Parameters related to the minimum rank problem

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Preliminary Exam

Minimum rank problem

$$\begin{bmatrix} ? & * & 0 \\ * & ? & * \\ 0 & * & ? \end{bmatrix}$$

Minimum rank problem

$$\begin{bmatrix} ? & * & 0 \\ * & ? & * \\ 0 & * & ? \end{bmatrix}$$

Real, Symmetric

What is the smallest possible rank?

Minimum rank problem

$$\begin{bmatrix} ? & * & 0 \\ * & ? & * \\ 0 & * & ? \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

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Minimum rank problem

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Real, Symmetric

What is the smallest possible rank?

$$\downarrow$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{\text{rank}} 3$$

Minimum rank problem

$$\begin{bmatrix} ? & * & 0 \\ * & ? & * \\ 0 & * & ? \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{\text{spectrum}} 1,\ 1 \pm \sqrt{2}$$

Real, Symmetric

What is the smallest possible rank?

Minimum rank problem

$$\begin{bmatrix} ? & * & 0 \\ * & ? & * \\ 0 & * & ? \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} \color{red}{0} & 1 & 0 \\ 1 & \color{red}{0} & 1 \\ 0 & 1 & \color{red}{0} \end{bmatrix}$$

Real, Symmetric

What is the smallest possible rank?

Minimum rank problem

$$\begin{bmatrix} ? & * & 0 \\ * & ? & * \\ 0 & * & ? \end{bmatrix}$$

Real, Symmetric

What is the smallest possible rank?

$$\downarrow$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{\text{rank}} 2$$

Minimum rank problem

- Let G be a simple graph.
- Denote $\mathcal{S}^F(G)$ as the family of symmetric matrices over the field F whose i,j-entry, $i \neq j$, is nonzero if $i \sim j$ and zero otherwise. (Diagonal entries are free.)
- The minimum rank of G is defined as

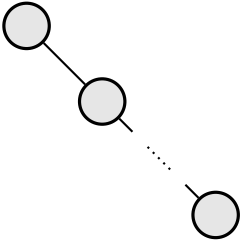
$$\mathrm{mr}^F(G) = \min\{\mathrm{rank}(A) : A \in \mathcal{S}^F(G)\}.$$

  The maximum nullity is

$$M^F(G) = \max\{\mathrm{null}(A) : A \in \mathcal{S}^F(G)\}.$$

- $M^F(G) + \mathrm{mr}^F(G) = |V(G)|$ for any G and F.

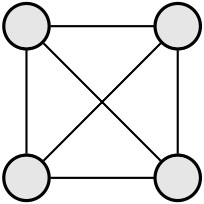
Example: Paths P_n



$$\begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & & \ddots & -2 & 1 \\ 0 & \cdots & 0 & 1 & -1 \end{bmatrix}$$

$M(G) \neq 0$ for all G.
$M(G) = 1$ iff G is a path.
[Fiedler (1969), Bento and Leal Duarte (2005)]

Example: Complete Graphs K_n



$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$M(G) = n \text{ iff } G = \overline{K_n}.$$
$$M(G) = n-1 \text{ iff } G = K_n \dot\cup \overline{K_m},\ n \geq 2.$$

Inverse eigenvalue problem

$$\begin{bmatrix} ? & * & 0 \\ * & ? & * \\ 0 & * & ? \end{bmatrix}$$

Real, Symmetric

What is the possible spectrum?

- We know $\mathrm{mr}(G) = 2$ and $M(G) = 1$ and Spec= $\{1, 1 \pm \sqrt{2}\}$ is possible.
- Can Spec= $\{1, 5, 5\}$?
- No, for otherwise $\mathrm{null}(A - 5I) = 2 > M(G)$.
- Largest possible multiplicity = $M(G)$.

Inverse eigenvalue problem

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Inverse eigenvalue problem

Theorem (K. H. Monfared, B. L. Shader 2013)

For a graph G and distinct real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$, there is a matrix $A \in \mathcal{S}^{\mathbb{R}}(G)$ such that the spectrum of A is $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Inverse eigenvalue problem

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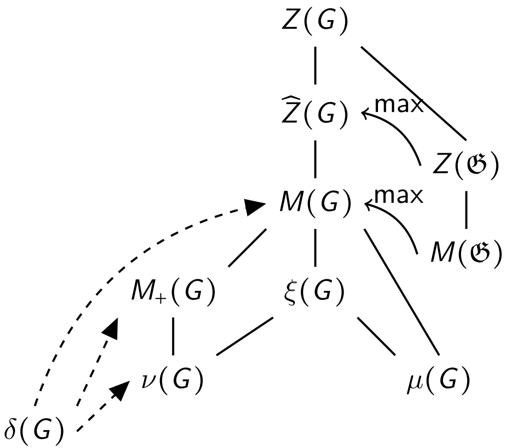
For the case multiplicity $\neq 1$, it is still unknown, but the minimum rank problem provides a restriction.

The landscape of minimum rank problems



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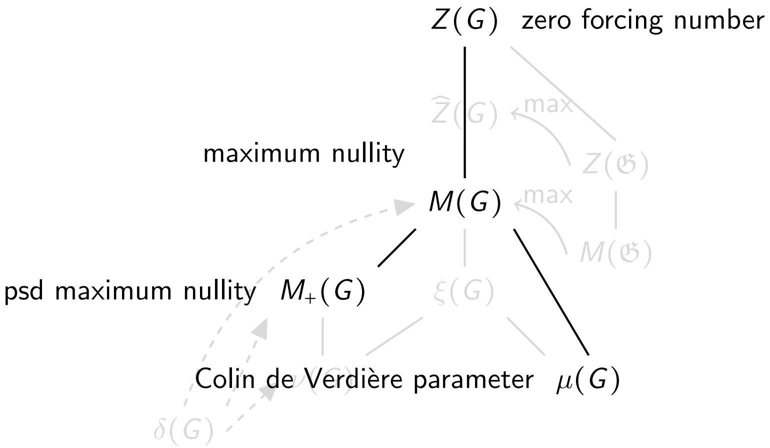
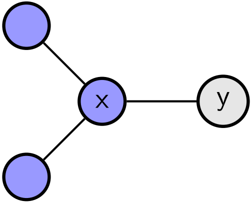


Zero forcing number

- A zero forcing game on a simple graph G starts by setting a set $B \subseteq V(G)$ of vertices blue and the others white, and then repeatedly applies the color-change rule (CCR):
  - if $y \in V(G)$ is the only white neighbor of $x \in V(G)$ and x is blue, then y turns blue.



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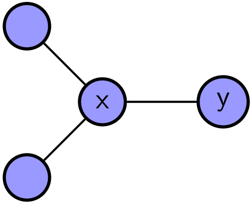

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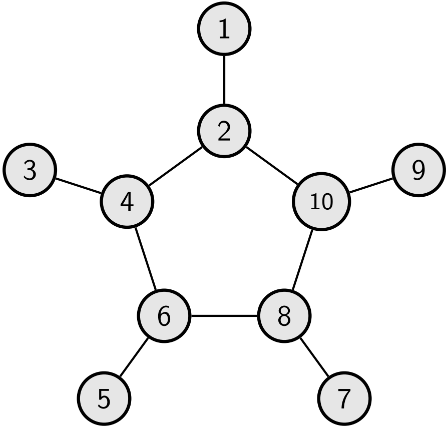
  - if $y \in V(G)$ is the only white neighbor of $x \in V(G)$ and x is blue, then y turns blue.
- The final coloring is the set of blue vertices when no more CCR applies.
- The initial set B is called a zero forcing set if its final coloring is $V(G)$.
- The zero forcing number of G, denoted as $Z(G)$, is the minimum cardinality of a zero forcing set on G.

Example: 5-sun H_5



$$Z(H_5) = 3.$$

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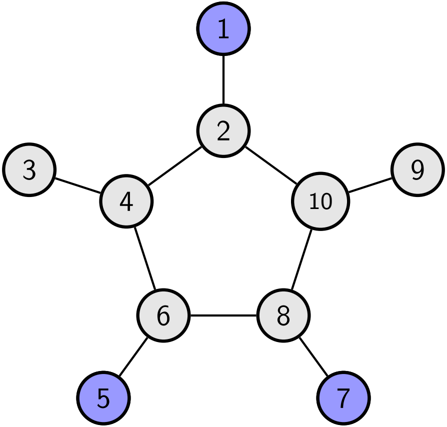

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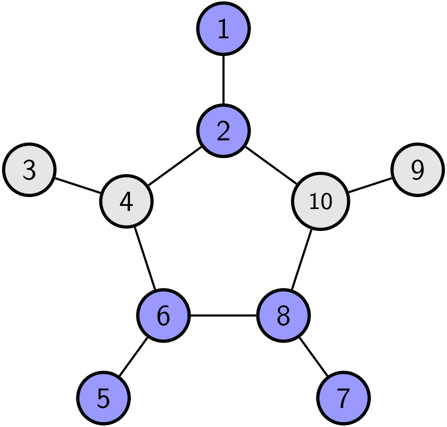

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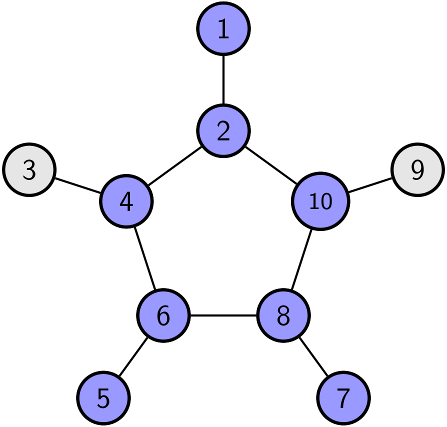

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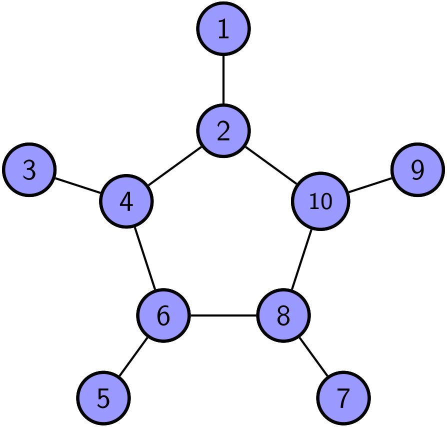

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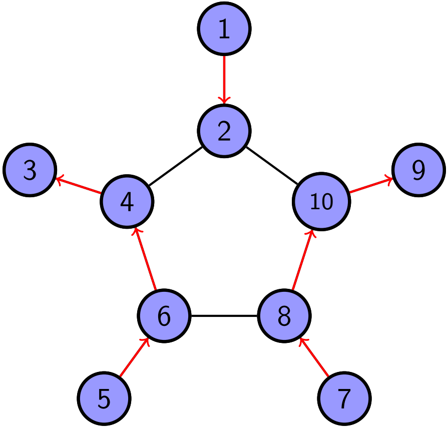


$$Z(H_5) = 3.$$

Triangle number

- Let Q be a pattern (a matrix with entries $\in \{0, *, ?\}$).
- An upper triangular subpattern is a square submatrix of Q such that the lower part is all 0, diagonals are $*$.
- The triangle number of Q, denoted as $\text{tri}(Q)$, is the largest size of an upper triangular subpattern that can be found in Q through row/column permutations.

$$\begin{bmatrix} * & 0 & 0 \\ ? & * & ? \\ 0 & 0 & * \end{bmatrix} \longrightarrow \begin{bmatrix} ? & * & ? \\ * & 0 & 0 \\ 0 & 0 & * \end{bmatrix} \longrightarrow \begin{bmatrix} * & ? & ? \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}$$

- If Q is the pattern of a graph G, then $\text{mr}(G) \geq \text{tri}(Q)$ and $M(G) \leq n - \text{tri}(Q)$.

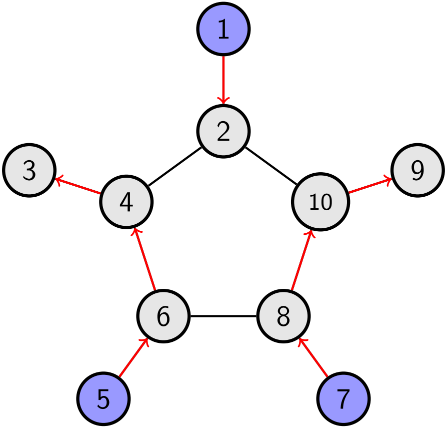
Triangle number of H_5?

The pattern Q below is the pattern for H_5. What is $\text{tri}(Q)$?

$$\begin{bmatrix}
? & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & ? & 0 & * & 0 & 0 & 0 & 0 & 0 & * \\
0 & 0 & ? & * & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & * & * & ? & 0 & * & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & ? & * & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & * & * & ? & 0 & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & ? & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & * & * & ? & 0 & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ? & * \\
0 & * & 0 & 0 & 0 & 0 & 0 & * & * & ?
\end{bmatrix}$$

Triangle number of H_5?

$1 \to 2$
$5 \to 6$
$7 \to 8$
$6 \to 4$
$8 \to 10$
$4 \to 3$
$10 \to 9$



Triangle number of H_5?

$\text{tri}(Q) = 7$ and $Z(H_5) = 3$.

	1	5	7	6	8	4	10	2	3	9
2	$*$	0	0	0	0	$*$	$*$	?	0	0
6	0	$*$	0	?	$*$	$*$	0	0	0	0
8	0	0	$*$	$*$	?	0	$*$	0	0	0
4	0	0	0	$*$	0	?	0	$*$	$*$	0
10	0	0	0	0	$*$	0	?	$*$	0	$*$
3	0	0	0	0	0	$*$	0	0	?	0
9	0	0	0	0	0	0	$*$	0	0	?
1	?	0	0	0	0	0	0	$*$	0	0
5	0	?	0	$*$	0	0	0	0	0	0
7	0	0	?	0	$*$	0	0	0	0	0

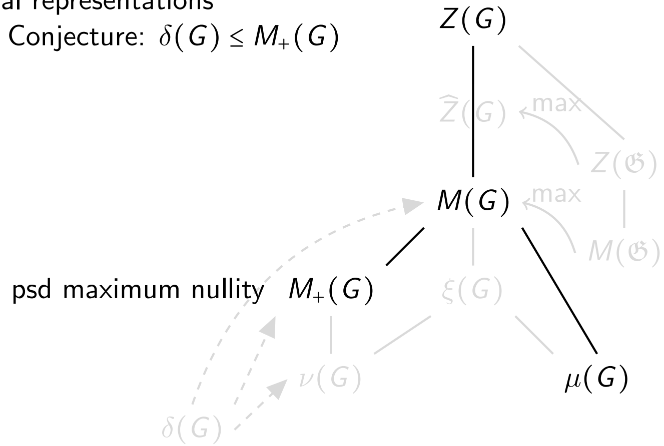
Zero forcing vs Triangle

- Number of forces $x_i \to y_i$ = size of triangle.
- $Z(G) = n - \text{tri}(Q)$, where Q is the pattern of G.
- $M^F(G) \leq Z(G)$, for any simple graph G, any field F [AIM Group 2007].
- It doesn't matter if $\mathcal{S}^F(G)$ is defined to be symmetric or not.
- $M(G) = Z(G)$ when $|V(G)| \leq 7$ or G is a tree, a cycle, a complete bipartite graph, ...
- $Z(H_5) = 3$ but $M(H_5) = 2$.

The landscape of minimum rank problems

orthogonal representations

Maehara Conjecture: $\delta(G) \leq M_+(G)$



PSD maximum nullity

- Denote $\mathcal{S}^{\mathbb{F}}(G)$ as the family of symmetric matrices over $\mathbb{F}$ whose i,j-entry, $i \neq j$, is nonzero if $i \sim j$ and zero otherwise. (Diagonal entries are free.)
- $\mathbb{F} = \mathbb{R}$, or $\mathbb{C}$.
- $\mathrm{mr}_{+}^{\mathbb{F}}(G) = \min\{\mathrm{rank}(A) : A \in \mathcal{S}^{\mathbb{F}}(G),\ A \text{ is psd}\}$.
- $M_{+}^{\mathbb{F}}(G) = \max\{\mathrm{null}(A) : A \in \mathcal{S}^{\mathbb{F}}(G),\ A \text{ is psd}\}$.

PSD Decomposition

- Let A be an $n \times n$ (symmetric) psd matrix with $\text{rank}(A) = r$.
- Then

$$S^*S = \begin{bmatrix} - & v_1^* & - \\ - & v_2^* & - \\ & \vdots & \\ - & v_n^* & - \end{bmatrix} \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} = [\langle v_i, v_j \rangle],$$

where $v_i \in \mathbb{F}^r$.

Orthogonal representation (faithful)

- $$S^*S = \begin{bmatrix} - & v_1^* & - \\ - & v_2^* & - \\ & \vdots & \\ - & v_n^* & - \end{bmatrix} \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} = [\langle v_i, v_j \rangle],$$

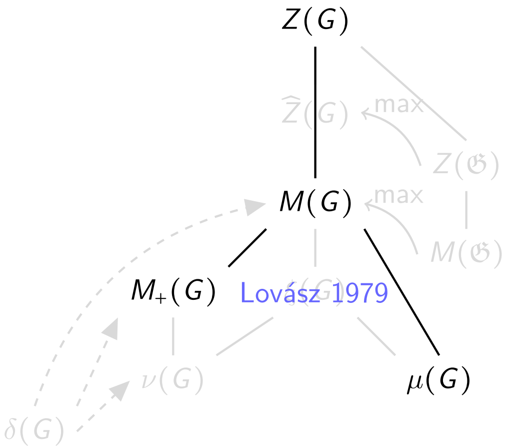
  where $v_i \in \mathbb{F}^r$.

- A (faithful) orthogonal representation is a function:

$$\begin{array}{ccc} V(G) & \longrightarrow & \mathbb{F}^d \\ i & \longmapsto & v_i \end{array} \quad \text{such that } \langle v_i, v_j \rangle \begin{cases} \neq 0 & \text{if } i \sim j \\ = 0 & \text{if } i \not\sim j. \end{cases}$$

- For a given graph G, $\min r = \min d$, so $M_+(G) = n - \min d$.

delta conjecture



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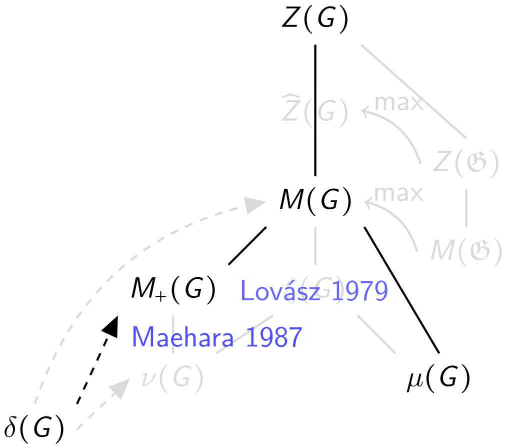

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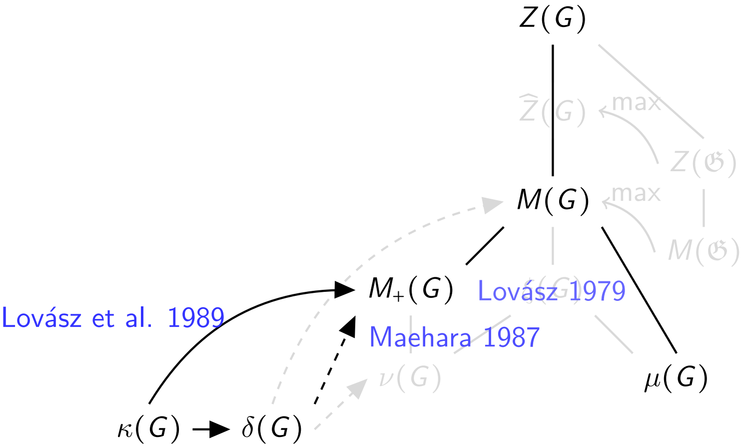

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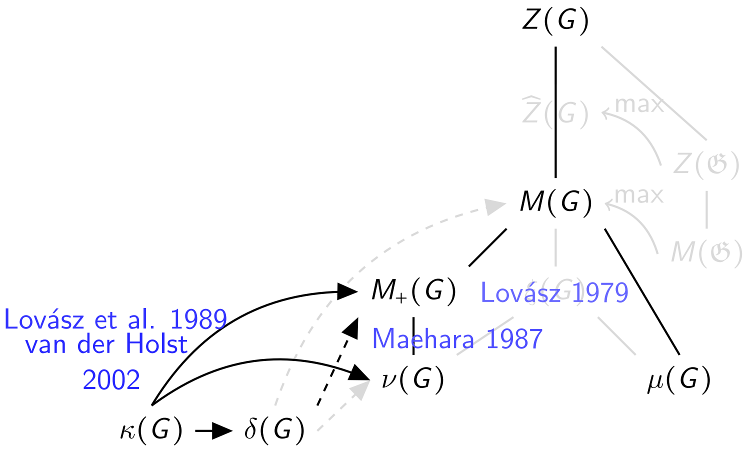

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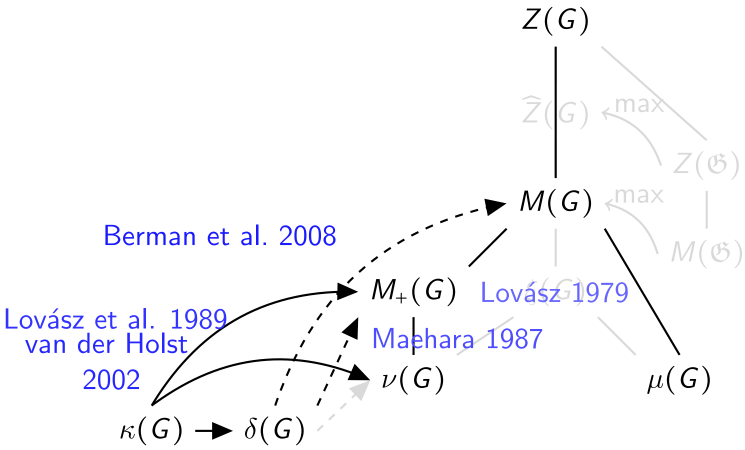

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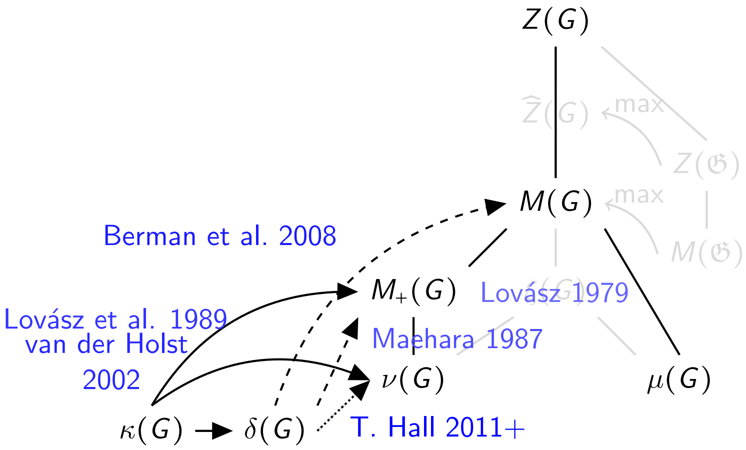


What is ν?

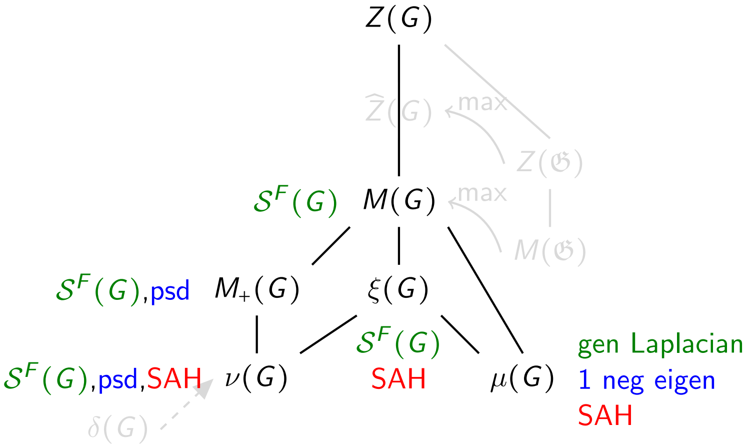
- We say a matrix A satisfies strong Arnol'd Hypothesis (SAH) if there is no nonzero symmetric matrix X satisfying

$$\begin{cases} I \circ X = O \\ A \circ X = O \\ AX = O \end{cases},$$

where $\circ$ is the Hadamard (entrywise) product.

- $\nu(G) = \max\{\text{null}(A) : A \in \mathcal{S}^{\mathbb{R}}(G),\ A \text{ is psd},\ \text{SAH}\}$
- Colin de Verdière (1998) proved that if H is a minor of G, then $\nu(H) \leq \nu(G)$.

Colin de Verdière type parameters



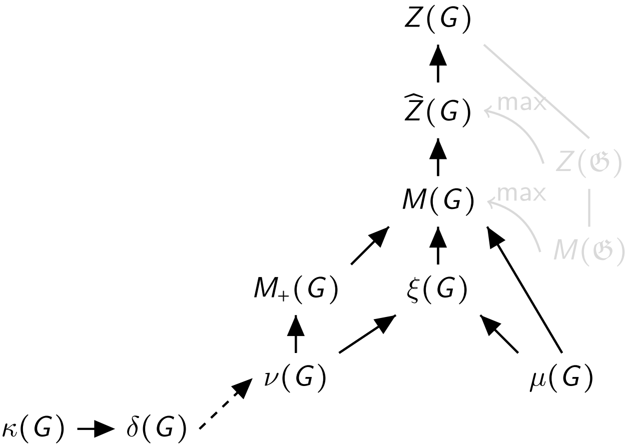
Colin de Verdière parameter μ

- $\mu(G)$ is defined as the maximum nullity among matrices M with the following properties:
  - Generalized Laplacian: $M_{ij} \begin{cases} < 0 & \text{if} \quad i \sim j, i \neq j \\ = 0 & \text{if} \quad i \nsim j, i \neq j \\ \text{free} & \text{if} \quad i = j \end{cases}$.
  - M has exactly one negative eigenvalue.
  - M satisfies SAH.
- $\mu(G)$ bridges algebraic and topological properties of a graph [Colin de Verdière, Robertson et al., Lovász et al.]:
  - $\mu(G) \leq 1$ iff G is a disjoint union of paths;
  - $\mu(G) \leq 2$ iff G is outerplanar;
  - $\mu(G) \leq 3$ iff G is planar;
  - $\mu(G) \leq 4$ iff G is linklessly embedable.
- Colin de Verdière conjectured $\chi(G) \leq \mu(G) + 1$.

Graph Complement Conjecture (GCC)

- Let G be a simple graph and $\beta(G)$ a parameter of G. Then GCC-β states that $\beta(G) + \beta(\overline{G}) \geq n - 2$.
  - Kotlov (1997) conjectured GCC-μ.
  - Brualdi et al. (2007) conjectured GCC-M.
  - Barioli et al. (2012) conjectured GCC-M_+ and GCC-ν.
  - ISU EGR group (2011) proved GCC-Z, GCC-Z_+, and GCC-tw.

Graph Complement Conjecture



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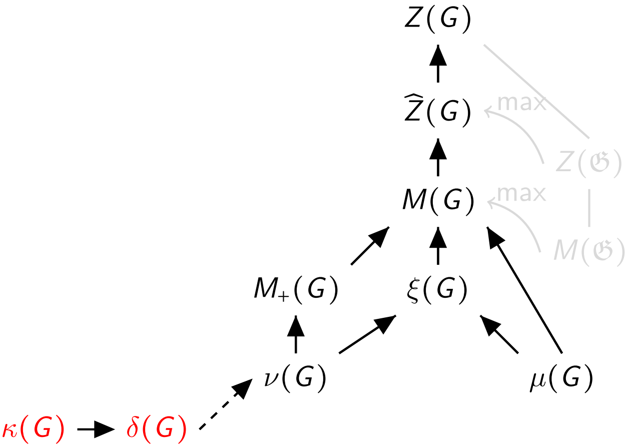

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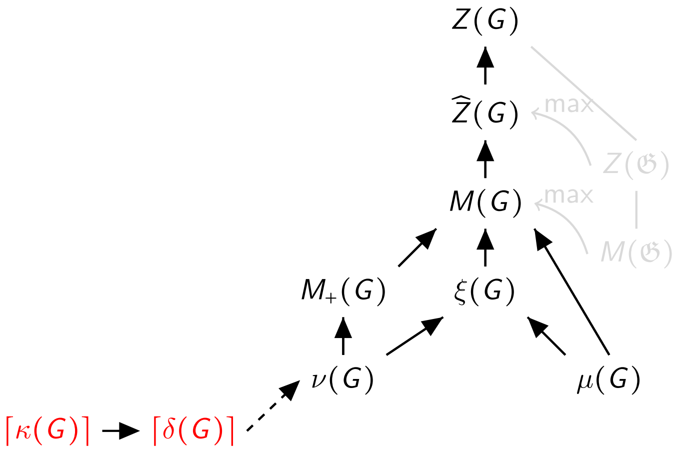

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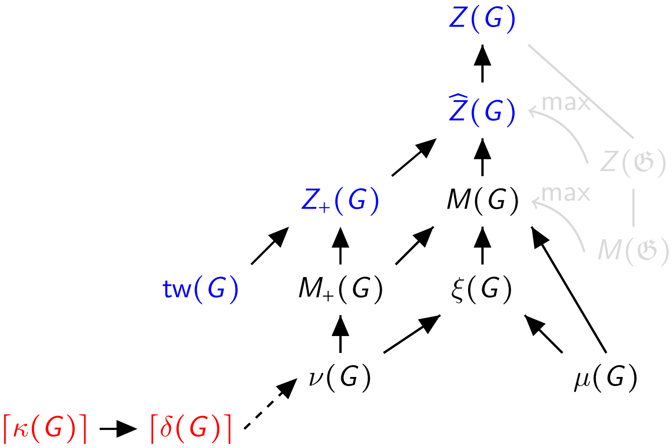
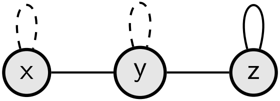


Loop graphs

- A loop graph $\mathfrak{G}$ is a graph where loops are allowed. (Each vertex has at most one loop.)
- A loop configuration $\mathfrak{G}$ of a simple graph G is a loop graph obtained from G by designating each vertex as having no loop or one loop. (There are 2^n possibilities.)
- $M^F(\mathfrak{G}) = \max\left\{ \text{null}(A) : A \in \mathcal{S}^F(G), A_{i,i} \begin{cases} \neq 0 & \text{if} \quad i \text{ has a loop;} \\ = 0 & \text{if} \quad i \text{ has no loop.} \end{cases} \right\}.$
- $M^F(G) = \max_{\mathfrak{G}} M^F(\mathfrak{G})$, where $\mathfrak{G}$ runs over all loop configurations of $\mathfrak{G}$.

$Z(\mathfrak{G})$ for loop graphs / $\widehat{Z}(G)$ for simple graphs

- The color-change rule for loop graphs is:
  - if $y \in V(\mathfrak{G})$ is the only white neighbor of $x \in V(\mathfrak{G})$ ~~and x is blue~~, then y turns blue. ($x = y$ is possible.)



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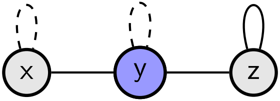

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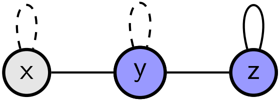

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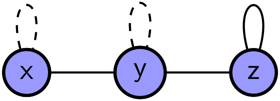
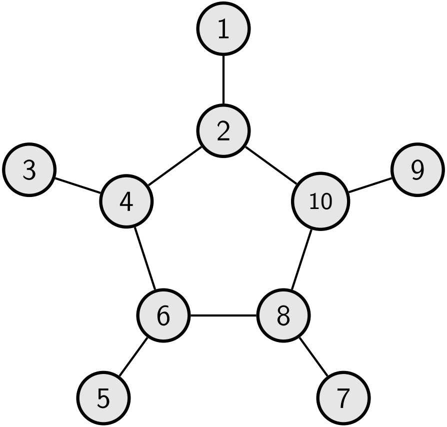


$Z(\mathfrak{G})$ for loop graphs / $\widehat{Z}(G)$ for simple graphs

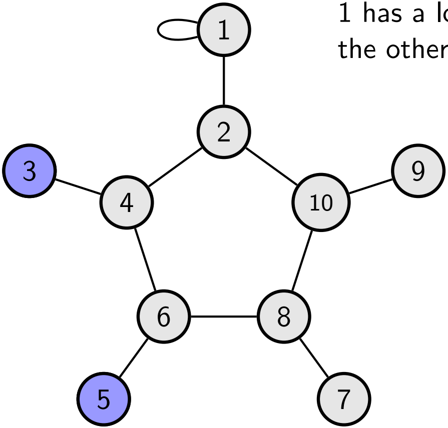
- The color-change rule for loop graphs is:
  - if $y \in V(\mathfrak{G})$ is the only white neighbor of $x \in V(\mathfrak{G})$ ~~and x is blue~~, then y turns blue. ($x = y$ is possible.)
- $Z(\mathfrak{G})$ is the smallest cardinality of a zero forcing set on $\mathfrak{G}$ using CCR for loop graphs.
- $M^F(\mathfrak{G}) \leq Z(\mathfrak{G})$ for all loop graphs $\mathfrak{G}$ and fields F [Hogben (2010)].
- If $\mathfrak{G}$ is a loop configuration of G, then $Z(\mathfrak{G}) \leq Z(G)$.
- The enhanced zero forcing number is defined as $\widehat{Z}(G) = \max_{\mathfrak{G}} Z(\mathfrak{G})$, where $\mathfrak{G}$ runs over all loop configurations of G.

H_5 revisited



$\widehat{Z}(H_5) = 2$ and $Z(H_5) = 3$.

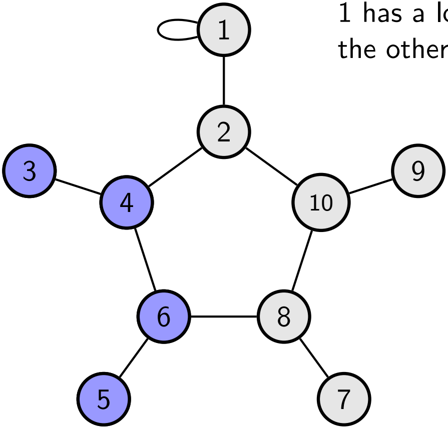
H_5 revisited



1 has a loop and
the others are unknown.

$$\widehat{Z}(H_5) = 2 \text{ and } Z(H_5) = 3.$$

H_5 revisited



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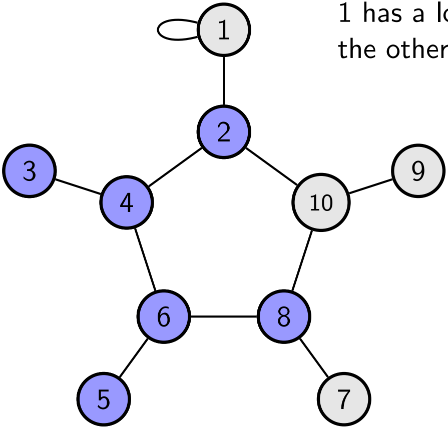

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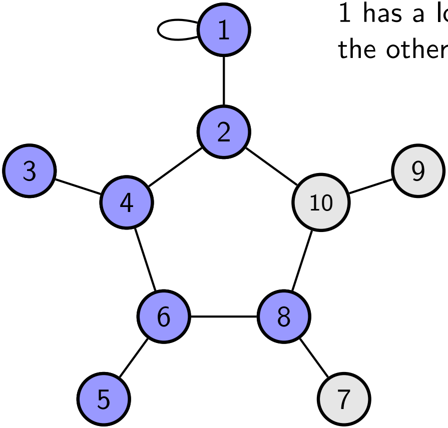

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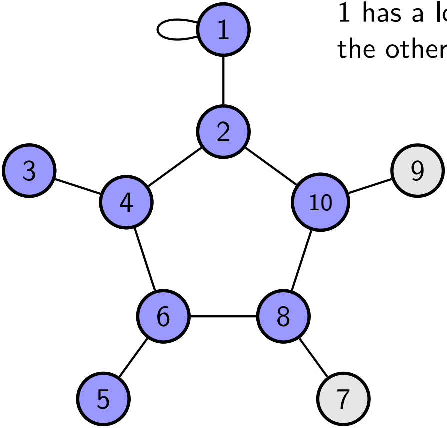
H_5 revisited



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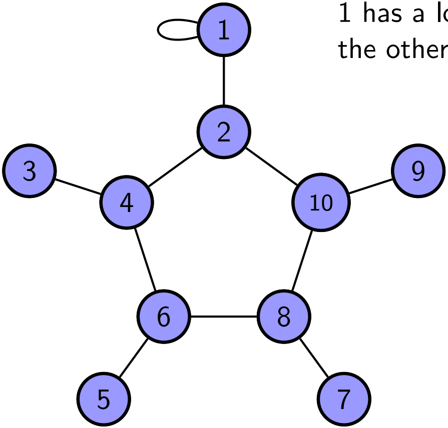
H_5 revisited



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the others are unknown.

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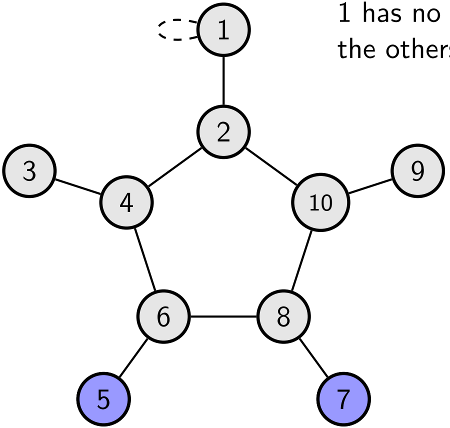
H_5 revisited



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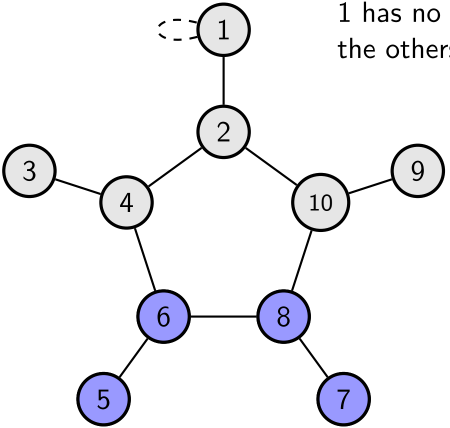
H_5 revisited



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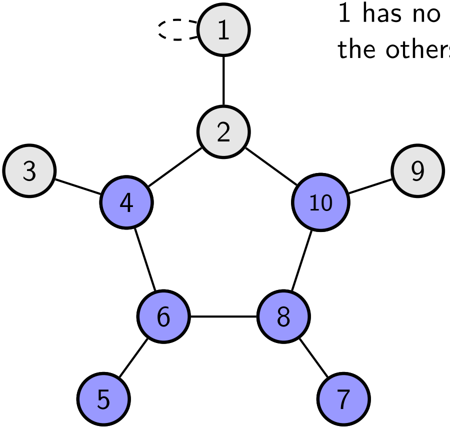
H_5 revisited



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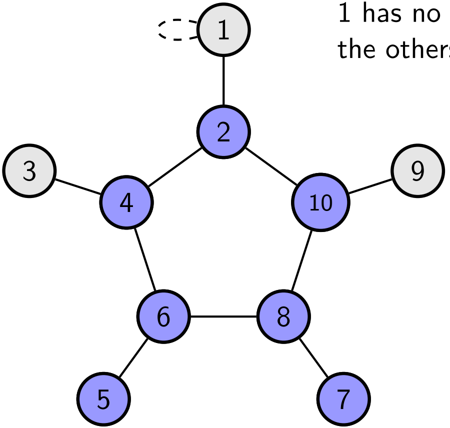
H_5 revisited



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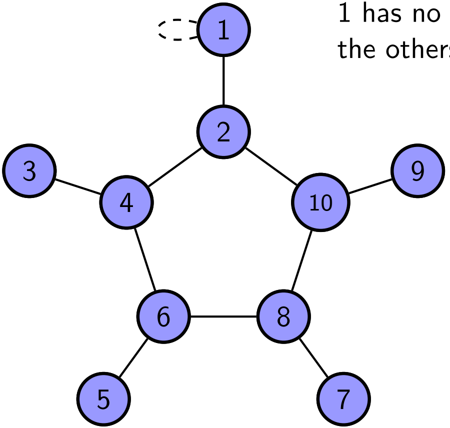
H_5 revisited



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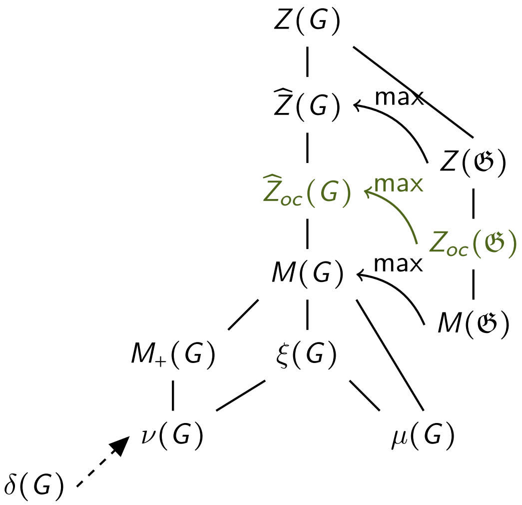
Sage Data

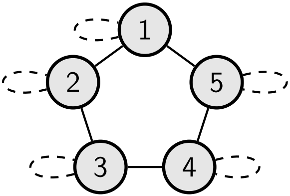
- $M(G) = \widehat{Z}(G) = Z(G)$ if $|V(G)| \leq 7$.
- For $n = 8$, there are 7 graphs with $\widehat{Z}(G) < Z(G)$.
- For $n = 9$, there are 412 graphs with $\widehat{Z}(G) < Z(G)$.
- For $n = 10$, there are 18700+ graphs with $\widehat{Z}(G) < Z(G)$.
- But $M(K_{3,3,3}) = 6$ and $Z(G) = \widehat{Z}(G) = 7$.

New parameters $\widehat{Z}_{oc}(G)$ and $Z_{oc}(\mathfrak{G})$



Odd cycles

- $\widehat{Z}(G)$ shows a bound for $M(\mathfrak{G})$ leads to a bound for $M(G)$; an improvement of bounds for loop graphs leads to an improvement for simple graphs.
- Let $\mathfrak{C}^0_{2k+1}$ be a loopless odd cycle, as a loop graph. Then $M(\mathfrak{C}^0_{2k+1}) = 0$ but $Z(\mathfrak{C}^0_{2k+1}) = 1$.



$$\det \begin{bmatrix} 0 & e_1 & & & e_{2k+1} \\ e_1 & 0 & e_2 & & \\ & e_2 & \ddots & \ddots & \\ & & \ddots & & e_{2k} \\ e_{2k+1} & & & e_{2k} & 0 \end{bmatrix} = 2\prod_{i=1}^{2k+1} e_i$$

Try to generalize triangle number

$$\text{rank}\begin{bmatrix} a_{1,1} & ? & ? & ? & ? \\ 0 & a_{2,2} & ? & ? & ? \\ 0 & 0 & a_{3,3} & ? & ? \\ ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \end{bmatrix} \geq 3$$

Try to generalize triangle number

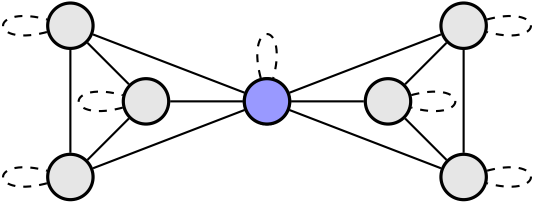
$$\operatorname{rank}\begin{bmatrix} A_{1,1} & ? & ? & ? & ? \\ O & A_{2,2} & ? & ? & ? \\ O & O & A_{3,3} & ? & ? \\ ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \end{bmatrix} \geq \sum_{i=1}^{3} \operatorname{rank}(A_{i,i})$$

Try to generalize triangle number

$$\text{rank}\begin{bmatrix} A(\mathfrak{C}_5^0) & ? & ? & ? & ? \\ O & A(\mathfrak{C}_7^0) & ? & ? & ? \\ O & O & A(\mathfrak{C}_3^0) & ? & ? \\ ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \end{bmatrix} \geq 5+7+3=15$$

Odd cycle zero forcing number

- The color-change rule CCR-Z_{oc} for loop graphs is:
  - if $y \in V(\mathfrak{G})$ is the only white neighbor of $x \in V(\mathfrak{G})$ ~~and x is blue~~, then y turns blue ($x = y$ is possible);
  - if W is the set of white vertices, and $\mathfrak{G}[W]$ has a connected component $\mathfrak{C}$ such that $\mathfrak{C} \cong \mathfrak{C}^0_{2k+1}$, then all vertices in $V(\mathfrak{C})$ turn blue.



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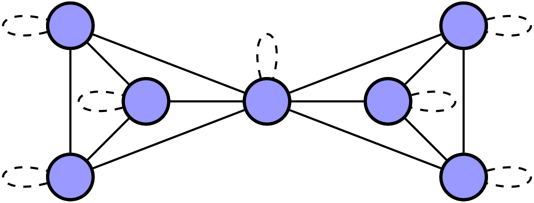

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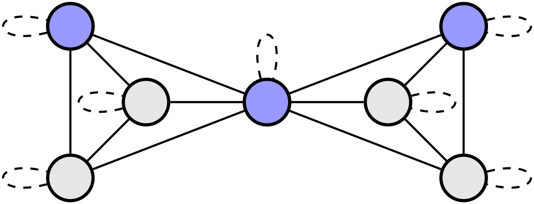

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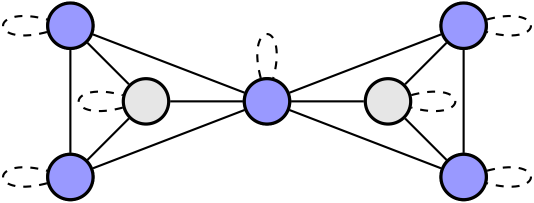

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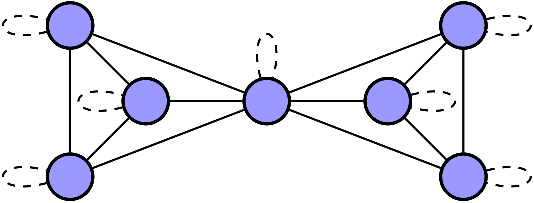

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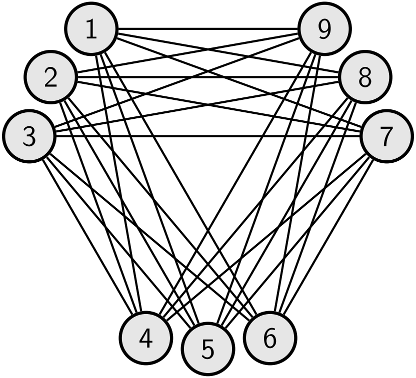
  - if W is the set of white vertices, and $\mathfrak{G}[W]$ has a connected component $\mathfrak{C}$ such that $\mathfrak{C} \cong \mathfrak{C}^0_{2k+1}$, then all vertices in $V(\mathfrak{C})$ turn blue.
- $Z_{oc}(\mathfrak{G})$ is the smallest cardinality of a zero forcing set on $\mathfrak{G}$ using CCR-Z_{oc} for loop graphs.
- $M^F(\mathfrak{G}) \le Z_{oc}(\mathfrak{G})$ whenever char $F \ne 2$ and matrices are symmetric.
- The enhanced odd cycle zero forcing number is defined as $\widehat{Z}_{oc}(G) = \max_{\mathfrak{G}} Z_{oc}(\mathfrak{G})$, where $\mathfrak{G}$ runs over all loop configurations of G.

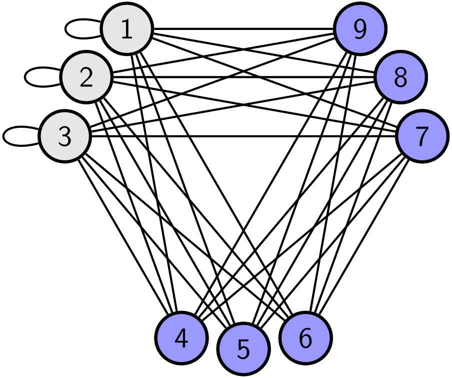
Example: $K_{3,3,3}$



$\widehat{Z}_{oc}(K_{3,3,3}) = 6$ and $\widehat{Z}(K_{3,3,3}) = Z(K_{3,3,3}) = 7$.

Example: $K_{3,3,3}$

1,2,3 have loops
others are unknown



$$\widehat{Z}_{oc}(K_{3,3,3}) = 6 \text{ and } \widehat{Z}(K_{3,3,3}) = Z(K_{3,3,3}) = 7.$$

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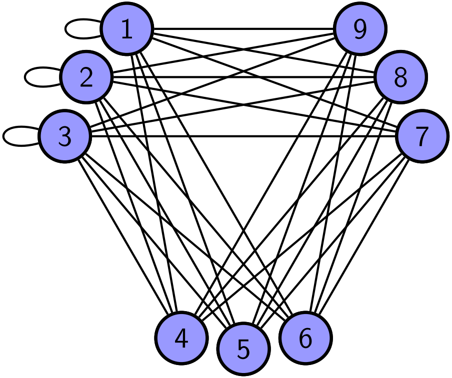

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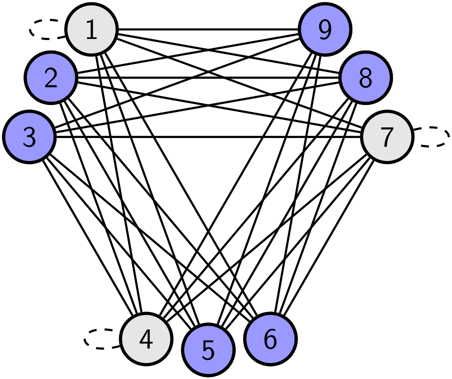
Example: $K_{3,3,3}$

1,4,7 have no loops
others are unknown



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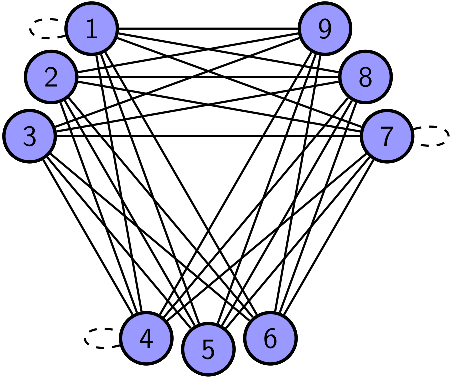
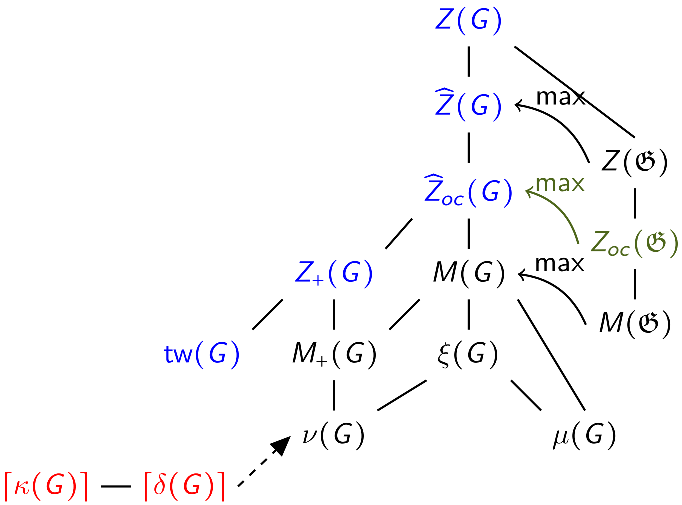
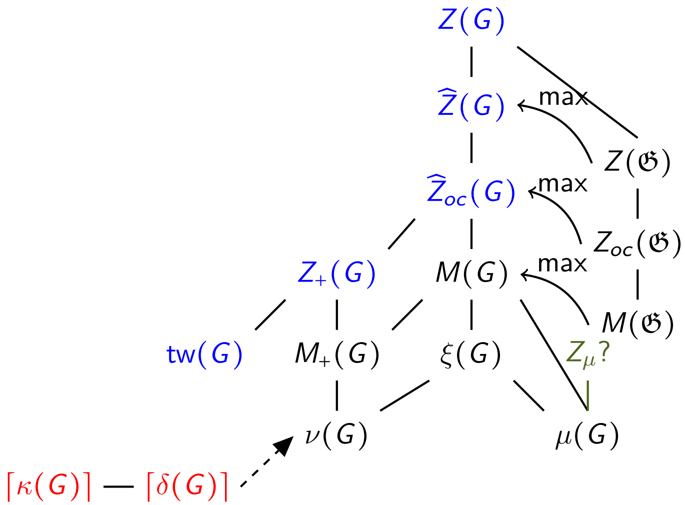


$$\widehat{Z}_{oc}(K_{3,3,3}) = 6 \text{ and } \widehat{Z}(K_{3,3,3}) = Z(K_{3,3,3}) = 7.$$

GCC-$\widehat{Z}_{oc}(G)$



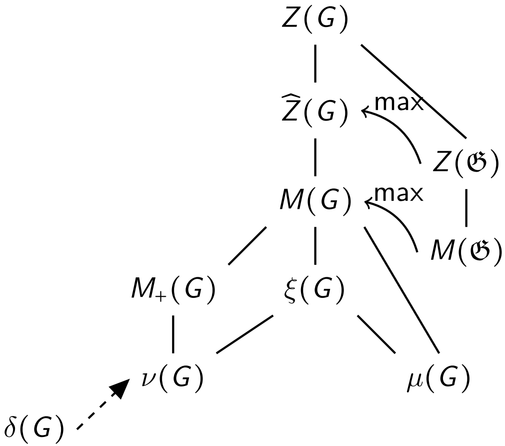
GCC-$\widehat{Z}_{oc}(G)$



Bound μ above

- Recall that $\mu(G)$ is the maximum nullity among matrices M with the following properties:
  - Generalized Laplacian: $M_{ij} \begin{cases} < 0 & \text{if} \quad i \sim j, i \neq j \\ = 0 & \text{if} \quad i \nsim j, i \neq j \\ \text{free} & \text{if} \quad i = j \end{cases}$.
  - M has exactly one negative eigenvalue.
  - M satisfies SAH.
- Goldberg and Berman (2014) found $Z_{\pm}$ to bound $M(Q_{\pm})$.
- Butler et al. (2014) found Z_q to bound $M_q(G)$.
- So $\mu(G) \leq \min\{Z_{\pm}(G), Z_1(G)\}$, but can we do better?
- If such Z_μ exists, is GCC-Z_μ true or not?

Transferring from ν to μ



GCC-ν

- A *k-tree* is formed by starting from K_{k+1} and repeatedly adding one vertex joined to an existing k-clique.
- Sinkovic and van der Holst (2011) showed that if G is a k-tree, then $\nu(\overline{G}) \geq n - 2 - k$.
- So if G is a subgraph of a k-tree T_k and $\nu(G) \geq k$, then GCC-ν holds.

$$\nu(G) + \nu(\overline{G}) \geq k + n - 2 - k = n - 2,$$

since $\overline{T_k}$ is a subgraph of $\overline{G}$.

- Can we replace ν by μ?

GCC-ν

- Barioli et al. (2012) showed that if either
  - G and H each have an edge, or
  - G has an edge and $H = \overline{K_r}$ with $\nu(G) \leq |V(G)| - r$,

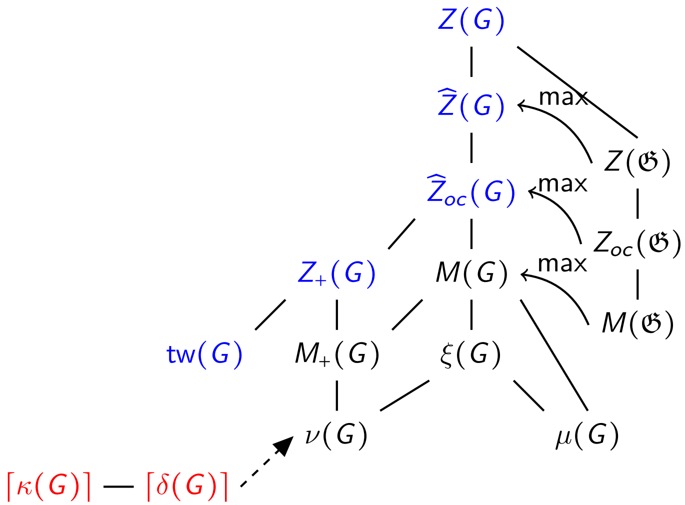
  then $\nu(G \vee H) = \min\{|V(G)| + \nu(H), \nu(G) + |V(H)|\}$;

  Otherwise,
  $\nu(G \vee H) = \min\{|V(G)| + \nu(H), \nu(G) + |V(H)|\} - 1$.
- Can we replace ν by μ?

Partial answers

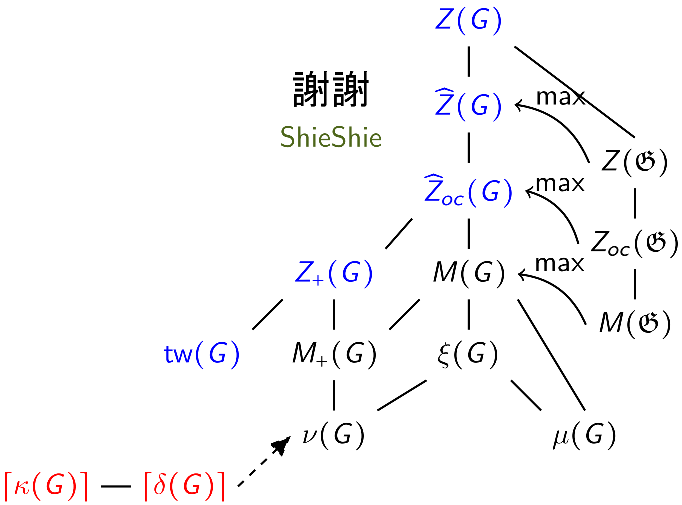
- $\mu(G) \leq \mu(G - v) + 1$.
- $\mu(G \vee H) \leq \min\{|V(G)| + \mu(H), \mu(G) + |V(H)|\}$.
- $\min\{|V(G)| + \mu(H), \mu(G) + |V(H)|\} - 1 \overset{?}{\leq} \mu(G \vee H)$.
- Up to $n \leq 7$, $\mu(G)$ can be determined.
  - $\mu(G) \leq 1$ iff G is a disjoint union of paths;
  - $\mu(G) \leq 2$ iff G is outerplanar;
  - $\mu(G) \leq 3$ iff G is planar;
  - $\mu(G) \leq 4$ iff G is linklessly embedable.
  - $\mu(G) \leq n - 1$, with the equality holds when G is $\overline{K_2}$ or K_n.
- The inequality holds for graphs with $n \leq 8$.

Keep going



$Z(G)$
$\widehat{Z}(G)$
max
$Z(\mathfrak{G})$
$\widehat{Z}_{oc}(G)$
max
$Z_{oc}(\mathfrak{G})$
$Z_{+}(G)$
$M(G)$
max
$M(\mathfrak{G})$
$\mathrm{tw}(G)$
$M_{+}(G)$
$\xi(G)$
$\nu(G)$
$\mu(G)$
$\lceil \kappa(G) \rceil$ — $\lceil \delta(G) \rceil$

Keep going



Z(G)
Ẑ(G)
max
謝謝
ShieShie
Z(𝔊)
Ẑ_oc(G)
max
Z_oc(𝔊)
Z_+(G)
M(G)
max
M(𝔊)
tw(G)
M_+(G)
ξ(G)
ν(G)
μ(G)
⌈κ(G)⌉ — ⌈δ(G)⌉

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