Zero forcing and eigenvalue multiplicities

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Applications of zero forcing number to the minimum rank problem

Advisor: Professor Gerard Jennhwa Chang, Ph.D.
Student: Chin-Hung Lin

Department of Mathematics, National Taiwan University

8/6 2011 in Tamkang University

Distance Spectra of Graphs

林晉宏 Jephian C.-H. Lin
Joint work supported by GRWC

Department of Mathematics, Iowa State University

Aug 4, 2016
2016 組合數學新苗研討會

System of linear equations

$$\begin{aligned} 2x &+3y &-z &= 4 \\ x &-y &+2z &= 3 \\ -3x &+2y &+z &= 2 \end{aligned}$$

Hard to know if the solution exists, or if the solution is unique.

I don't want to solve it!

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$$\begin{aligned} 2x \quad +y \quad -z &= 1 \\ 2y \qquad &= 4 \\ +2y \quad +3z &= 7 \end{aligned}$$

Easy to see $y = 2$, then $z = 1$, and then $x = 0$.
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Main philosophy

$$2x + y + 3z = 7$$

$$x = 1, y = 2 \implies z = 1$$

In a linear equation, if all but one variable are known, then this remaining variable is also known.

$$2x + y + 3z = 0$$

$$x = 0, y = 0 \implies z = 0$$

In a homogeneous linear equation, if all but one variable are zero, then this remaining variable is also zero.

Hidden triangle in a system

$$\begin{array}{llllllll}
1. & x & & +z & & +u & =0 \\
2. & & y & +z & & & =0 \\
3. & x & +y & +z & +w & +u & =0 \\
4. & & & z & +w & & =0 \\
5. & x & & +z & & +u & =0
\end{array}$$

Given information: $x = y = 0$. Then

$$2. \implies z = 0,$$

$$1. \implies u = 0,$$

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$$\begin{array}{lcccccl} 2. & & y & +z & & & = 0 \\ 1. & x & & +z & & +u & = 0 \\ 3. & x & +y & +z & +w & +u & = 0 \\ 1. & x & & +z & & +u & = 0 \\ 5. & x & & +z & & +u & = 0 \end{array}$$

Given information: $x = y = 0$. Then

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$$2. \implies z = 0,$$
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As long as the red terms has nonzero coefficients and the orange terms are zero, the same argument always works.

Application to algebra

Find the inverse of a formal power series.

$$\begin{array}{rcccccc} & 1 & +2x & +3x^2 & +4x^3 & +5x^4 & +\cdots \\ \times) & b_0 & +b_1x & +b_2x^2 & +b_3x^3 & +b_4x^4 & +\cdots \\ \hline & 1 & & & & & \end{array}$$

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$$\begin{array}{lllll} 1b_0 & & & & = 1 \\ 2b_0 & +1b_1 & & & = 0 \\ 3b_0 & +2b_1 & +1b_2 & & = 0 \\ 4b_0 & +3b_1 & +2b_2 & +1b_3 & = 0 \\ & & \vdots & & \end{array}$$

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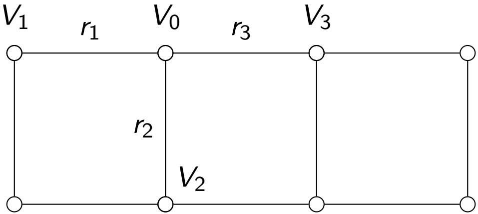
A formal power series has an inverse if and only if the constant term is nonzero.

Electronic circuit

$$\frac{1}{r_1}(V_1 - V_0) + \frac{1}{r_2}(V_2 - V_0) + \frac{1}{r_3}(V_3 - V_0) + \epsilon V_0 = 0$$



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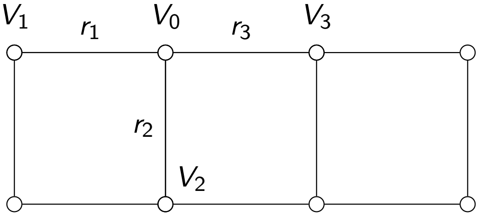
$$\frac{1}{r_1}V_1 + \frac{1}{r_2}V_2 + \frac{1}{r_3}V_3 + (\epsilon - \frac{1}{r_1} - \frac{1}{r_2} - \frac{1}{r_3})V_0 = 0$$



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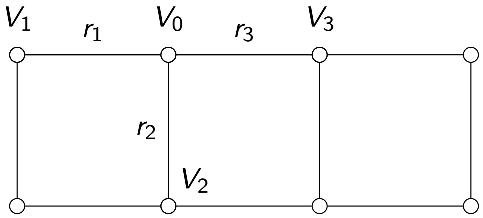
$$a_1 V_1 + a_2 V_2 + a_3 V_3 + a_0 V_0 = 0$$

nonzero   zero or nonzero



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$$a_1 V_1 + a_2 V_2 + a_3 V_3 + a_0 V_0 = 0$$

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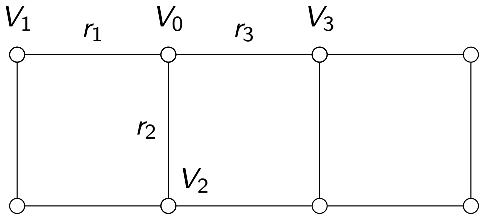
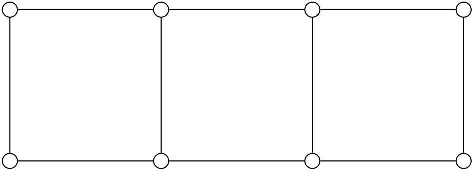
The conservation law leads to a linear equation on each node; itself and its neighbors represent the variables.



How many sensors required to monitor the voltages?



- Each vertex represents a linear equation; variables are itself and its neighbors (closed neighborhood).
- If in a closed neighborhood, all but one voltages are known, then this remaining one are also known.

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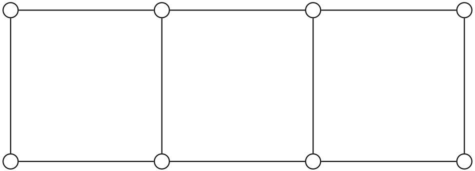

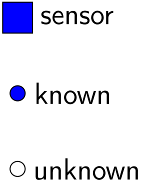

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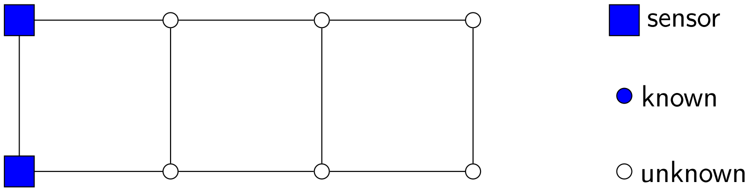

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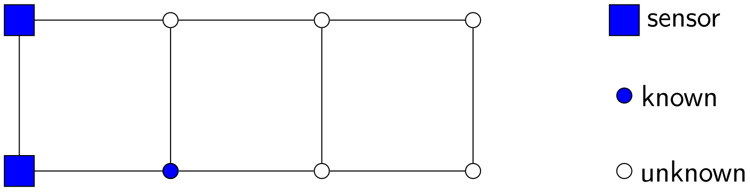

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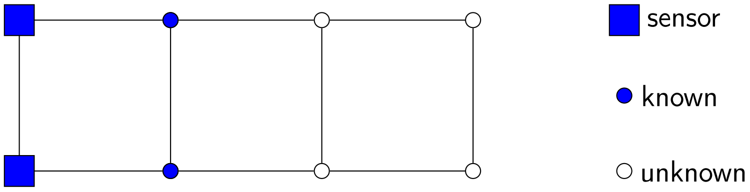

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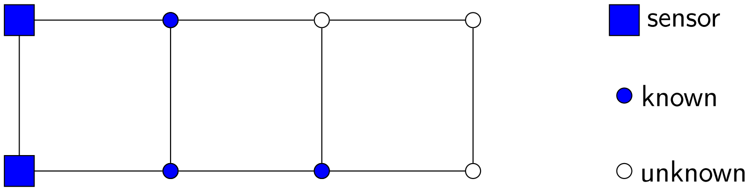

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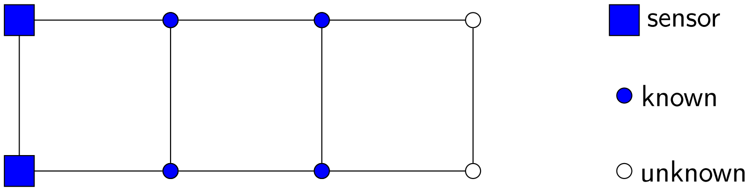

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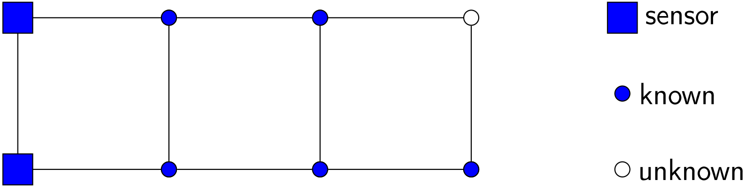

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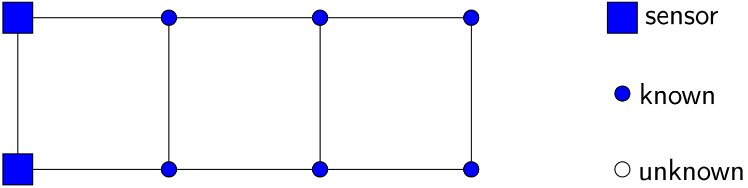

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Model by graphs and matrices

- A electronic circuit can be represented by a graph; each vertex represents a node, and each edge represents a connection.
- The linear equations can be recorded into a matrix; each row represents a equation, and each column represents an unknown voltage.
- This is a symmetric matrix where rows and columns are both indexed by the vertices.

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Let G be a simple graph on n vertices. The family $\mathcal{S}(G)$ consists of all $n \times n$ real symmetric matrix $M = \begin{bmatrix} M_{i,j} \end{bmatrix}$ with

$$\begin{cases} M_{i,j} = 0 & \text{if } i \neq j \text{ and } \{i,j\} \text{ is not an edge,} \\ M_{i,j} \neq 0 & \text{if } i \neq j \text{ and } \{i,j\} \text{ is an edge,} \\ M_{i,j} \in \mathbb{R} & \text{if } i = j. \end{cases}$$

$$\mathcal{S}(\circ\!\!-\!\!\circ\!\!-\!\!\circ) \ni \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0.1 & 0 \\ 0.1 & 1 & \pi \\ 0 & \pi & 0 \end{bmatrix}, \cdots$$

Zero forcing

Zero forcing process:

- Start with a given set of blue vertices (sensors).
- If for some x, the closed neighborhood $N_G[x]$ are all blue except for one vertex y and $y \neq x$, then y turns blue.

An initial blue set that can make the whole graph blue is called a zero forcing set. The zero forcing number $Z(G)$ of a graph G is the minimum size of a zero forcing set.

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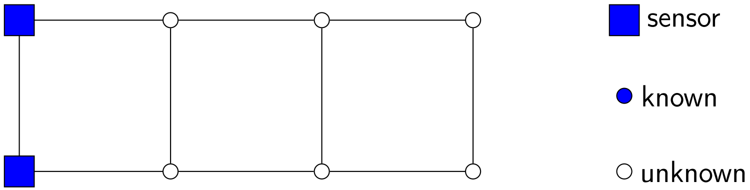

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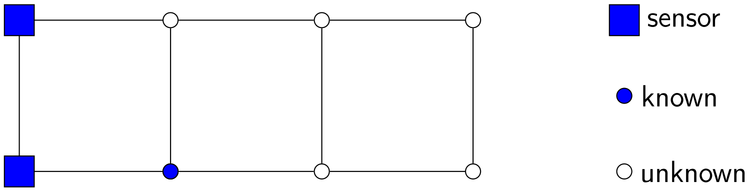

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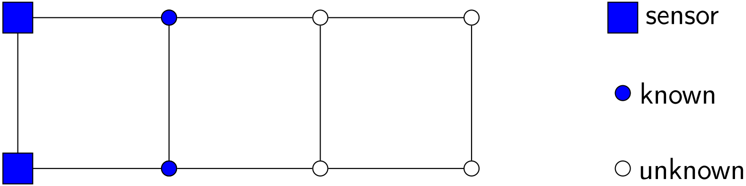

How to deploy the sensors?

Any zero forcing set is a good deployment of sensors that can monitor the whole graph.

The zero forcing number is the minimum number of sensors required.

Zero forcing sets suggest a good deployment before knowing the details of the network.

Many studies are done on zero forcing and its variation power domination.

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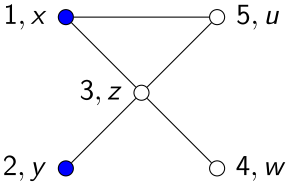
Hidden triangle revisit



$$
\begin{array}{lccccccc}
1. & x & & +z & & +{\color{red}u} & = 0 \\
2. & & y & +{\color{red}z} & & & = 0 \\
3. & x & +y & +z & +{\color{red}w} & +u & = 0 \\
4. & & & z & +w & & = 0 \\
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$$

Given blue vertices: 1 and 2.
Then

$$
\begin{aligned}
2 &\rightarrow 3, \\
1 &\rightarrow 5, \\
3 &\rightarrow 4.
\end{aligned}
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Given information: $x = y = 0$.
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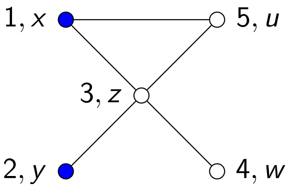

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Proposition (Kenter and L 2018)

Let G be a graph on the vertex set V. The following are equivalent:

1. *B is a zero forcing set.*
2. *For any $A \in \mathcal{S}(G)$, the columns corresponding to $V \setminus B$ hides a lower triangular matrix.*
3. *For any $A \in \mathcal{S}(G)$, the columns corresponding to $V \setminus B$ are linearly independent.*

Theorem (AIM Work Group 2008)

Let G be a graph on n vertices. Then for any matrix $A \in \mathcal{S}(G)$, $n - Z(G) \leq \operatorname{rank}(A)$.

More zero forcing

- Same argument works for non-symmetric matrices.
- When more information are known on the matrices, the design of the zero forcing process can be improved.
- For example, nonnegative matrices, zero diagonal entries, or nonzero diagonal entries.
- They all follow the same philosophy.
- Zero forcing is related to the minimum rank problem (Math), quantum control (Physics), building logic circuit (Physics), the graph searching problem (ComS).

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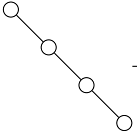
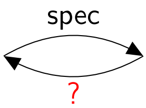
Inverse eigenvalue problem of a graph (IEP-G)

Let G be a graph. Define $\mathcal{S}(G)$ as the family of all real symmetric matrices $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ such that

$$a_{ij} \begin{cases} \neq 0 & \text{if } ij \in E(G), i \neq j; \\ = 0 & \text{if } ij \notin E(G), i \neq j; \\ \in \mathbb{R} & \text{if } i = j. \end{cases}$$



$$\longrightarrow \begin{bmatrix} ? & * & 0 & 0 \\ * & ? & * & 0 \\ 0 & * & ? & * \\ 0 & 0 & * & ? \end{bmatrix}$$



$\{1, 2, 3, 4\}$

IEP-G: What are the possible spectra of a matrix in $\mathcal{S}(G)$?

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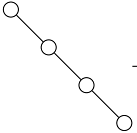
Let G be a graph. Define $\mathcal{S}(G)$ as the family of all real symmetric matrices $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ such that

$$a_{ij} \begin{cases} \neq 0 & \text{if } ij \in E(G), i \neq j; \\ = 0 & \text{if } ij \notin E(G), i \neq j; \\ \in \mathbb{R} & \text{if } i = j. \end{cases}$$



$$\longrightarrow \begin{bmatrix} ? & * & 0 & 0 \\ * & ? & * & 0 \\ 0 & * & ? & * \\ 0 & 0 & * & ? \end{bmatrix} \underset{?}{\overset{\text{spec}}{\rightleftarrows}} \{1, 2, 3, 4\}$$

IEP-G: What are the possible spectra of a matrix in $\mathcal{S}(G)$?

Generic case

Theorem (Monfared and Shader 2013)

Let G be a graph on n vertices and $\lambda_1 < \cdots < \lambda_n$ distinct real numbers. Then there is a matrix $A \in \mathcal{S}(G)$ such that $\operatorname{spec}(A) = \{\lambda_1, \ldots, \lambda_n\}$.

Maximum multiplicity

Theorem (AIM Work Group 2008)

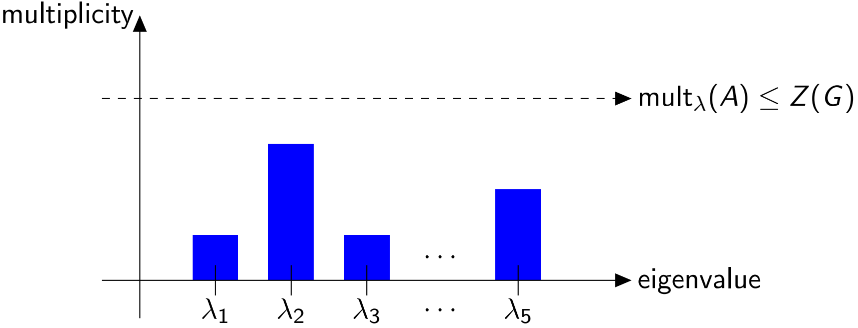
Let G be a graph on n vertices. Then or any matrix $A \in \mathcal{S}(G)$,

- $n - Z(G) \leq \text{rank}(A)$,
- $\text{null}(A) \leq Z(G)$, *and*
- $\text{mult}_\lambda(A) \leq Z(G)$.

Recall that $\text{mult}_\lambda(A) = \text{null}(A - \lambda I)$ and $A - \lambda I \in \mathcal{S}(G)$ if and only if $A \in \mathcal{S}(G)$.

Ordered multiplicity list

Suppose the spectrum of a real symmetric matrix A is $\{\lambda_1^{(m_1)}, \ldots, \lambda_q^{(m_q)}\}$. Then the ordered multiplicity list of A is $(m_1, \ldots, m_q)$.



IEP-P_n

$$\begin{bmatrix} ? & * & & & \\ * & ? & * & & \\ & * & \ddots & \ddots & \\ & & \ddots & & * \\ & & & * & ? \end{bmatrix}$$

$Z(P_n) = 1$ $\lambda_1 < \cdots < \lambda_n$

Hochstadt 1974, Gray and Wilson 1976, and Hald 1976 gave stronger results.

IEP-C_n

$$\begin{bmatrix} ? & * & & & * \\ * & ? & * & & \\ & * & \ddots & \ddots & \\ & & \ddots & & * \\ * & & & * & ? \end{bmatrix}$$

$$Z(C_n) = 2$$

$$\lambda_1 \le \lambda_2 < \lambda_3 \le \lambda_4 < \cdots, \text{ or}$$

$$\lambda_1 < \lambda_2 \le \lambda_3 < \lambda_4 \le \lambda_5 < \cdots$$

For example, $(2, 1, 2)$ is not possible for C_5.

Ferguson 1980 proved stronger results, which uses the eigenvector-eigenvalue identity (see, e.g., DPTZ 2019 for a survey on this identity).

IEP-$K_{1,n-1}$

$$\begin{bmatrix} ? & * & * & \cdots & * \\ * & ? & & & \\ * & & \ddots & & \\ \vdots & & & & \\ * & & & & ? \end{bmatrix}$$

$Z(K_{1,n-1}) = n-2$

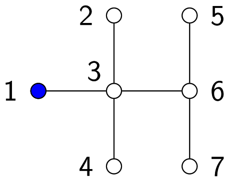
$Z_+(K_{1,n-1}) = 1$

any spectrum with ordered multiplicity list $(m_1, \ldots, m_q)$ satisfying $m_i \leq n-2$ and $m_1 = m_q = 1$

For example, $(1,2,1)$ is achievable for $K_{1,3}$ but $(2,1,1)$ is not.

PSD zero forcing

- Rule for Z: If for some x, the closed neighborhood $N_G[x]$ are all blue except for one vertex y and $y \neq x$, then y turns blue.
- Rule for Z_+: Apply the same rule to each component of the white vertices.



Theorem (BBFHHSvdDvdH 2010)

Let G be a graph. Then for any matrix $A \in \mathcal{S}(G)$ with ordered multiplicity list $(m_1, \ldots, m_q)$,

$$m_1 \leq Z_+(G), \text{ and } m_q \leq Z_+(G).$$

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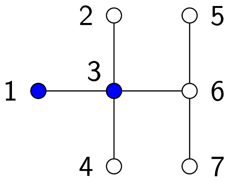

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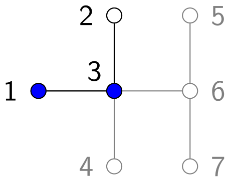

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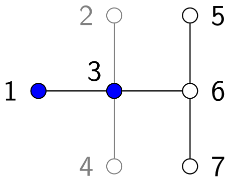

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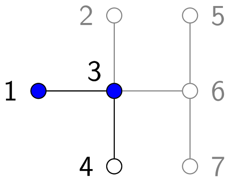

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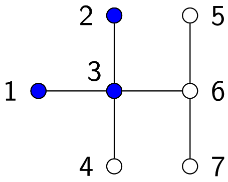

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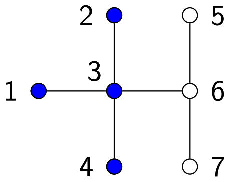

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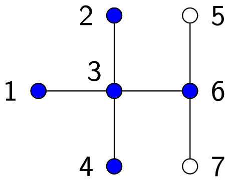

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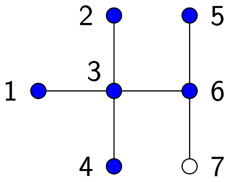

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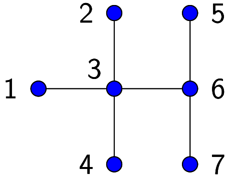

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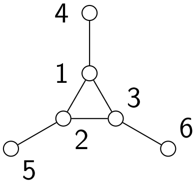
IEPG for 3-sun



$$\begin{bmatrix} ? & * & * & * & & \\ * & ? & * & & * & \\ * & * & ? & & & * \\ * & & & ? & & \\ & * & & & ? & \\ & & * & & & ? \end{bmatrix}$$

$Z(G) = 2$

$Z_+(G) = 2$

any spectrum with ordered multiplicity list $(m_1, \ldots, m_q)$ satisfying $m_i + m_{i+1} \leq 3$ and one of m_1 and m_q is 1

For example, $(2, 1, 2, 1)$ is achievable for G but $(1, 2, 2, 1)$ is not.

This behavior was found BBFHHLSY 2020.

Why $m_i + m_j \leq 3$?

- Let G be the 3-sun.
- Suppose $A \in \mathcal{S}(G)$ has ordered multiplicity list $(1,2,2,1)$.
- Then $B = (A - \lambda_2 I)(A - \lambda_3 I)$ has ordered multiplicity list $(4,2)$ or $(4,1,1)$.
- The graph of B is a multigraph Γ

- $Z_+(\Gamma) = 3$, so $(4,2)$ or $(4,1,1)$ are not possible.

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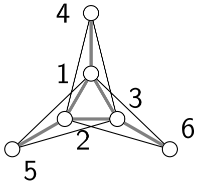

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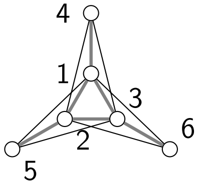

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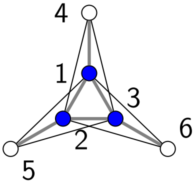
Multigraphs

Let Γ be a multigraph.
If there are two or more edges between vertices i, j, then

- the i, j-entry of a matrix in $\mathcal{S}(\Gamma)$ can be zero or nonzero, and
- j is counted as 2 (or more) neighbors of i, so $i \to j$ is not allowed in zero forcing.



Graph power

Let G be a simple graph.
A lazy walk from i to j of length r on G is a sequence

$$i = v_0 \to v_1 \to \cdots \to v_r = j$$

such that v_i and v_{i+1} are adjacent or the same.

Definition

Define $\Gamma(G, r)$ as the multigraph on $V(G)$ such that the number of edges between i, j is the number of lazy walks from i to j of length r.

Power zero forcing

Proposition (Kenter and L 2021)

Let $A \in \mathcal{S}(G)$ with spectrum $\{\lambda_1^{(m_1)}, \ldots, \lambda_q^{(m_q)}\}$. Then

- $B = (A - \lambda_i I)(A - \lambda_j I)$ *has nullity* $m_i + m_j$ *and is in* $\mathcal{S}(\Gamma(G, 2))$.
- $B = (A - \lambda_i I)(A - \lambda_{i+1} I)$ *has nullity* $m_i + m_{i+1}$*, is PDS, and is in* $\mathcal{S}(\Gamma(G, 2))$.

Define $Z^{(r)}(G) = Z(\Gamma(G, r))$ and $Z_+^{(r)}(G) = Z_+(\Gamma(G, r))$.

Theorem (Kenter and L 2021)

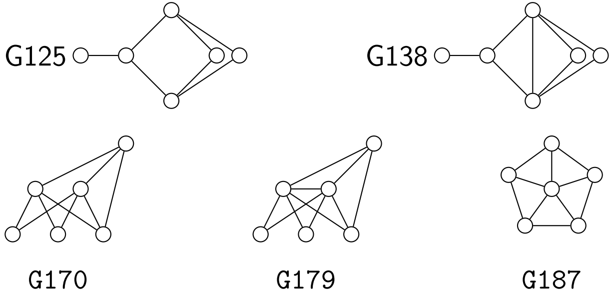
Let $A \in \mathcal{S}(G)$ with spectrum $\{\lambda_1^{(m_1)}, \ldots, \lambda_q^{(m_q)}\}$. Then

- $m_i + m_j \leq Z^{(2)}(G)$*, and*
- $m_i + m_{i+1} \leq Z_+^{(2)}(G)$.

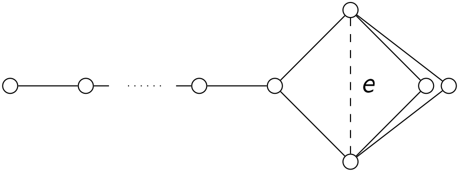
Similar theorems can be built for other powers.

Highlights

- Using $Z^{(r)}$ and $Z_+^{(r)}$ is enough to get the correct ordered multiplicity lists for 100/113 graphs on 6 vertices (using the answers in AABBCGKMW 2021).
- The power zero forcing provides a unified theory for the following graphs:



$K_{2,3}$ and $K_{2,3}^e$ with a trailing path

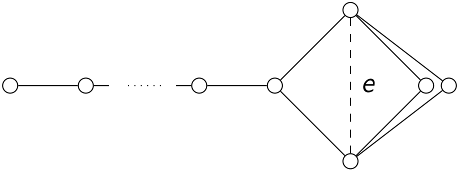


Theorem (Kenter and L 2021)

Let G be G_ℓ or G_ℓ^e and $A \in \mathcal{S}(G)$. Then the sum of any two eigenvalue multiplicities of A is at most 4.

Thanks!

$K_{2,3}$ and $K_{2,3}^e$ with a trailing path



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