Graphs whose distance matrices have the same determinant

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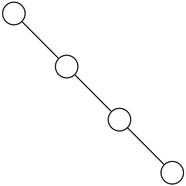
Distance matrix

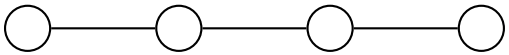
- Let G be a connected graph. The distance between two vertices i and j is the length of the shortest path connecting them, denoted as $\mathrm{dist}_G(i, j)$.
- The distance matrix of G is

$$\mathcal{D}(G) = \left[\mathrm{dist}_G(i, j)\right].$$

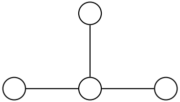
- For short, let $\det_{\mathcal{D}}(G) = \det(\mathcal{D}(G))$.



$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix}$$



$\det_{\mathcal{D}}(P_4) = -12$



$\det_{\mathcal{D}}(K_{1,3}) = -12$

Theorem (Graham and Pollak 1971)

For every tree T on n vertices,

$$\det_{\mathcal{D}}(T) = (-1)^{n-1}(n-1)2^{n-2}.$$

History

- Graham and Pollak 1971: $\det_{\mathcal{D}}(T)$ of a tree T only depends on n. [Yan and Yeh gave a simpler proof in 2006.]
- Graham, Hoffman, and Hosoya 1977: $\det_{\mathcal{D}}(G)$ only depends on its blocks, but not how blocks attached together.
- Bapat, Kirkland, and Neumann: weighted distance matrix of a tree.
- Bapat, Lal, and Pati; Yan and Yeh: q-analog and the q-exponential distance matrix of a tree.

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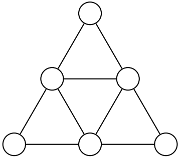
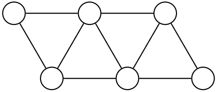
How about graphs without a cut vertex?

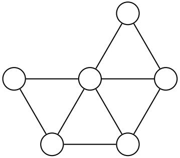
How about k-trees?



$\det_{\mathcal{D}}(G_1) = -8$

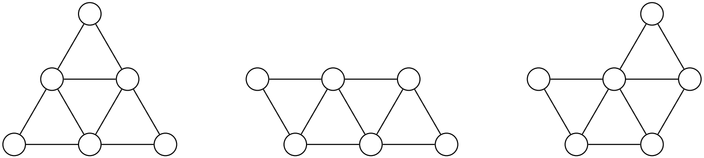


$\det_{\mathcal{D}}(G_2) = -9$



$\det_{\mathcal{D}}(G_3) = -9$

How about k-trees?



$\det_{\mathcal{D}}(G_1) = -8$ $\quad$ $\det_{\mathcal{D}}(G_2) = -9$ $\quad$ $\det_{\mathcal{D}}(G_3) = -9$

Linear 2-trees seems promising.

Linear 2-trees

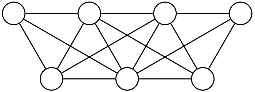
A linear k-tree is a graph obtained from K_{k+1} by adding a vertex each time and join it to the previously added vertex and $k-1$ of its neighbors.

Theorem (Cheng and L 2018+)

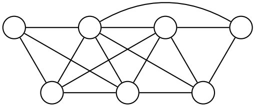
For every linear 2-tree G on n vertices,

$$\det_{\mathcal{D}}(G) = (-1)^{n-1}\left(1+\left\lfloor\frac{n-2}{2}\right\rfloor\right)\left(1+\left\lceil\frac{n-2}{2}\right\rceil\right).$$

How about linear k-tree?



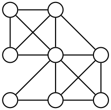
$\det_{\mathcal{D}}(G_1) = 4$



$\det_{\mathcal{D}}(G_2) = 6$

2-clique paths

Given $p_1, \ldots, p_m \geq 3$, a 2-clique path is obtained from a sequence of complete graphs $K_{p_1}, \ldots, K_{p_m}$ by gluing an edge of K_{p_i} to an edge of $K_{p_{i+1}}$, $i = 1, \ldots, m$; an edge cannot be glued twice. The family $\mathcal{CP}_{2:p_1,\ldots,p_m}$ collects all such graphs.



$G \in \mathcal{CP}_{2:3,4,3,4}$
$\det_{\mathcal{D}}(G) = (1+1+1)(1+2+2) = 15$

Theorem (Cheng and L 2018+)

For every graph $G \in \mathcal{CP}_{2:p_1,\ldots,p_m}$ on n vertices,

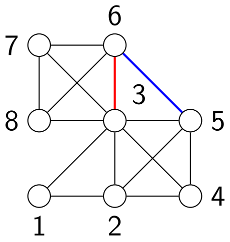
$$\det_{\mathcal{D}}(G) = (-1)^{n-1}\left(1 + \sum_{k \text{ odd}} (p_k - 2)\right)\left(1 + \sum_{k \text{ even}} (p_k - 2)\right).$$

Alternative way to construct a 2-clique path

- Decide the backward degree $q_1, \ldots, q_n$; e.g. $0, 1, 2, 2, 3, 2, 2, 3$
- Define $b_k = k - q_k + 1$ so that

$$[b_k, k-1] = \{b_k, \ldots, k-1\}$$

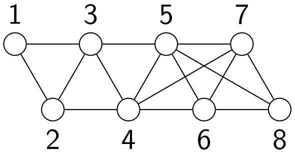
  are the previous $q_k - 1$ vertices before k.
- Start with K_2 on vertices 1 and 2. For $k = 3, \ldots, n$, add a new vertex k, then join it with the $q_k - 1$ vertices in $[b_k, k-1]$ and another neighbor a_k of $k - 1$.

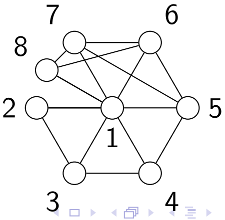


$k = 6, q_k = 2$
$b_k = 5, [b_k, k-1] = \{5\}$
a_k can be chosen from $\{2, 3, 4\}$

The CP graphs

- A sequence $0, 1, q_3, \ldots, q_n$ is called a non-leaping sequence if $2 \leq q_k \leq q_{k-1} + 1$ for $k \geq 3$. (So $q_3 = 2$ if $n \geq 3$.)
- The CP graphs $\mathcal{CP}_{q_1,\ldots,q_n}$ consists of any graphs constructed by the following way:
  - $b_k = k - q_k + 1$ so that $[b_k, k-1]$ has $q_k - 1$ elements.
  - Start with K_2 on vertices 1 and 2. For $k = 3, \ldots, n$, add a new vertex k, then join it with the $q_k - 1$ vertices in $[b_k, k-1]$ and another neighbor a_k of $k - 1$.
- Examples of $\mathcal{CP}_{0,1,2,2,2,2,3,3}$:

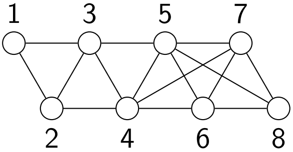




Reducing matrix

- The reducing matrix E of a CP graph is an $n \times n$ matrix whose k-th column is

$$\begin{cases} \mathbf{e}_k & \text{if } k \in \{1,2\}, \\ \mathbf{e}_k - \mathbf{e}_{a_k} - \mathbf{e}_{k-1} + \mathbf{e}_{a_{k-1}} & \text{if } k \geq 3. \end{cases}$$



$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Theorem (Cheng and L 2018+)

Let s be a non-leaping sequence. For any $G \in \mathcal{CP}_s$ with the distance matrix $\mathcal{D}$ and the reducing matrix E, the matrix

$$E^\top \mathcal{D} E$$

only depends on s.

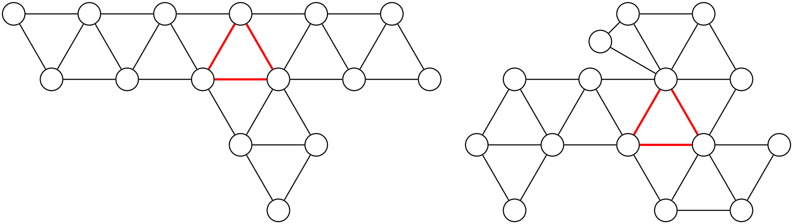
- Note that E is an upper triangular matrix with every diagonal entry equal to 1.

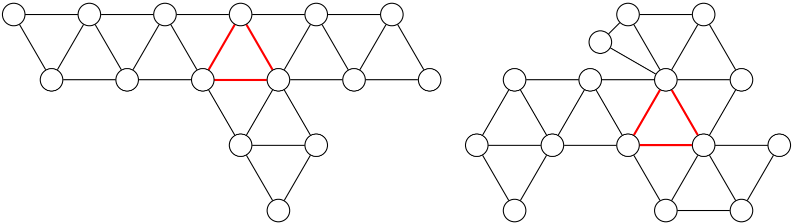
Corollary (Cheng and L 2018+)

Let s be a non-leaping sequence. Then

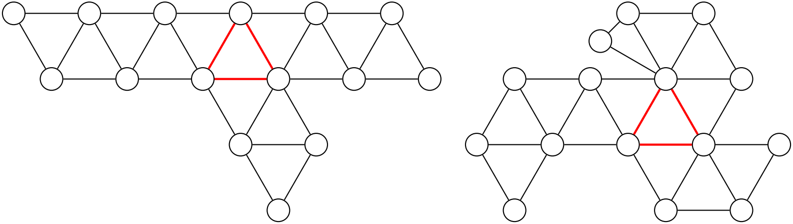
$$\det_{\mathcal{D}}(G) \text{ and } \mathrm{inertia}_{\mathcal{D}}(G)$$

are independent of the choice of $G \in \mathcal{CP}_s$.





$$\det_{\mathcal{D}}(G_1) = \det_{\mathcal{D}}(G_2) = 56$$



$$\det_{\mathcal{D}}(G_1) = \det_{\mathcal{D}}(G_2) = 56$$

Thank you!

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