Distance Spectra of Graphs

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Rocky Mountain-Great Plains Graduate Research Workshop in Combinatorics

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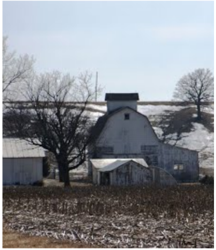

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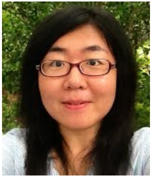

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Distance matrix

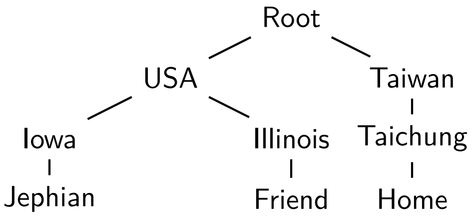
- Let G be a connected simple graph on vertex set $V = \{1, \ldots, n\}$.
- The distance $d_G(i,j)$ between two vertices i, j on G is the length of the shortest path.
- The distance matrix of G is an $n \times n$ matrix

$$\mathcal{D} = \left[d_G(i,j)\right].$$

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 2 & 3 \\ 2 & 1 & 0 & 1 & 2 \\ 3 & 2 & 1 & 0 & 1 \\ 4 & 3 & 2 & 1 & 0 \end{bmatrix}$$

Motivation: Pierce's loop switching scheme

- How two build a phone call between two persons?
  - Root>USA>Iowa>Jephian
  - Root>USA>Illinois>Friend
  - Root>Taiwan>Taichung>Home



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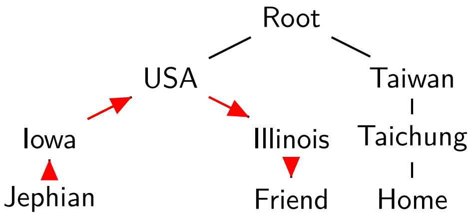

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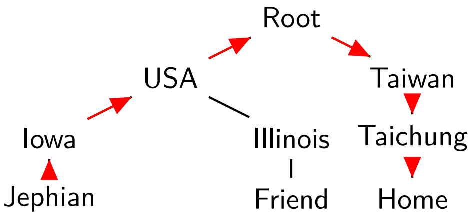
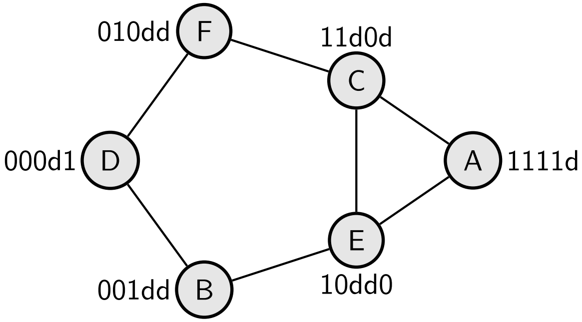


Graham and Pollak's model

- A model works for all graphs, not limited to trees.
- Each vertex is assigned with an address, and the distance between two vertices is the Hamming distance of the address.
- Find the neighbor that decrease the Hamming distance.



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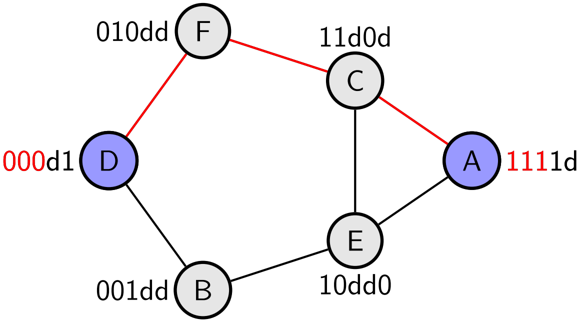

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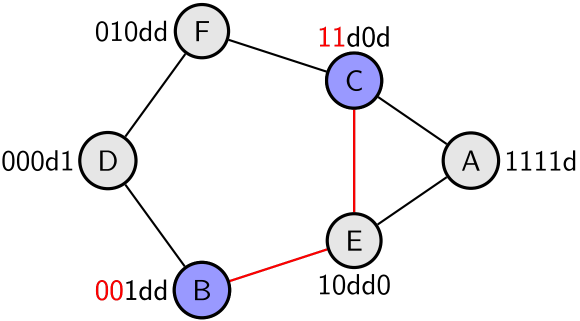

Length of the address

Theorem (Graham and Pollak 1971)

Let G be a graph and $\mathcal{D}$ its distance matrix. Then the length of the address is at least

$$\max\{n_-, n_+\},$$

where n_-, n_+ are the negative and positive inertia.

Corollary (Graham and Pollak 1971)

When G is a complete graph or a tree, then the minimum length of the address is $|V(G)| - 1$.

Number of distinct eigenvalues

- Suppose A is a matrix. Let $q(A)$ be the number of distinct eigenvalues.
- If A is the adjacency matrix of graph G, then

$$q(A) \geq \mathrm{diam}(G) + 1.$$

- Key: When a matrix M is diagonalizable, then

$$q(M) = \text{degree of min polynomial.}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, A^2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, A^3 = \begin{bmatrix} 0 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 0 & 2 \\ 1 & 0 & 2 & 0 \end{bmatrix}.$$

How about distance matrices?

- Distance matrices are dense (all off-diagonal entries are non-zero).
- Let Q_d be the d-dimensional hypercube. Then $q(\mathcal{D}(Q_d)) = 3$ and $\text{diam}(Q_d) = d$ for $d \geq 2$.
- What is the relation between $q(\mathcal{D}(G))$ and $\text{diam}(G)$?

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Theorem (Aalipour et al 2016)

Let T be a tree and $\mathcal{D}$ its distance matrix. Then

$$q(\mathcal{D}) \geq \left\lceil \frac{\text{diam}(T)}{2} \right\rceil.$$

Proof.

- Let $L(T)$ be the line graph of T and A the adjacency matrix of $L(T)$.
- $\text{spec}(-2(2I + A)^{-1})$ interlaces $\text{spec}(\mathcal{D})$. [Merris 1990]
- $q(A) \geq \text{diam}(L(T)) + 1 = \text{diam}(T)$.
- $q(\mathcal{D}) \geq \left\lceil \frac{q(A)}{2} \right\rceil$.

□

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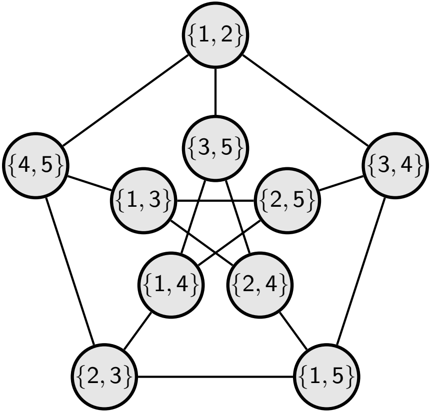
Kneser graph

The Kneser graph $K(n,r)$ has vertices $V = \binom{[n]}{r}$ and two vertices S, T are adjacent if $|S \cap T| = 0$. The Petersen graph is $K(5,2)$.



generalized Johnson graph

- The generalized Johnson graph $J(n;r;i)$ has vertices $V = \binom{[n]}{r}$ and two vertices S, T are adjacent if $|S \cap T| = r - i$. The Kneser graph $K(n,r)$ is $J(n;r;r)$.
- For two vertices S, T of $K(n,r)$, the distance only depends on $r - i = |S \cap T|$ and is known as

$$d(S,T) = \min\left\{2\left\lceil\frac{i}{n-2r}\right\rceil, 2\left\lceil\frac{r-1}{n-2r}\right\rceil + 1\right\} =: f(i).$$

[Valencia-Pabon and Vera 2005]

Decomposition

- Let $A_0 = I$. Let A_i be the adjacency matrix of $J(n; r; i)$. The distance matrix of $K(n, r)$ is

$$\mathcal{D} = \sum_{i=0}^{r} f(i) A_i.$$

- Not hard to show $\{A_i\}_{i=0}^{r}$ is a commuting family, which means they are simultaneousely diagonalizable.
- What is the eigenvalues of each of A_i and how do they correlated?

Johnson Scheme

- $\{A_i\}_{i=0}^r$ is called Johnson scheme a commuting family: [Bannai and Ito 1984]
  - $\mathbb{R}^{\binom{n}{r}} = \oplus_{j=0}^r V_j$.
  - For each A_i,

$$p_i(j) = \sum_{t=0}^{j} (-1)^t \binom{j}{t} \binom{r-j}{i-t} \binom{n-r-j}{i-t}$$

is the eigenvalue corresponding to V_j, with the multiplicity

$$m_j = \frac{n-2j+1}{n-j+1} \binom{n}{j}.$$

- The function $p_i(j)$ is known as Eberlein polynomial.
- The distance spectrum of $K(n,r)$ is

$$\theta_j = \sum_{i=0}^{r} f(i) p_i(j) \text{ with multiplicity } m_j.$$

Side stories...

By equitable partition, A_i and B_i has the same eigenvalue. Let $Q = \left[\binom{i}{j}\right]$. Then the j, ℓ-entry of $Q^{-1}B_iQ$ is

$$f(n,r,i,j,\ell) = \sum_{a,b,x} (-1)^{j+a} \binom{j}{a} \binom{r-a}{x} \binom{a}{r-i-x} \binom{a}{r-b-x} \binom{n-r-a}{b-r+i+x} \binom{b}{\ell}.$$

All computation indicates $Q^{-1}B_iQ$ is a upper-triangular matrix.

$$f(n,r,i,j,\ell) = 0 \text{ when } j > \ell?$$

$$f(n,r,i,j,j) = p_i(j)?$$

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Thank you!

References I

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