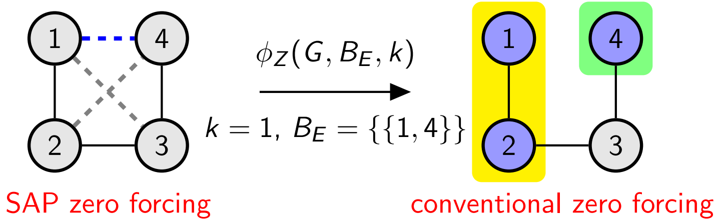
Idea: If the zero forcing number is zero, then every matrix has the SAP.

SAP zero forcing

- In an SAP zero forcing game, every non-edge has color either blue or white.
- If B_E is the set of blue non-edges, the local game on a given vertex k is a conventional zero forcing game on G, with blue vertices

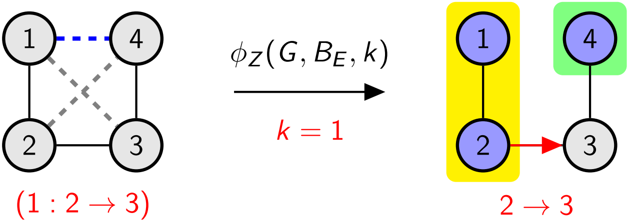
$$\phi_k(G, B_E) := N_G[k] \cup N_{\langle B_E \rangle}(k).$$

The local game is denoted by $\phi_Z(G, B_E, k)$.



SAP zero forcing

- Color change rule-Z_{SAP}:
  - Forcing triple $(k : i \to j)$: If $i \to j$ in $\phi_Z(G, B_E, k)$, then $\{j, k\}$ turns blue.
  - Odd cycle rule $(i \to C)$: Let G_W be the graph whose edges are the white non-edges. If $G_W[N_G(i)]$ contains a component that is an odd cycle C. Then $E(C)$ turns blue.
- $Z_{\mathrm{SAP}}(G)$ is the minimum number of blue non-edges such that all non-edges can turn blue eventually by CCR-Z_{SAP}.



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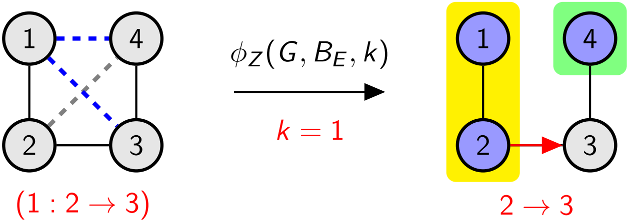

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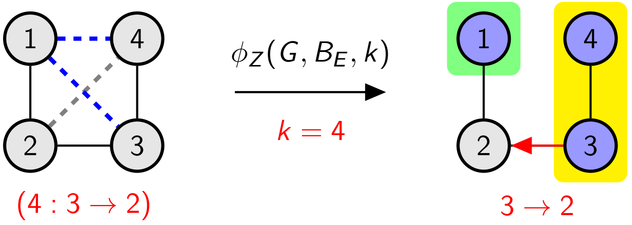

SAP zero forcing

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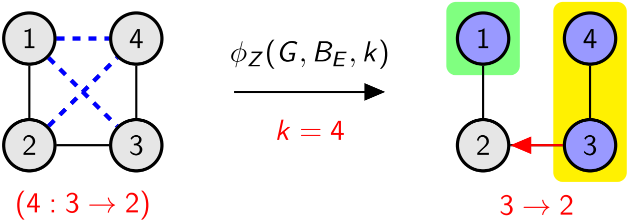
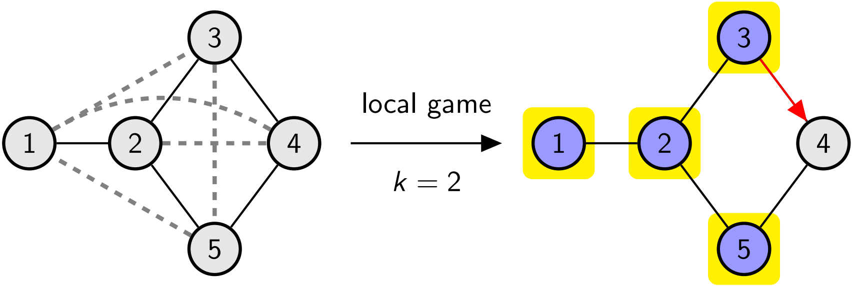
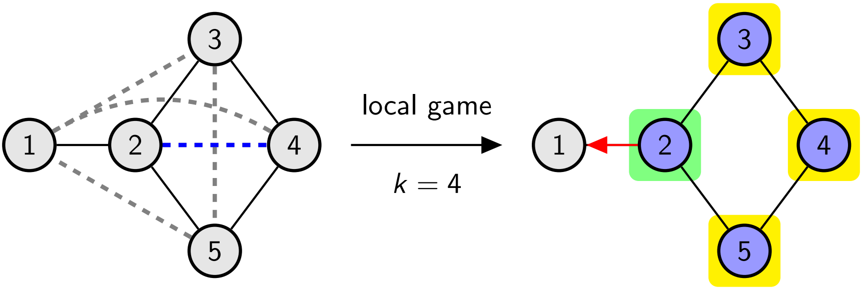


Example of $Z_{\mathrm{SAP}}(G) = 0$



Step	Forcing triple	Forced non-edge
1	$(2 : 3 \to 4)$	$\{2, 4\}$
2	$(4 : 2 \to 1)$	$\{4, 1\}$
3	$(5 : 4 \to 3)$	$\{5, 3\}$
4	$(3 : 2 \to 1)$	$\{3, 1\}$
5	$(5 : 2 \to 1)$	$\{5, 1\}$

Example of $Z_{\mathrm{SAP}}(G) = 0$



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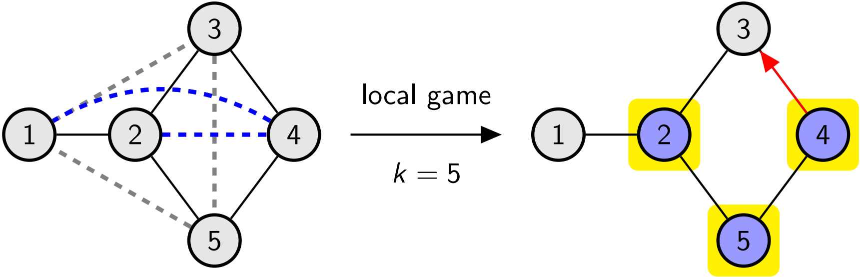
Example of $Z_{\text{SAP}}(G) = 0$



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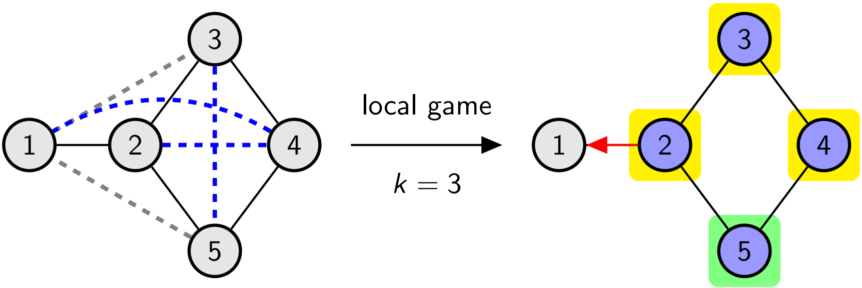
Example of $Z_{\mathrm{SAP}}(G) = 0$



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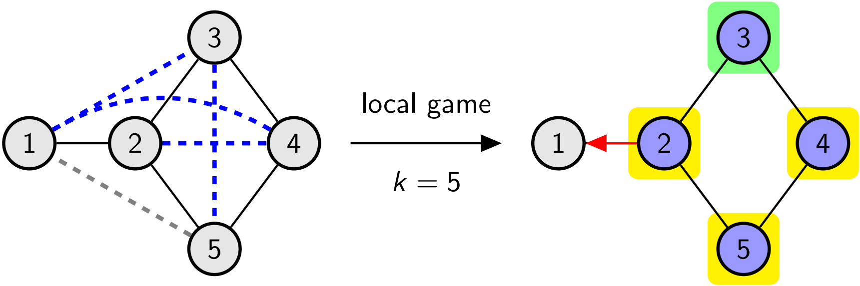

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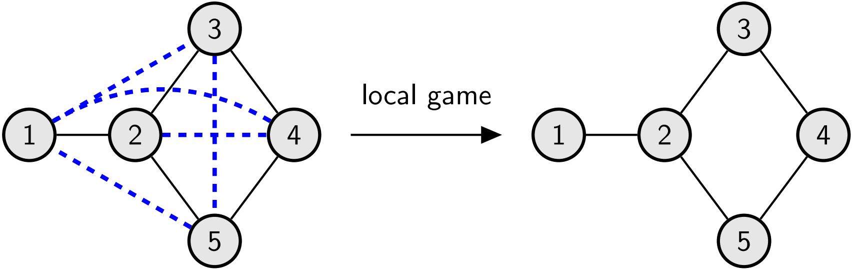
Example of $Z_{\mathrm{SAP}}(G) = 0$



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3	$(5 : 4 \to 3)$	$\{5, 3\}$
4	$(3 : 2 \to 1)$	$\{3, 1\}$
5	$(5 : 2 \to 1)$	$\{5, 1\}$

Theorem (L '16)

If $Z_{\mathrm{SAP}}(G) = 0$*, then every matrix* $A \in \mathcal{S}(G)$ *has the SAP. Therefore,* $\xi(G) = M(G)$*,* $M_+(G) = \nu(G)$*, and* $M_\mu(G) = \mu(G)$*.*

Computational results

How many graphs have the property $Z_{\mathrm{SAP}}(G) = 0$? The table shows for fixed n the proportion of graphs with $Z_{\mathrm{SAP}}(G) = 0$ in all connected graphs. (Isomorphic graphs count only once.)

n	$Z_{\mathrm{SAP}} = 0$
1	1.0
2	1.0
3	1.0
4	1.0
5	0.86
6	0.79
7	0.74
8	0.73
9	0.76
10	0.79

Applications

Theorem (L '16)

For every graph G, $M(G) - \xi(G) \leq Z_{\mathrm{vc}}(G)$.

Theorem (L '16)

The value of $\xi(G)$ can be computed for graphs G up to 7 vertices.

Conclusion

- Zero forcing controls the nullity of a linear system.
- Apply on patterns of graphs:
  - $M(\mathfrak{G}) \leq Z_{oc}(\mathfrak{G})$ for loop graphs;
  - $M(G) \leq \widehat{Z}_{oc}(G)$ for simple graphs.
- Apply on pattern of the SAP matrix:
  - A has the SAP $\Leftrightarrow$ the SAP matrix is full-rank;
  - when $Z_{\mathrm{SAP}}(G) = 0$, every matrix of G has the SAP.

Future work I

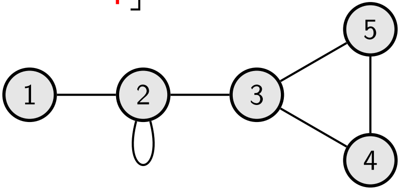
- $\widehat{Z}_{oc}(G)$ provides an upper bound for $M(G)$. How about the lower bounds?
- Davila and Kenter (2015) conjectured
$$(g-3)(\delta-2)+\delta \leq Z(G)$$
for graphs with girth $g \geq 3$ and minimum degree $\delta \geq 2$.
- Davila, Kalinowski, and Stephen (2017) posted a proof of the conjecture.
- Future work: Is it true that
$$(g-3)(\delta-2)+\delta \leq M(G)?$$
- Note that when $g=3$ or $\delta=2$, this is the delta conjecture/theorem.

Future work II

- The SAP allow us to perturb a matrix while preserving the rank.
- The Strong Spectral Property (SSP) and the Strong Multiplicity Property (SMP) preserves the spectrum and the multiplicity list, respectively.
- The SAP/SMP/SSP should have a counterpart where matrices do not require the symmetry.
- The counterpart of the SSP is called the Nilpotent Centralizer method in the field of sign patterns.
- Future work: use the zero forcing to control these properties, and find their applications.

$$
\begin{array}{c} \\ 1 \\ 3 \\ 4 \\ 5 \\ 2 \end{array}
\begin{array}{c}
\begin{array}{ccccc} x_2 & x_3 & x_4 & x_5 & x_1 \end{array} \\
\begin{bmatrix}
* & 0 & 0 & 0 & 0 \\
* & 0 & a & b & 0 \\
0 & a & 0 & c & 0 \\
0 & b & c & 0 & 0 \\
* & * & 0 & 0 & *
\end{bmatrix}
\end{array}
$$

Thank you!



References I

AIM Minimum Rank – Special Graphs Work Group (F. Barioli, W. Barrett, S. Butler, S. M. Cioabă, D. Cvetković, S. M. Fallat, C. Godsil, W. Haemers, L. Hogben, R. Mikkelson, S. Narayan, O. Pryporova, I. Sciriha, W. So, D. Stevanović, H. van der Holst, K. Vander Meulen, and A. Wangsness).
Zero forcing sets and the minimum rank of graphs.
Linear Algebra Appl., 428:1628–1648, 2008.

F. Barioli, S. M. Fallat, and L. Hogben.
A variant on the graph parameters of Colin de Verdière: Implications to the minimum rank of graphs.
Electron. J. Linear Algebra, 13:387–404, 2005.

References II

Y. Colin de Verdière.
Sur un nouvel invariant des graphes et un critère de planarité.
J. Combin. Theory Ser. B, 50:11–21, 1990.

Y. Colin de Verdière.
On a new graph invariant and a criterion for planarity.
In *Graph Structure Theory*, pp. 137–147, American Mathematical Society, Providence, RI, 1993.

Y. Colin de Verdière.
Multiplicities of eigenvalues and tree-width graphs.
J. Combin. Theory Ser. B, 74:121–146, 1998.

References III

R. Davila, T. Kalinowski, and S. Stephen.
The zero forcing number of graphs with a given girth.
https://arxiv.org/abs/1611.06557, 2017.

R. Davila and F. H. J. Kenter.
Bounds for the zero forcing number of graphs with large girth.
Theory Appl. Graphs, 2, 2015.

L. Hogben.
Minimum rank problems.
Linear Algebra Appl., 432:1961–1974, 2010.

References IV

J. C.-H. Lin.
Odd cycle zero forcing parameters and the minimum rank of graph blowups.
Electron. J. Linear Algebra, 31:42–59, 2016.

J. C.-H. Lin.
Using a new zero forcing process to guarantee the Strong Arnold Property.
Linear Algebra Appl., 507:229–250, 2016.