Applications of zero forcing number to the minimum rank problem

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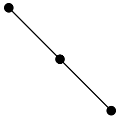
Abstract

- Introduction
- Exhaustive zero forcing number
- Sieving process

$\mathcal{G}$:real symmetric matrices $\rightarrow$ graphs.

$$\begin{pmatrix} -3 & 3 & 0 \\ 3 & -5 & 2 \\ 0 & 2 & -2 \end{pmatrix} \xrightarrow{\mathcal{G}}$$



Relation between Matrices and Graphs

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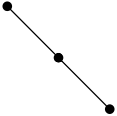

$$\mathcal{S}(G) = \{A \in M_{n\times n}(\mathbb{R}) : A = A^t, \mathcal{G}(A) = G\}.$$

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- The minimum rank problem of a graph G is to determine the number $\mathrm{mr}(G)$ or $M(G)$.

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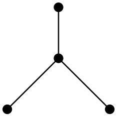
- The zero forcing number $Z(G)$ of a graph G is the minimum size of a zero forcing set.
- For all graph G, $M(G) \leq Z(G)$.[1]

Example for These Parameters

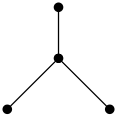
$$\begin{pmatrix} ? & * & * & * \\ * & ? & 0 & 0 \\ * & 0 & ? & 0 \\ * & 0 & 0 & ? \end{pmatrix} \xrightarrow{\mathcal{G}}$$



- rank ≥ 2.

Example for These Parameters

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\mathcal{G}}$$



- $\text{rank} \geq 2$.
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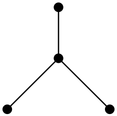

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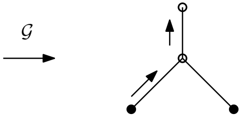

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- $Z(G) = 2$.

General Strategy to Determine $M(G)$

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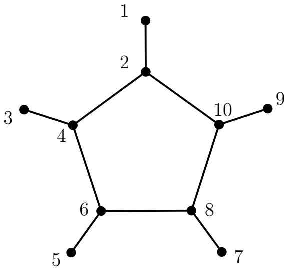

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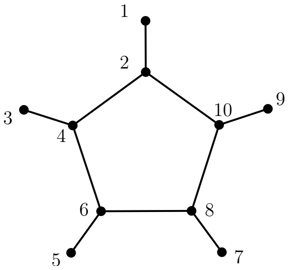

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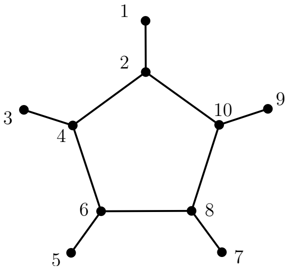

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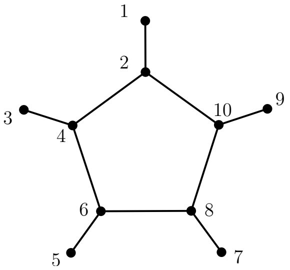

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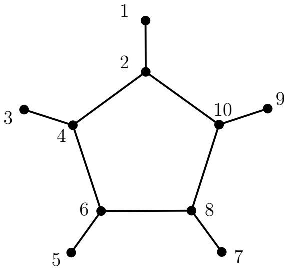

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$$\mathrm{mr}(Q) = \min\{\mathrm{rank} A \colon A \cong Q\}.$$

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- The rank 2 is achievable. Hence $\mathrm{mr}(Q) = 2$.

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- Zero forcing number banned by B with support W $Z_W(G,B)$: minimum size of $F \supseteq W$.
- When W and B is empty, $Z_W(G,B) = Z(G)$.

- Q is a given $m \times n$ pattern. $G = (X \cup Y, E)$ is the related bipartite defined by

$$X = \{a_1, a_2, \ldots, a_m\}, \ \ Y = \{b_1, b_2, \ldots, b_n\}, \ \ E = \{a_i b_j : Q_{ij} \neq 0\}.$$

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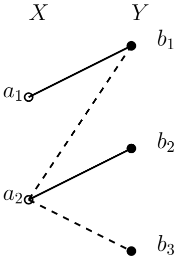

Natural Relation between Patterns and Bipartites

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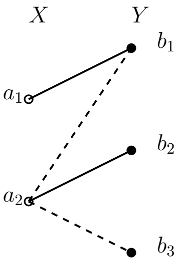

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For a given $m \times n$ pattern matrix Q, If $G = (X \cup Y, E)$ is the graph and B is the set of banned edges defined above, then

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- For $n \times n$ square pattern Q, it becomes

$$n - \mathrm{mr}(Q) \leq Z_Y(G, B) - n$$

The Exhaustive Zero Forcing Number

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- The inequality become

$$M(G) \le n - \mathrm{mr}(Q) = \max_{I \subseteq [n]} [n - \mathrm{mr}(Q_I)] \le \max_{I \subseteq [n]} Z_Y(\widetilde{G}_I) - n.$$

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- It can be proven that $M(G) \leq \widetilde{Z}(G) \leq Z(G)$.

Example of Exhaustive Zero Forcing Number

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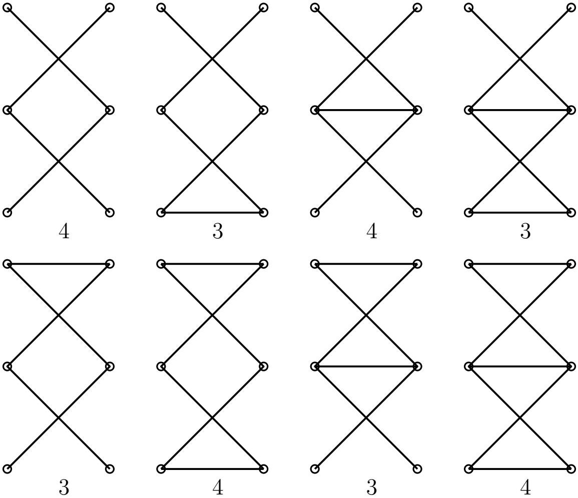
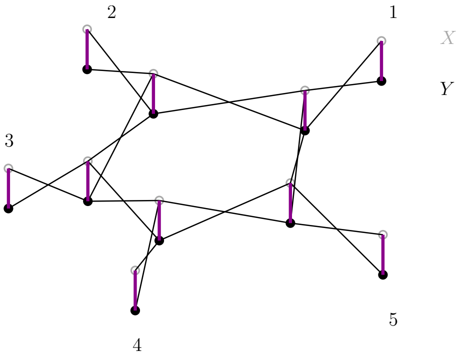
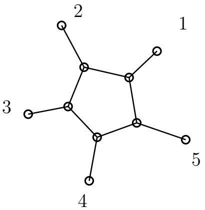
- $1 = M(P_3) \leq \widetilde{Z}(P_3) \leq Z(P_3) = 1$. Hence $\widetilde{Z}(G) = 1$.

Bipartites related to P_3



- If $Z(\widetilde{H_5}_I) - 10 = 3$ for some I, then $1 \in I$ and $2 \notin I$, a contradiction.
- $$\widetilde{Z}(G) = 12 - 10 = 2.$$





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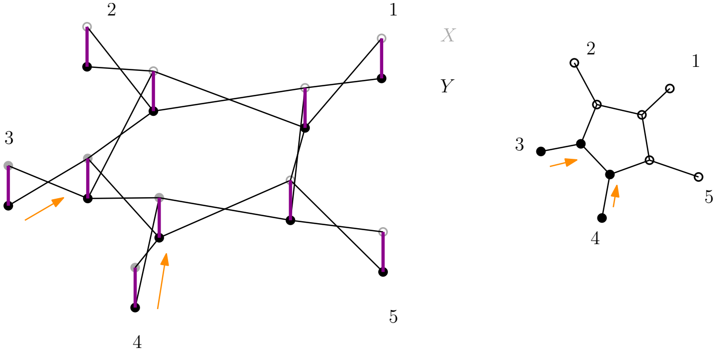

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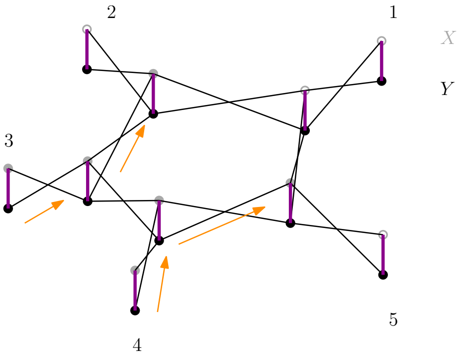

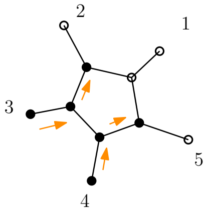

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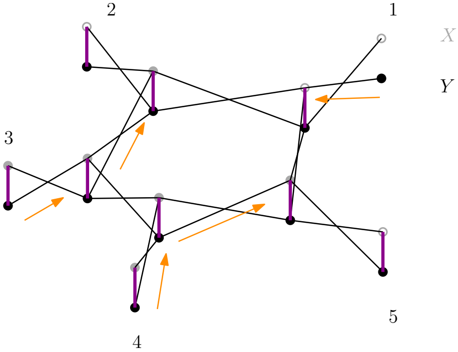

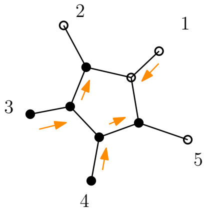

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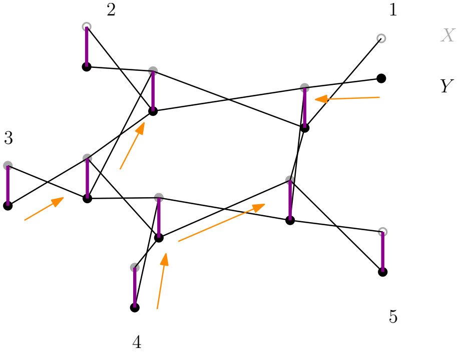

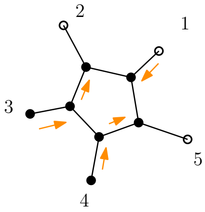

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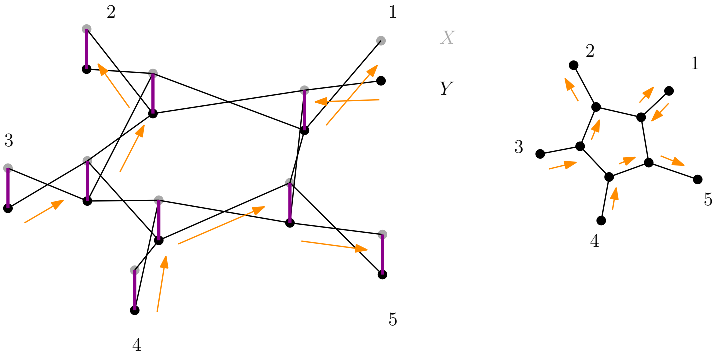

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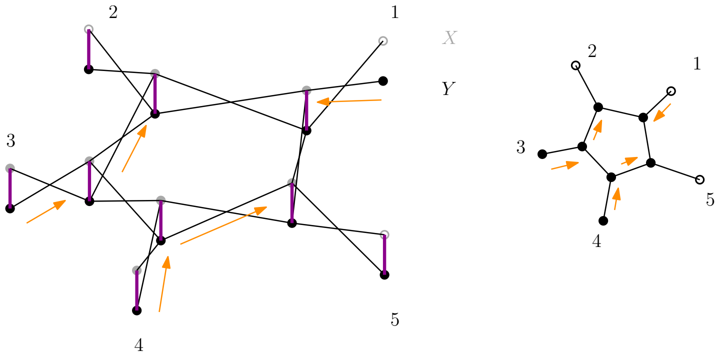

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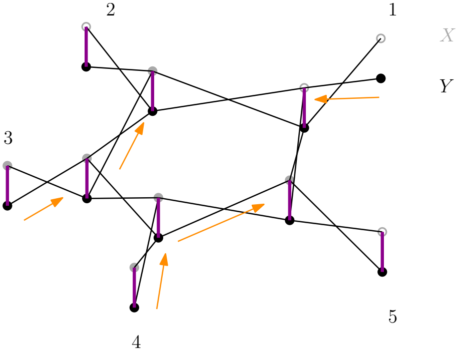
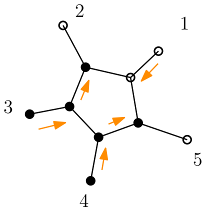
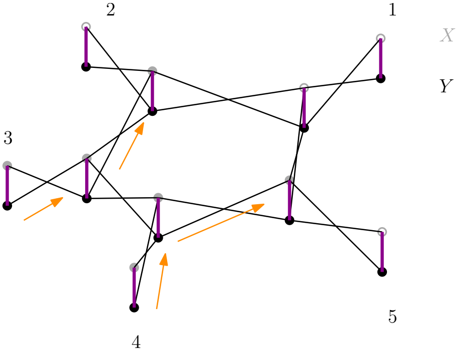
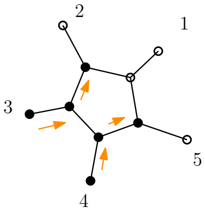




- If $Z(\widetilde{H_{5}}_I) - 10 = 3$ for some I, then $1 \in I$ and $2 \notin I$, a contradiction.
-
$$\widetilde{Z}(G) = 12 - 10 = 2.$$





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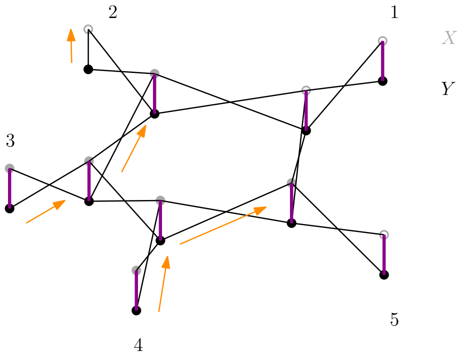

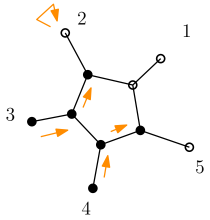

Example for Sieving Process

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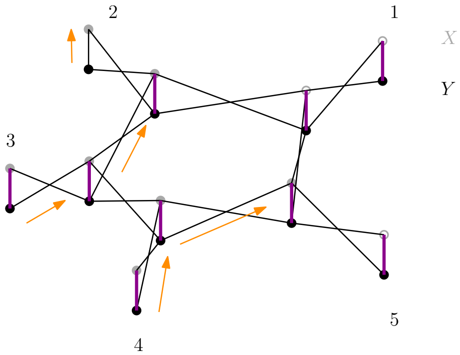

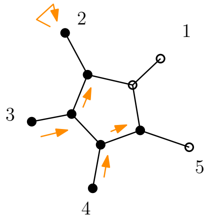

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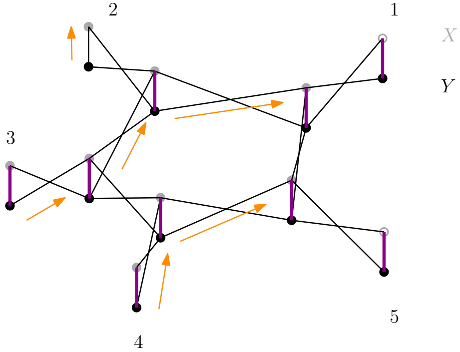

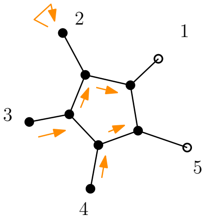

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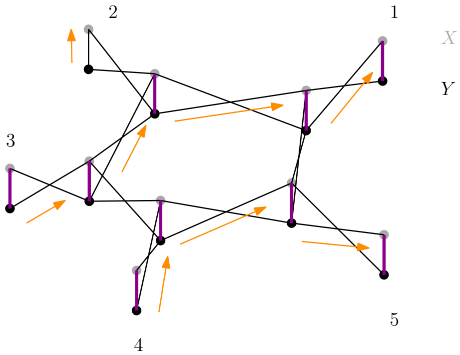

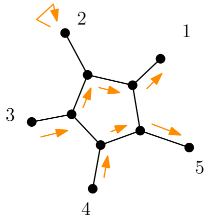

Sieving Process

- Rewrite

$$\widetilde{Z}(G) = \max_{I \subseteq [n]} Z_Y(\widetilde{G}_I) - n = \max\{k \colon k = Z_Y(\widetilde{G}_I) - n \text{ for some } I\}.$$

Sieving Process

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$$\widetilde{Z}(G) = \max_{I \subseteq [n]} Z_Y(\widetilde{G}_I) - n = \max\{k : k = Z_Y(\widetilde{G}_I) - n \text{ for some } I\}.$$

- Let $\mathcal{I}_k(G) = \{I \subseteq [n] : Z_Y(\widetilde{G}_I) - n \geq k\}$.

Sieving Process

- Rewrite

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- Let $\mathcal{I}_k(G) = \{I \subseteq [n] : Z_Y(\widetilde{G}_I) - n \geq k\}$.
- $\widetilde{Z}(G) = \max\{k : \mathcal{I}_k \neq \varnothing\}$.

Sieving Process

- Rewrite

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- $\widetilde{Z}(G) = \max\{k : \mathcal{I}_k \neq \varnothing\}$.
- Each $F \supseteq Y$ with size $n + k - 1$ is a sieve for $\mathcal{I}_k(G)$ to delete impossible index sets.

Nonzero-vertex and Zero-vertex

- If $i \in I$ for all $I \in \mathcal{I}_k(G)$, then i is called a nonzero-vertex.

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- Nonzero:

$$\begin{pmatrix} 1 & a^t & 0 \\ a & \widehat{A}_{11} & \widehat{A}_{12} \\ 0 & \widehat{A}_{21} & \widehat{A}_{22} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & \widehat{B}_{11} & \widehat{A}_{12} \\ 0 & \widehat{A}_{21} & \widehat{A}_{22} \end{pmatrix}.$$

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- Zero:

$$\begin{pmatrix} \alpha & a^t & O \\ a & \widehat{A}_{11} & \widehat{A}_{12} \\ O & \widehat{A}_{21} & \widehat{A}_{22} \end{pmatrix} \to \begin{pmatrix} \alpha & O & O \\ O & \widehat{B}_{11} & \widehat{A}_{12} \\ O & \widehat{A}_{21} & \widehat{A}_{22} \end{pmatrix}.$$

Here α has the form $\begin{pmatrix} 0 & * \\ * & u \end{pmatrix}$ and α^{-1} has the form $\begin{pmatrix} u & * \\ * & 0 \end{pmatrix}$.

Thank my advisor.

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I will be in army in 8/8.

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Thank you.

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