Zero forcing process and strong Arnold property

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Zero forcing

Zero forcing process:

- Start with a given set of blue vertices.
- If for some x, the closed neighbourhood $N_G[x]$ are all blue except for one vertex y and $y \neq x$, then y turns blue.

An initial blue set that can make the whole graph blue is called a zero forcing set. The zero forcing number $Z(G)$ of a graph G is the minimum size of a zero forcing set.

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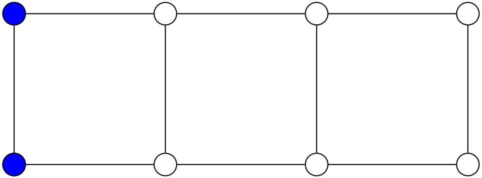

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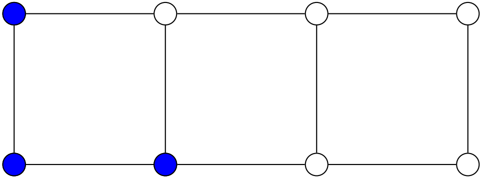

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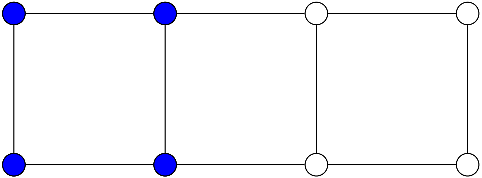

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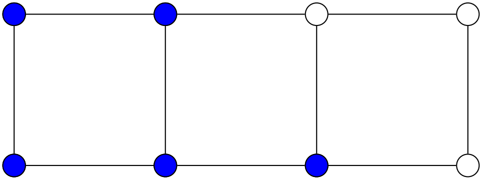

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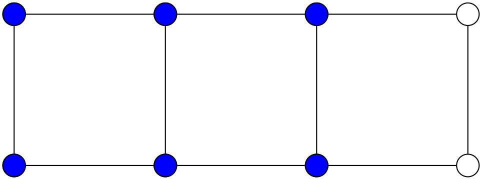

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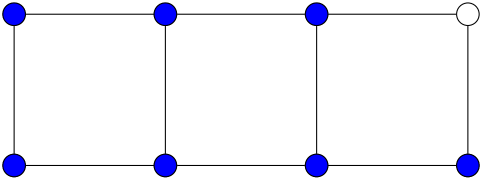

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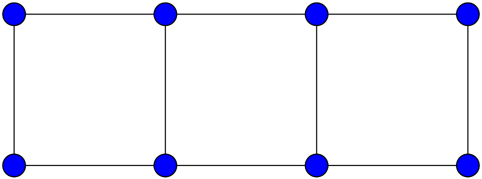
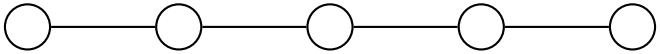


$Z(G) = 1$



$Z(G) = 1$ if and only if G is a path.

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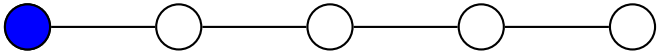

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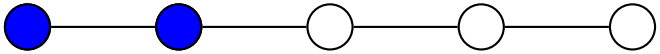

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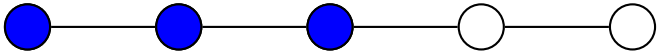

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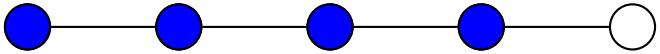

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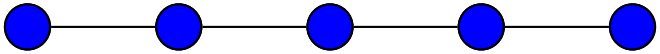


$Z(G) = 1$ if and only if G is a path.

$Z(G) = n$ or $n - 1$

$Z(G) = n \implies P_2$-free $\qquad Z(G) = n - 1 \implies P_3$-free

Let G be a graph on n vertices.

- Then $Z(G) = n$ if and only if G is the union of isolated vertices.
- And $Z(G) = n - 1$ if and only if G is $K_r \dot{\cup} \overline{K_{n-r}}$, $r \neq 1$.

Generalised adjacency matrix

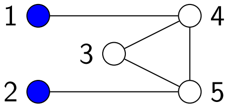
Let G be a simple graph on n vertices. The family $\mathcal{S}(G)$ consists of all $n \times n$ real symmetric matrix $M = \left[M_{i,j}\right]$ with

$$\begin{cases} M_{i,j} = 0 & \text{if } i \neq j \text{ and } \{i,j\} \text{ is not an edge,} \\ M_{i,j} \neq 0 & \text{if } i \neq j \text{ and } \{i,j\} \text{ is an edge,} \\ M_{i,j} \in \mathbb{R} & \text{if } i = j. \end{cases}$$

$$\mathcal{S}(\circ\!-\!\circ\!-\!\circ) \ni \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0.1 & 0 \\ 0.1 & 1 & \pi \\ 0 & \pi & 0 \end{bmatrix}, \cdots$$

Why zero forcing?

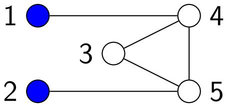
$$\begin{bmatrix} -2 & 0 & 0 & 7 & 0 \\ 0 & 1 & 0 & 0 & -9 \\ 0 & 0 & 0 & 3 & 4 \\ 7 & 0 & 3 & -4 & 5 \\ 0 & -9 & 4 & 5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



- Pick a matrix $A \in \mathcal{S}(G)$ and consider $A\mathbf{x} = \mathbf{0}$.
- Each vertex represents a variable. Each vertex also represents an equation where appearing variables are the neighbours and possibly itself.
- Blue means zero. White means unknown.

Hidden triangle in a system

$$
\begin{array}{lrrrrrl}
1. & -2x_1 & & & +7x_4 & & = 0 \\
2. & & 1x_2 & & & -9x_5 & = 0 \\
3. & & & & 3x_4 & +4x_5 & = 0 \\
4. & 7x_1 & & +3x_3 & -4x_4 & +5x_5 & = 0 \\
5. & & -9x_2 & +4x_3 & +5x_4 & & = 0
\end{array}
$$



Given $x_1 = x_2 = 0$,

$$
\begin{aligned}
1. &\implies x_4 = 0, \\
2. &\implies x_5 = 0, \\
4. &\implies x_3 = 0.
\end{aligned}
$$

Given 1 and 2 blue,

$$
\begin{aligned}
1 &\to 4, \\
2 &\to 5, \\
4 &\to 3.
\end{aligned}
$$

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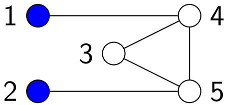

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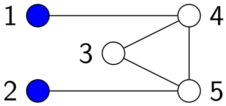

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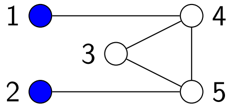
\end{array}$$



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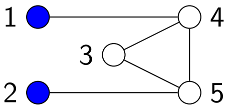

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As long as the red terms has nonzero coefficients and the orange terms are zero, the same argument always works.

Triangle number

- A pattern is a matrix whose entries are in $\{0, *, ?\}$.
- A triangle is a submatrix of a pattern that can be permuted to a lower triangular matrix with $*$ on the diagonal.

$$\begin{bmatrix} ? & 0 & 0 & * & 0 \\ 0 & ? & 0 & 0 & * \\ 0 & 0 & ? & * & * \\ * & 0 & * & ? & * \\ 0 & * & * & * & ? \end{bmatrix} \qquad \begin{bmatrix} 0 & * & 0 \\ 0 & 0 & * \\ * & ? & * \end{bmatrix} \to \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ ? & * & * \end{bmatrix} \qquad \text{triangle}$$

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$$\begin{bmatrix} ? & 0 & 0 & * & 0 \\ 0 & ? & 0 & 0 & * \\ 0 & 0 & ? & * & * \\ * & 0 & * & ? & * \\ 0 & * & * & * & ? \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & * \\ * & ? & * \\ 0 & * & ? \end{bmatrix} \to \begin{bmatrix} * & 0 & 0 \\ ? & * & 0 \\ * & ? & * \end{bmatrix} \qquad \text{triangle}$$

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- The triangle number $\mathrm{tri}(\mathcal{P})$ of a pattern $\mathcal{P}$ is the largest size of a triangle in $\mathcal{P}$.
- Define $\mathrm{tri}(G) = \mathrm{tri}(\mathcal{P})$, where $\mathcal{P}$ is the pattern of the generalized adjacency matrix of G.

Triangle number and zero forcing

Theorem

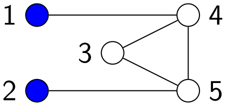
For any simple graph G on n vertices, $\text{tri}(G) = n - Z(G)$.

Proof.

Record all the forces in order. Find the rows of the "forc-ers", find the columns of the "forc-ees", then you find the triangle. □

$$\begin{bmatrix} ? & 0 & 0 & * & 0 \\ 0 & ? & 0 & 0 & * \\ 0 & 0 & ? & * & * \\ * & 0 & * & ? & * \\ 0 & * & * & * & ? \end{bmatrix}$$

$1 \to 4$
$2 \to 5$
$4 \to 3$



Proposition (Kenter and L 2018)

Let G be a graph on the vertex set V. The following are equivalent:

1. *B is a zero forcing set.*
2. *For any $A \in \mathcal{S}(G)$, the columns corresponding to $V \setminus B$ hides a lower triangular matrix.*
3. *For any $A \in \mathcal{S}(G)$, the columns corresponding to $V \setminus B$ are linearly independent.*

Theorem (AIM Work Group 2008)

Let G be a graph on n vertices. For any matrix $A \in \mathcal{S}(G)$, $n - Z(G) \leq \operatorname{rank}(A)$.

Corollary tridiagonal

Corollary

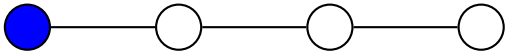
Any symmetric irreducible tridiagonal matrix has all its eigenvalues distinct.

$$\begin{bmatrix} ? & * & 0 & \cdots & 0 \\ * & ? & * & \ddots & \vdots \\ 0 & * & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & & * \\ 0 & \cdots & 0 & * & ? \end{bmatrix}$$

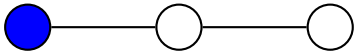
Proof.

For any $A \in \mathcal{S}(P_n)$, $\text{null}(A) \leq Z(P_n) = 1$ and $\text{null}(A - \lambda I) \leq Z(P_n) = 1$. □

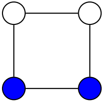
$Z(G) - 1 \leq Z(G - v) \leq Z(G) + 1$



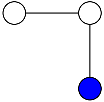
$Z(G) = 1$



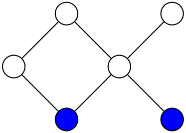
$Z(G - v) = 1$



$Z(G) = 2$



$Z(G - v) = 1$



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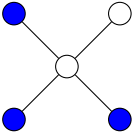

$Z(G - v) = 3$

tri(G) is induced subgraph monotone

- If H is an induced subgraph of G, then $\mathrm{tri}(H) \leq \mathrm{tri}(G)$.
- For each k, let $\mathbf{Forb}_{\mathrm{tri}(G)\leq k}$ be the set of minimal induced subgraph of $\{H : \mathrm{tri}(H) \geq k+1\}$.
- Then $\mathrm{tri}(G) \leq k$ if and only if G is $\mathbf{Forb}_{\mathrm{tri}(G)\leq k}$-free.

$$\mathbf{Forb}_{\mathrm{tri}(G)\leq 0} = \{P_2\}$$

$$\mathbf{Forb}_{\mathrm{tri}(G)\leq 1} = \{P_3, 2P_2\}$$

$\mathbf{Forb}_{\mathrm{tri}(G)\leq 2} = \{P_4,$  ,  $, P_2 \dot{\cup} P_3, 3P_2\}$

Is $|\mathbf{Forb}_{\text{tri}(G)\leq k}|$ always finite?

Proposition

Any graph with $\text{tri}(G) \geq k+1$ *contains an induced subgraph with* $\text{tri}(G) \geq k+1$ *and of order at most* $2k+2$.

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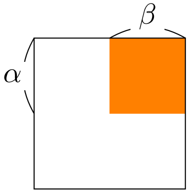

$$|\alpha|, |\beta| = k+1$$
$$|\alpha \cup \beta| \leq 2k+2$$

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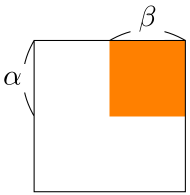

$$|\alpha|, |\beta| = k+1$$
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Corollary

Any graph in $\mathbf{Forb}_{\text{tri}(G)\leq k}$ *has order at most* $2k+2$*.*

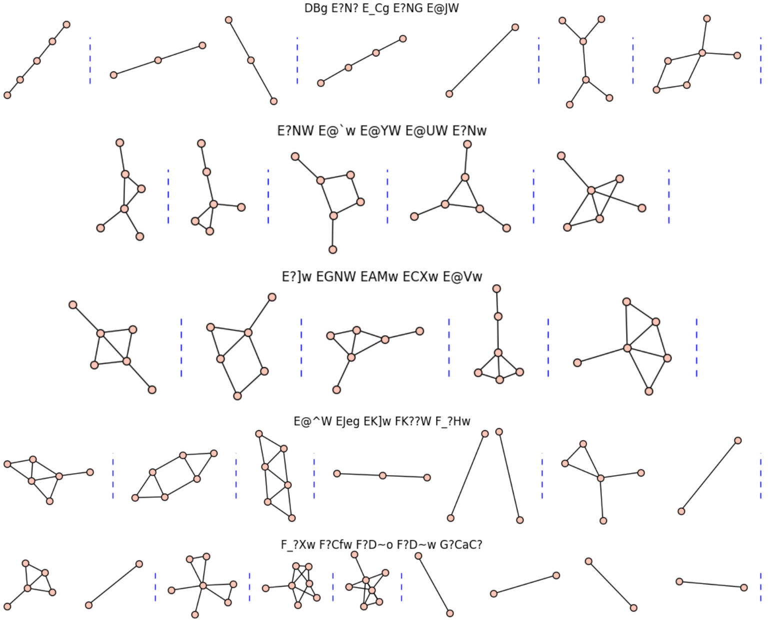
$$\mathbf{Forb}_{\mathrm{tri}(G)\leq 0} = \{P_2\}$$

$$\mathbf{Forb}_{\mathrm{tri}(G)\leq 1} = \{P_3, 2P_2\}$$

$\mathbf{Forb}_{\mathrm{tri}(G)\leq 2} = \{P_4,$ , , $P_2 \dot\cup P_3, 3P_2\}$

$$\mathbf{Forb}_{\mathrm{tri}(G)\leq 3} = \{19 \text{ connected}, 6 \text{ disconnected}\}$$

$$|\mathbf{Forb}_{\mathrm{tri}(G)\leq 4}| = 263, \ldots$$


DBg E?N? E_Cg E?NG E@JW
E?NW E@`w E@YW E@UW E?Nw
E?]w EGNW EAMw ECXw E@Vw
E@^W EJeg EK]w FK??W F_?Hw
F_?Xw F?Cfw F?D~o F?D~w G?CaC?

Triangle number on any pattern

The definition of the triangle number does not require the pattern to be symmetric or to be a square pattern.

$$\begin{bmatrix} 0 & * & ? & * & 0 & 0 & 0 & 0 & ? & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & ? & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ? & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * & ? & 0 & 0 & 0 & 0 & * & ? & * & 0 \end{bmatrix}$$

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Strong Arnold property

A matrix M is said to have the strong Arnold property (SAP) if $X = O$ is the only symmetric matrix that satisfies

- $X \circ M = X \circ I = O$,

  [That is, $(X)_{i,j} = 0$ when $i = j$ and when $(M)_{i,j} \neq 0$.]
- $MX = O$.

Here $\circ$ is the entry-wise product.

Matrices with SAP

$$\text{SAP: } X \circ M = X \circ I = O \text{ and } MX = O \implies X = O$$

- If $M \in \mathcal{S}(K_n)$, then M has the SAP.
- If M is nonsingular, then M has the SAP.
- The matrix $M \in \mathcal{S}(P_n)$ below has the SAP. [Will verify later.]

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -3 & 2 & 0 \\ 0 & 2 & -5 & 3 \\ 0 & 0 & 3 & -3 \end{bmatrix}$$

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Matrices without SAP

$$\text{SAP: } X \circ M = X \circ I = O \text{ and } MX = O \implies X = O$$

- The matrix M and X below show that M does not have the SAP.

$$M = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \end{bmatrix}$$

Colin de Verdiére parameter

In 1990, Colin de Verdière defined a parameter $\mu(G)$ as the maximum nullity over matrices M in $\mathcal{S}(G)$ with the following properties:

- M is a generalized Laplacian matrix of G; (off-diagonal entries ≤ 0)
- M has exactly one negative eigenvalue; (counting multiplicity)
- M has the SAP.

It was shown in the same paper:

- $\mu(H) \leq \mu(G)$ if H is a minor of G. (minor monotone)
- $\mu(G) \leq 1$ if and only if G is a disjoint union of paths.
- $\mu(G) \leq 2$ if and only if G is outer planar.
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Conjecture: $\mu(G) + 1 \geq \chi(G)$ for any graph.

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How to test the SAP?

$$M = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -3 & 2 & 0 \\ 0 & 2 & -5 & 3 \\ 0 & 0 & 3 & -3 \end{bmatrix} \qquad X = \begin{bmatrix} 0 & 0 & a & b \\ 0 & 0 & 0 & c \\ a & 0 & 0 & 0 \\ b & c & 0 & 0 \end{bmatrix}$$

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$$X = a \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

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$$X = aX_{1,3} + bX_{1,4} + cX_{2,4}$$

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SAP if and only if the linear system has only trivial solution.

$$a\begin{bmatrix}0&0&-1&0\\2&0&1&0\\-5&0&0&0\\3&0&0&0\end{bmatrix}+b\begin{bmatrix}0&0&0&-1\\0&0&0&1\\3&0&0&0\\-3&0&0&0\end{bmatrix}+c\begin{bmatrix}0&0&0&1\\0&0&0&-3\\0&3&0&2\\0&-3&0&0\end{bmatrix}=O$$

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SAP if and only if full row-rank.

$$\begin{bmatrix} 0 & 2 & -5 & 3 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & -3 & 0 & 0 & 0 & 0 & 1 & -3 & 2 & 0 \end{bmatrix}$$

SAP if and only if full row-rank.

This matrix adopts the pattern from M.

$$\begin{bmatrix} 0 & * & ? & * & 0 & 0 & 0 & 0 & ? & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & ? & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ? & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * & ? & 0 & 0 & 0 & 0 & * & ? & * & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 & -5 & 3 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & -3 & 0 & 0 & 0 & 0 & 1 & -3 & 2 & 0 \end{bmatrix}$$

SAP if and only if full row-rank.

$$\begin{bmatrix} 0 & * & ? & * & 0 & 0 & 0 & 0 & ? & \color{red}{*} & 0 & 0 & 0 & \color{red}{0} & \color{red}{0} & 0 \\ 0 & 0 & * & ? & 0 & 0 & 0 & 0 & 0 & \color{red}{0} & 0 & 0 & ? & \color{red}{*} & \color{red}{0} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * & ? & 0 & \color{red}{0} & 0 & 0 & * & \color{red}{?} & \color{red}{*} & 0 \end{bmatrix}$$

Always full row-rank regardless the choice of M! (That is, any matrix $M \in \mathcal{S}(P_4)$ has the SAP.)

Main idea

- Each graph G has a pattern $\mathcal{P}$.
- Use this pattern to compute the rectangular pattern $\mathcal{Q}$ for testing SAP.
- $\mathcal{Q}$ has $\overline{m}$ rows, which is the number of non-edges.
- If $\mathcal{Q}$ has a large triangle of order $\overline{m}$, than every matrix $A \in \mathcal{S}(G)$ has the SAP.

Will define $Z_{\mathrm{SAP}}(G)$ such that

$$Z_{\mathrm{SAP}}(G) = 0 \iff \mathcal{Q} \text{ has a triangle of order } \overline{m}.$$

Theorem (L '16)

If $Z_{\mathrm{SAP}}(G) = 0$, then every matrix $A \in \mathcal{S}(G)$ has the SAP.

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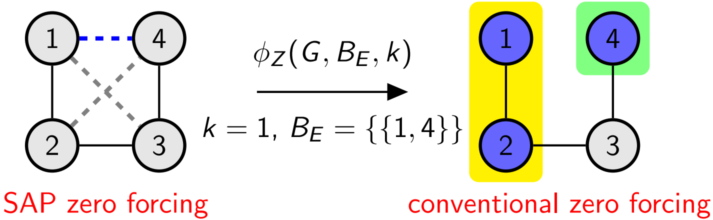
SAP zero forcing

- In an SAP zero forcing game, every non-edge has color either blue or white.
- If B_E is the set of blue non-edges, the local game on a given vertex k is a conventional zero forcing game on G, with blue vertices

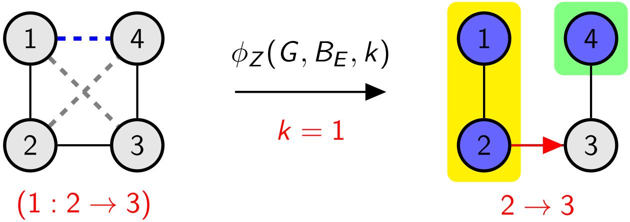
$$\phi_k(G, B_E) := N_G[k] \cup N_{\langle B_E \rangle}(k).$$

The local game is denoted by $\phi_Z(G, B_E, k)$.



SAP zero forcing

- Color change rule-Z_{SAP}:
  - Forcing triple $(k : i \to j)$: If $i \to j$ in $\phi_Z(G, B_E, k)$, then $\{j, k\}$ turns blue.
  - Odd cycle rule $(i \to C)$: Let G_W be the graph whose edges are the white non-edges. If $G_W[N_G(i)]$ contains a component that is an odd cycle C. Then $E(C)$ turns blue.
- $Z_{\mathrm{SAP}}(G)$ is the minimum number of blue non-edges such that all non-edges can turn blue eventually by CCR-Z_{SAP}.



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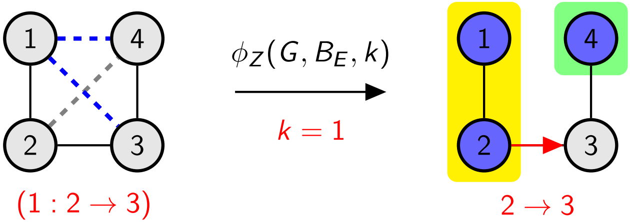

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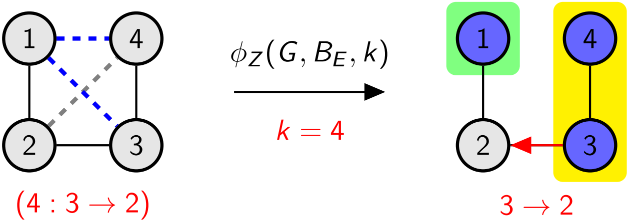

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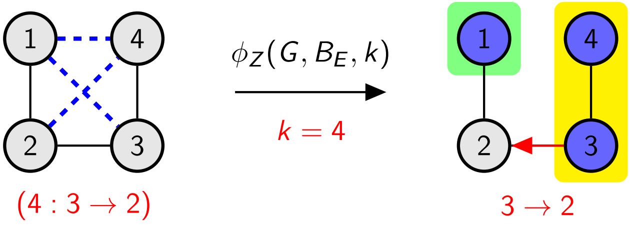
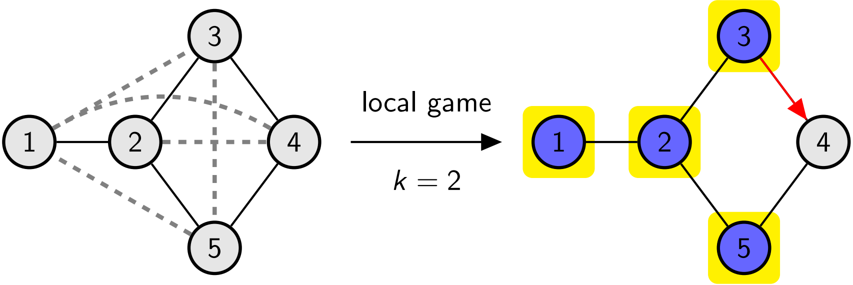


Example of $Z_{\mathrm{SAP}}(G) = 0$



Step	Forcing triple	Forced non-edge
1	$(2 : 3 \to 4)$	$\{2, 4\}$
2	$(4 : 2 \to 1)$	$\{4, 1\}$
3	$(5 : 4 \to 3)$	$\{5, 3\}$
4	$(3 : 2 \to 1)$	$\{3, 1\}$
5	$(5 : 2 \to 1)$	$\{5, 1\}$

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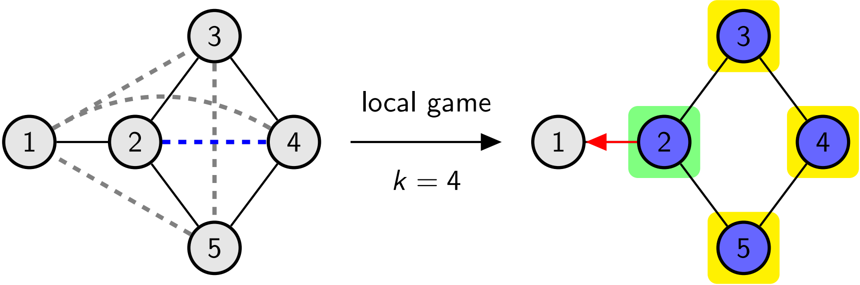

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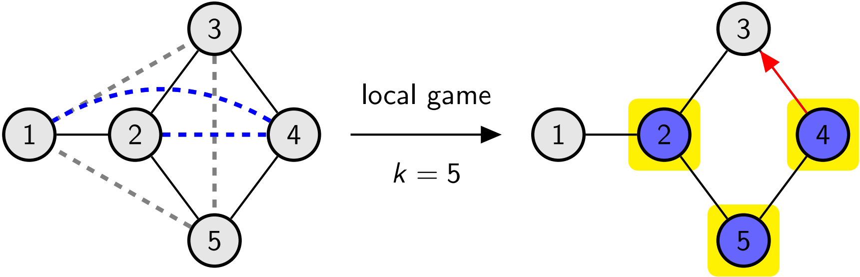

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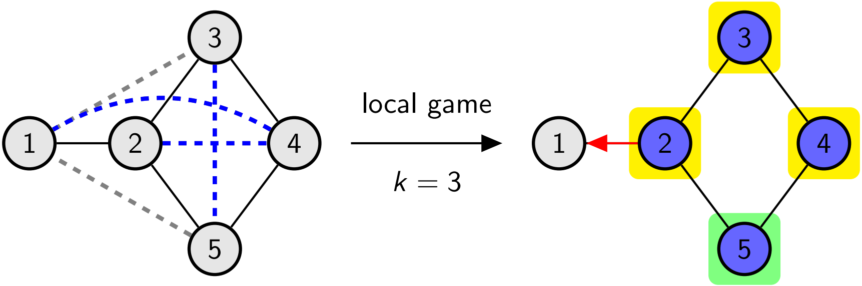

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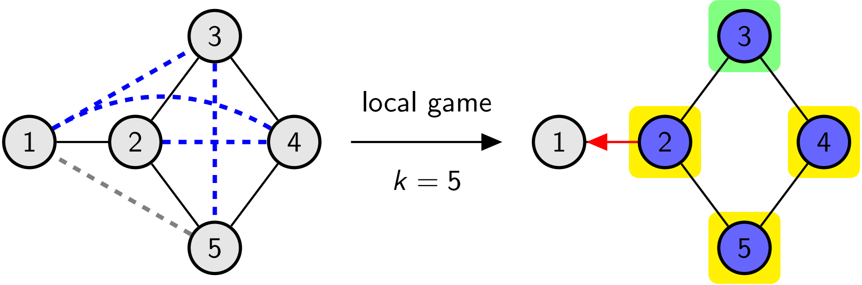

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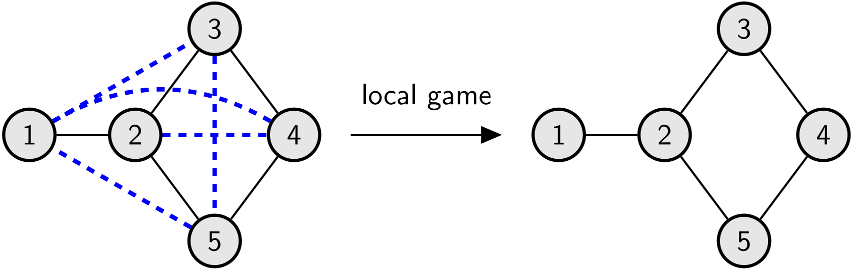

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5	$(5:2 \to 1)$	$\{5,1\}$

Theorem (L '16)

If $Z_{\mathrm{SAP}}(G) = 0$, then every matrix $A \in \mathcal{S}(G)$ has the SAP.

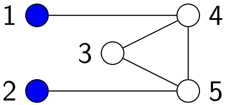
How many graphs have the property $Z_{\mathrm{SAP}}(G) = 0$?

The table shows for fixed n the proportion of graphs with $Z_{\mathrm{SAP}}(G) = 0$ in all connected graphs. (Isomorphic graphs count only once.)

n	$Z_{\mathrm{SAP}} = 0$
1	1.0
2	1.0
3	1.0
4	1.0
5	0.86
6	0.79
7	0.74
8	0.73
9	0.76
10	0.79

$$\begin{bmatrix} ? & 0 & 0 & * & 0 \\ 0 & ? & 0 & 0 & * \\ 0 & 0 & ? & * & * \\ * & 0 & * & ? & * \\ 0 & * & * & * & ? \end{bmatrix}$$

$1 \to 4$
$2 \to 5$
$4 \to 3$



$$\begin{bmatrix} 0 & * & ? & * & 0 & 0 & 0 & 0 & ? & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & ? & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ? & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * & ? & 0 & 0 & 0 & 0 & * & ? & * & 0 \end{bmatrix}$$

$$\begin{bmatrix} ? & 0 & 0 & * & 0 \\ 0 & ? & 0 & 0 & * \\ 0 & 0 & ? & * & * \\ * & 0 & * & ? & * \\ 0 & * & * & * & ? \end{bmatrix}$$

$1 \to 4$
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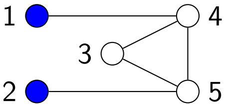

Thank you!

$$\begin{bmatrix} 0 & * & ? & * & 0 & 0 & 0 & 0 & ? & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & ? & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ? & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * & ? & 0 & 0 & 0 & 0 & * & ? & * & 0 \end{bmatrix}$$

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