Sign patterns requiring a unique inertia

Jephian C.-H. Lin 林晉宏

Department of Applied Mathematics, National Sun Yat-sen University

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Joint work with

Pauline van den Driessche
University of Victoria

D. Dale Olesky
University of Victoria

Sign pattern

- A sign pattern is a matrix whose entries are in $\{+, -, 0\}$.
- The qualitative class of a sign pattern $\mathcal{P} = \left[p_{i,j}\right]$ is the family of matrices $A = \left[a_{i,j}\right]$ such that $\text{sign}(a_{i,j}) = p_{i,j}$.

$$Q\left(\begin{bmatrix} + & + & 0 \\ 0 & - & + \end{bmatrix}\right) \ni \begin{bmatrix} 1 & 2 & 0 \\ 0 & -3 & 0.5 \end{bmatrix}, \begin{bmatrix} 5 & 3 & 0 \\ 0 & -1 & \pi \end{bmatrix}, \ldots$$

Require and allow

- Let $\mathcal{P}$ be a sign pattern.
- Let R be a property of a matrix. E.g., being invertible, being nilpotent, etc.
- $\mathcal{P}$ requires property R if every matrix in $Q(\mathcal{P})$ has property R.
- $\mathcal{P}$ allows property R if at least a matrix in $Q(\mathcal{P})$ has property R.

$$\begin{bmatrix} + & 0 & 0 \\ - & + & 0 \\ 0 & - & + \end{bmatrix} \text{ requires a positive determinant.}$$

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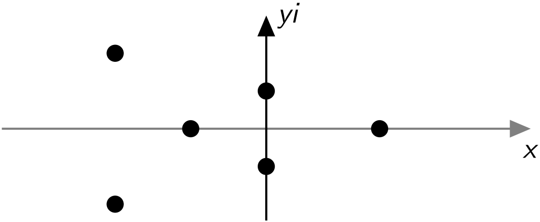
Inertia

Let A be a matrix.

- $n_+(A) =$ number of eigenvalues with positive real part.
- $n_-(A) =$ number of eigenvalues with negative real part.
- $n_0(A) =$ number of eigenvalues with zero real part.
- $n_z(A) =$ number of eigenvalues that equals zero.

The inertia of A is the triple $(n_+(A), n_-(A), n_0(A))$.



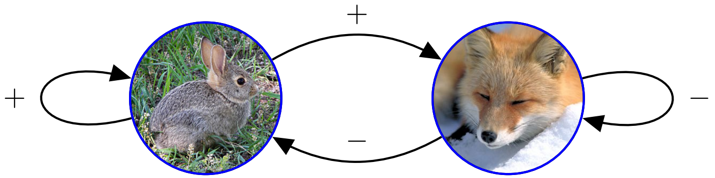
$n_- = 3$ $n_0 = 2$ $n_+ = 1$

Question:

- Which sign pattern requires a unique inertia?

Outlines:

- Motivations from dynamical systems
- Sign patterns requiring a unique inertia

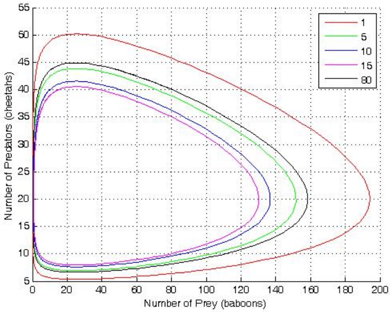


Predator-Prey Model

$$\frac{dx}{dt} = \alpha x - \beta xy$$

$$\frac{dy}{dt} = \delta xy - \gamma y$$

(pictures from Wikipedia)



General form

Let $\mathcal{P} = [p_{i,j}]$ be a sign pattern. Let $x_{i,j}$ be variables for $i, j \in [n]$. The general form of $\mathcal{P}$ is a variable matrix X with

$$(X)_{i,j} = \begin{cases} x_{i,j} & \text{if } p_{i,j} = +; \\ -x_{i,j} & \text{if } p_{i,j} = -; \\ 0 & \text{if } p_{i,j} = 0. \end{cases}$$

$$\mathcal{P} = \begin{bmatrix} + & + \\ - & - \end{bmatrix} \qquad X = \begin{bmatrix} x_{1,1} & x_{1,2} \\ -x_{2,1} & -x_{2,2} \end{bmatrix}$$

Write $\det(zI - X) = S_0 z^n - S_1 z^{n-1} + S_2 z^{n-2} + \cdots + (-1)^n S_n$.
Then each S_k is a multivariate polynomial in $x_{i,j}$'s.

Sign of a polynomial

- Let p be a polynomial.
- p can be expanded into a linear combination of non-repeated monomials.

$$\operatorname{sign}(p) = \begin{cases} 0 & \text{if all coefficients} = 0; \\ + & \text{if all nonzero coefficients} > 0 \text{ and } \operatorname{sign}(p) \neq 0; \\ - & \text{if all nonzero coefficients} < 0 \text{ and } \operatorname{sign}(p) \neq 0; \\ \# & \text{otherwise.} \end{cases}$$

Minor sequence

- Let X be the general form a sign pattern $\mathcal{P}$. The minor sequence of $\mathcal{P}$ is $s_0, s_1, \ldots, s_n$, where $s_k = \text{sign}(S_k)$.

Theorem (JL, Olesky, and van den Driessche 2018)

If $s_n = \#$, then $\mathcal{P}$ does not require a unique inertia. When $\mathcal{P}$ is a 2×2 sign pattern, $\mathcal{P}$ require a unique inertia if and only if $s_2 \neq \#$.

$$\begin{bmatrix} 0 & + \\ - & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & + \\ + & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & - \\ - & + \end{bmatrix} \quad \begin{bmatrix} 0 & - \\ + & + \end{bmatrix}$$

[+,0,+] [+,0,−] [+,+,−] [+,+,+]

$$\begin{bmatrix} - & - \\ - & + \end{bmatrix} \quad \begin{bmatrix} - & - \\ + & + \end{bmatrix} \quad \begin{bmatrix} + & - \\ - & + \end{bmatrix} \quad \begin{bmatrix} + & + \\ - & + \end{bmatrix}$$

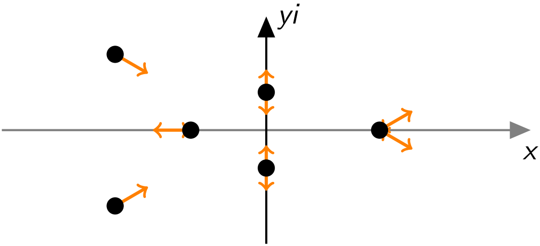
[+,#,−] [+,#,#] [+,+,#] [+,+,+]

Equivalence conditions

Theorem (JL, Olesky, and van den Driessche 2018)

Let $\mathcal{P}$ be a sign pattern. The following are equivalent:

- *$\mathcal{P}$ requires a unique inertia.*
- *$\mathcal{P}$ requires a fixed n_0.*
- *$\mathcal{P}$ requires a fixed n_z and a fixed number of nonzero pure imaginary eigenvalues.*



Number of nonzero pure imaginary roots

Substitute z by ti (with $t \neq 0$):

$$\begin{aligned} p(z) &= x^5 + x^4 + 6x^3 + 2x^2 + 9x - 3 \\ &= (t^4 - 6t^2 + 9)ti + (t^4 - 2t^2 - 3) \end{aligned}$$

$$\text{odd part} = x^2 - 6x + 9$$
$$\text{even part} = x^2 - 2x - 3$$

$\#$ of nonzero pure imaginary roots

$= 2 \cdot \#$ of common positive roots of the odd and the even parts

For $\det(zI - X)$, $\begin{array}{l} \text{odd part} : S_0, -S_2, S_4, \ldots \\ \text{even part} : S_1, -S_3, S_5, \ldots \end{array}$

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Descartes' rule of signs

Theorem (Descartes' rule of signs)

Suppose $p(x) \neq 0$ is a polynomial whose coefficients has t sign changes (ignoring the zeros). Then $p(x)$ has $t - 2k$ positive roots for some $k \geq 0$.

For example

- $x^2 - 6x + 9$ has 2 or 0 positive roots, and
- $x^2 + 0x - 4$ has 1 positive root.

[Key: No sign changes, no positive roots!]

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Resultant

Let $p_1(x) = \sum_{k=0}^{\ell} c_k x^{\ell-k}$ and $p_2(x) = \sum_{k=0}^{m} d_k x^{m-k}$.
The Sylvester matrix of p_1 and p_2 is an $(m+\ell) \times (m+\ell)$ matrix

$$S(p_1, p_2) = \begin{bmatrix} c_0 & & & & d_0 & & & \\ c_1 & c_0 & & & d_1 & d_0 & & \\ c_2 & c_1 & \ddots & & d_2 & d_1 & \ddots & \\ \vdots & & \ddots & c_0 & \vdots & & \ddots & d_0 \\ & \vdots & & c_1 & & \vdots & & d_1 \\ c_\ell & & & & d_m & & & \\ & c_\ell & & \vdots & & d_m & & \vdots \\ & & \ddots & & & & \ddots & \\ & & & c_\ell & & & & d_m \end{bmatrix}.$$

The resultant of p_1 and p_2 is

$$\text{Res}(p_1, p_2) = \det(S(p_1, p_2)).$$

Theorem

$\text{Res}(p_1, p_2) = 0$ *if and only if* p_1 *and* p_2 *have a common factor.*

Suppose $\mathcal{P}$ is a sign pattern with general form X.

- $\text{Res}(\mathcal{P}) = \text{Res}(\text{even part}, \text{odd part})$ with the two parts from $\det(zI - X)$.

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$$\begin{bmatrix} 0 & x_{1,2} & 0 \\ -x_{2,1} & 0 & -x_{2,3} \\ 0 & -x_{3,2} & x_{3,3} \end{bmatrix},$$

$$S_0(\mathcal{P}) = 1 \qquad S_2(\mathcal{P}) = x_{1,2}x_{2,1} - x_{2,3}x_{3,2}$$
$$S_1(\mathcal{P}) = x_{3,3} \qquad S_3(\mathcal{P}) = x_{1,2}x_{2,1}x_{3,3}$$

$$\begin{aligned} \mathrm{Res}(\mathcal{P}) &= x_{3,3}(x_{1,2}x_{2,1} - x_{2,3}x_{3,2}) - x_{1,2}x_{2,1}x_{3,3} \\ &= x_{3,3}x_{1,2}x_{2,1} - x_{3,3}x_{2,3}x_{3,2} - x_{1,2}x_{2,1}x_{3,3} \\ &= x_{3,3}x_{2,3}x_{3,2}. \end{aligned}$$

$\mathrm{sign}(\mathrm{Res}(\mathcal{P})) = + \implies$ never has common positive roots

So, $\mathcal{P}$ does not allow any nonzero pure imaginary eigenvalues.

Exceptional, not exceptional

A 3×3 sign pattern $\mathcal{P}$ is in $\mathcal{E}$ if its minor sequence is $[+, \#, \#, +]$ or $[+, \#, \#, -]$

3×3 sign patterns not in $\mathcal{E}$

Theorem (JL, Olesky, and van den Driessche 2018)

Let $\mathcal{P}$ be a 3×3 irreducible sign pattern that is not in $\mathcal{E}$. Then $\mathcal{P}$ requires a unique inertia if and only if

1. $s_{k_0} \in \{+, -\}$ *and* $s_k = 0$ *for all* $k > k_0$ *(fixed* n_z*), and*
2. *At least one of the following holds: (fixed* $n_0 - n_z = 0$*)*
   - 2.1 $s_2 = -$. *(no sign changes in even part)*
   - 2.2 $s_1, s_3 \in \{+, -, 0\}$ *and* $s_1 \neq s_3$. *(no sign changes in odd part)*
   - 2.3 $\mathrm{Res}(\mathcal{P})$ *has a fixed sign.*

Embedded $\mathcal{T}_2$

$\mathcal{T}_2 = \begin{bmatrix} + & + \\ - & - \end{bmatrix}$ allows two inertias $(2,0,0)$ and $(0,2,0)$

$\begin{bmatrix} + & + & 0 \\ - & - & 0 \\ 0 & 0 & 0 \end{bmatrix}$ allows two inertias $(2,0,1)$ and $(0,2,1)$

Lemma (JL, Olesky, and van den Driessche 2018)

If $\mathcal{P}$ is a 3×3 sign pattern with $\mathcal{T}_2$ (or $\mathcal{T}_2^\top$) embedded in $\mathcal{P}$ as a principal subpattern, then $\mathcal{P}$ does not require a unique inertia.

$$\begin{bmatrix} 0 & 0 & + \\ - & + & + \\ 0 & - & - \end{bmatrix} \text{ has minor sequence } [+, \#, \#, +]$$

3×3 sign patterns in $\mathcal{E}$

Theorem (JL, Olesky, and van den Driessche 2018)

Let $\mathcal{P}$ be a 3×3 sign pattern in $\mathcal{E}$. Then $\mathcal{P}$ requires a unique inertia if and only if $\mathcal{T}_2$ is not embedded in $\mathcal{P}$ as a principal subpattern.

Enumerations

All 2×2 and 3×3 sign patterns are characterized.

2×2:

- 8 sign patterns in total
- 6 UI; 2 not UI

3×3:

	UI	not UI	subtotal
not in $\mathcal{E}$	51	118	169
in $\mathcal{E}$	12	6	18
subtotal	63	124	187

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Thank you!

References I

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