On the zero forcing process

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Zero forcing

Zero forcing process:

- Start with a given set of blue vertices.
- If for some x, the closed neighbourhood $N_G[x]$ are all blue except for one vertex y and $y \neq x$, then y turns blue.

An initial blue set that can make the whole graph blue is called a zero forcing set. The zero forcing number $Z(G)$ of a graph G is the minimum size of a zero forcing set.

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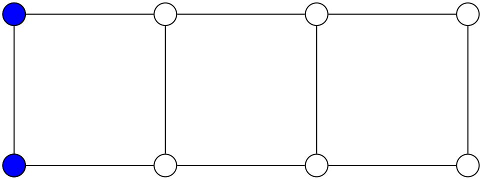

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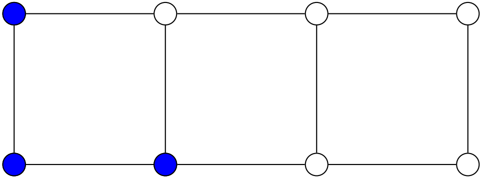

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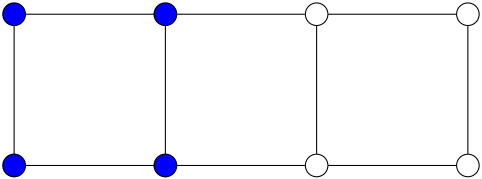

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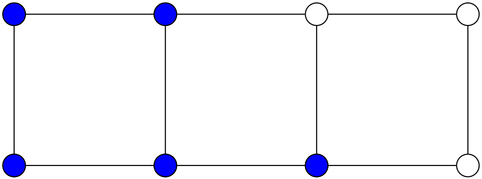

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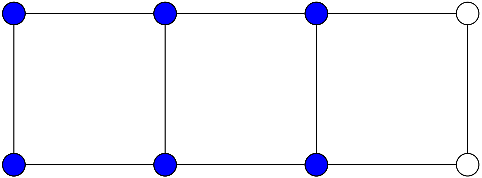

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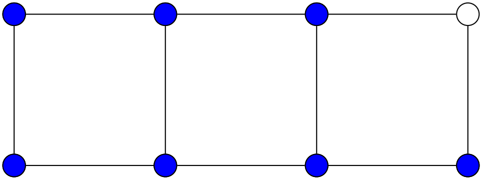

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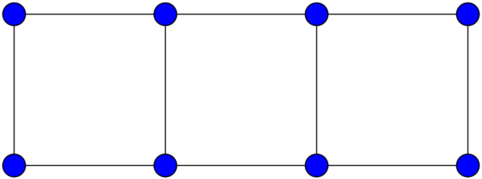
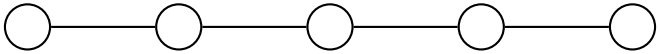


$Z(G) = 1$



$Z(G) = 1$ if and only if G is a path.

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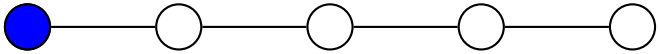

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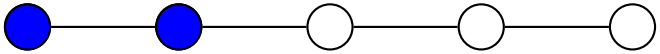

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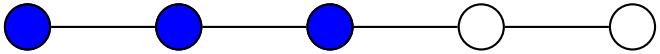

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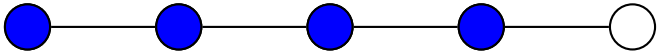

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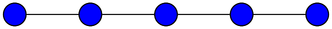
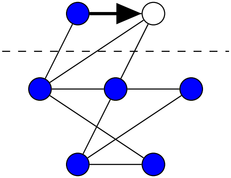
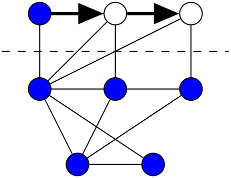


$Z(G) = 1$ if and only if G is a path.

$Z(G) = n$ or $n - 1$

$Z(G) = n \implies P_2$-free $\qquad Z(G) = n - 1 \implies P_3$-free





Let G be a graph on n vertices.

- Then $Z(G) = n$ if and only if G is the union of isolated vertices.
- And $Z(G) = n - 1$ if and only if G is $K_r \dot\cup \overline{K_{n-r}}$, $r \neq 1$.

Generalised adjacency matrix

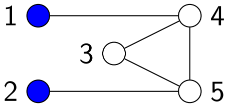
Let G be a simple graph on n vertices. The family $\mathcal{S}(G)$ consists of all $n \times n$ real symmetric matrix $M = \left[M_{i,j}\right]$ with

$$\begin{cases} M_{i,j} = 0 & \text{if } i \neq j \text{ and } \{i,j\} \text{ is not an edge,} \\ M_{i,j} \neq 0 & \text{if } i \neq j \text{ and } \{i,j\} \text{ is an edge,} \\ M_{i,j} \in \mathbb{R} & \text{if } i = j. \end{cases}$$

$$\mathcal{S}(\circ\!-\!\circ\!-\!\circ) \ni \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0.1 & 0 \\ 0.1 & 1 & \pi \\ 0 & \pi & 0 \end{bmatrix}, \cdots$$

Why zero forcing?

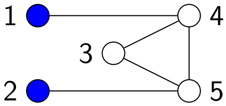
$$\begin{bmatrix} -2 & 0 & 0 & 7 & 0 \\ 0 & 1 & 0 & 0 & -9 \\ 0 & 0 & 0 & 3 & 4 \\ 7 & 0 & 3 & -4 & 5 \\ 0 & -9 & 4 & 5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



- Pick a matrix $A \in \mathcal{S}(G)$ and consider $A\mathbf{x} = \mathbf{0}$.
- Each vertex represents a variable. Each vertex also represents an equation where appearing variables are the neighbours and possibly itself.
- Blue means zero. White means unknown.

Hidden triangle in a system

$$\begin{array}{lrrrrrl}
1. & -2x_1 & & & +7x_4 & & = 0 \\
2. & & 1x_2 & & & -9x_5 & = 0 \\
3. & & & & 3x_4 & +4x_5 & = 0 \\
4. & 7x_1 & & +3x_3 & -4x_4 & +5x_5 & = 0 \\
5. & & -9x_2 & +4x_3 & +5x_4 & & = 0
\end{array}$$



Given $x_1 = x_2 = 0$,

$$1. \implies x_4 = 0,$$
$$2. \implies x_5 = 0,$$
$$4. \implies x_3 = 0.$$

Given 1 and 2 blue,

$$1 \to 4,$$
$$2 \to 5,$$
$$4 \to 3.$$

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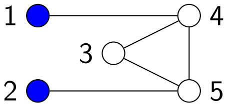

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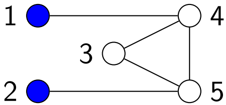

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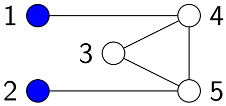

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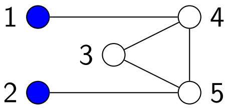

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As long as the red terms has nonzero coefficients and the orange terms are zero, the same argument always works.

Triangle number

- A pattern is a matrix whose entries are in $\{0, *, ?\}$.
- A triangle is a submatrix of a pattern that can be permuted to a lower triangular matrix with $*$ on the diagonal.

$$\begin{bmatrix} ? & 0 & 0 & * & 0 \\ 0 & ? & 0 & 0 & * \\ 0 & 0 & ? & * & * \\ * & 0 & * & ? & * \\ 0 & * & * & * & ? \end{bmatrix} \qquad \begin{bmatrix} 0 & * & 0 \\ 0 & 0 & * \\ * & ? & * \end{bmatrix} \rightarrow \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ ? & * & * \end{bmatrix} \qquad \text{triangle}$$

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$$\begin{bmatrix} ? & 0 & 0 & * & 0 \\ 0 & ? & 0 & 0 & * \\ 0 & 0 & ? & * & * \\ * & 0 & * & ? & * \\ 0 & * & * & * & ? \end{bmatrix} \qquad \begin{bmatrix} ? & 0 & 0 \\ 0 & ? & 0 \\ 0 & 0 & ? \end{bmatrix} \quad \text{not a triangle}$$

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$$\begin{bmatrix} ? & 0 & 0 & * & 0 \\ 0 & ? & 0 & 0 & * \\ 0 & 0 & ? & * & * \\ * & 0 & * & ? & * \\ 0 & * & * & * & ? \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & * \\ * & ? & * \\ 0 & * & ? \end{bmatrix} \to \begin{bmatrix} * & 0 & 0 \\ ? & * & 0 \\ * & ? & * \end{bmatrix} \qquad \text{triangle}$$

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- The triangle number $\text{tri}(\mathcal{P})$ of a pattern $\mathcal{P}$ is the largest size of a triangle in $\mathcal{P}$.
- Define $\text{tri}(G) = \text{tri}(\mathcal{P})$, where $\mathcal{P}$ is the pattern of the generalized adjacency matrix of G.

Triangle number and zero forcing

Theorem

For any simple graph G on n vertices, $\mathrm{tri}(G) = n - Z(G)$.

Proof.

Record all the forces in order. Find the rows of the "forc-ers", find the columns of the "forc-ees", then you find the triangle. □

$$\begin{bmatrix} ? & 0 & 0 & * & 0 \\ 0 & ? & 0 & 0 & * \\ 0 & 0 & ? & * & * \\ * & 0 & * & ? & * \\ 0 & * & * & * & ? \end{bmatrix}$$

$1 \to 4$
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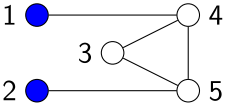


Proposition (Kenter and L 2018)

Let G be a graph on the vertex set V. The following are equivalent:

1. *B is a zero forcing set.*
2. *For any $A \in \mathcal{S}(G)$, the columns corresponding to $V \setminus B$ hides a lower triangular matrix.*
3. *For any $A \in \mathcal{S}(G)$, the columns corresponding to $V \setminus B$ are linearly independent.*

Theorem (AIM Work Group 2008)

Let G be a graph on n vertices. For any matrix $A \in \mathcal{S}(G)$, $n - Z(G) \leq \operatorname{rank}(A)$.

Corollary tridiagonal

Corollary

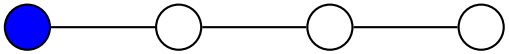
Any symmetric irreducible tridiagonal matrix has all its eigenvalues distinct.

$$\begin{bmatrix} ? & * & 0 & \cdots & 0 \\ * & ? & * & \ddots & \vdots \\ 0 & * & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & & * \\ 0 & \cdots & 0 & * & ? \end{bmatrix}$$

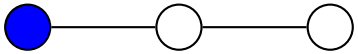
Proof.

For any $A \in \mathcal{S}(P_n)$, $\mathrm{null}(A) \leq Z(P_n) = 1$ and $\mathrm{null}(A - \lambda I) \leq Z(P_n) = 1$. □

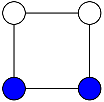
$Z(G) - 1 \leq Z(G - v) \leq Z(G) + 1$



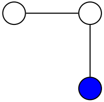
$Z(G) = 1$



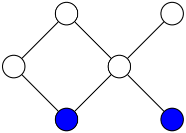
$Z(G - v) = 1$



$Z(G) = 2$



$Z(G - v) = 1$



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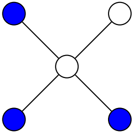

$Z(G - v) = 3$

$\mathrm{tri}(G)$ is induced subgraph monotone

- If H is an induced subgraph of G, then $\mathrm{tri}(H) \leq \mathrm{tri}(G)$.
- For each k, let $\mathbf{Forb}_{\mathrm{tri}(G)\leq k}$ be the set of minimal induced subgraph of $\{H : \mathrm{tri}(H) \geq k+1\}$.
- Then $\mathrm{tri}(G) \leq k$ if and only if G is $\mathbf{Forb}_{\mathrm{tri}(G)\leq k}$-free.

$$\mathbf{Forb}_{\mathrm{tri}(G)\leq 0} = \{P_2\}$$

$$\mathbf{Forb}_{\mathrm{tri}(G)\leq 1} = \{P_3, 2P_2\}$$

$\mathbf{Forb}_{\mathrm{tri}(G)\leq 2} = \{P_4,$  ,  , $P_2 \dot\cup P_3, 3P_2\}$

Is $|\mathbf{Forb}_{\mathrm{tri}(G)\leq k}|$ always finite?

Proposition

Any graph with $\mathrm{tri}(G) \geq k+1$ *contains an induced subgraph with* $\mathrm{tri}(G) \geq k+1$ *and of order at most* $2k+2$.

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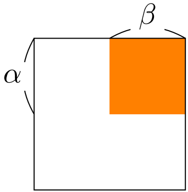

$$|\alpha|, |\beta| = k+1$$
$$|\alpha \cup \beta| \leq 2k+2$$

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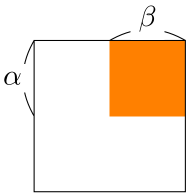


$$|\alpha|, |\beta| = k+1$$
$$|\alpha \cup \beta| \leq 2k+2$$

Corollary

Any graph in $\mathbf{Forb}_{\text{tri}(G)\leq k}$ *has order at most* $2k+2$*.*

$$\mathbf{Forb}_{\mathrm{tri}(G)\leq 0} = \{P_2\}$$

$$\mathbf{Forb}_{\mathrm{tri}(G)\leq 1} = \{P_3, 2P_2\}$$

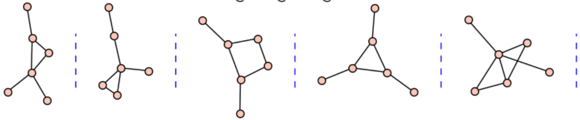
$$\mathbf{Forb}_{\mathrm{tri}(G)\leq 2} = \{P_4,$$  ,  $$, P_2 \,\dot\cup\, P_3, 3P_2\}$$

$$\mathbf{Forb}_{\mathrm{tri}(G)\leq 3} = \{19 \text{ connected}, 6 \text{ disconnected}\}$$

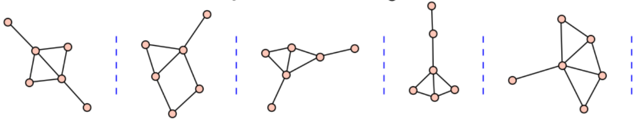
$$\left|\mathbf{Forb}_{\mathrm{tri}(G)\leq 4}\right| = 263, \ldots$$

DBg E?N? E_Cg E?NG E@JW

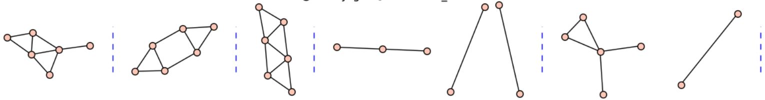
E?NW E@`w E@YW E@UW E?Nw



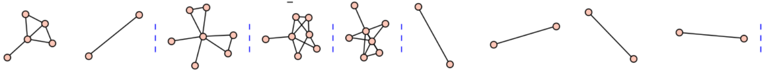
E?]w EGNW EAMw ECXw E@Vw



E@^W EJeg EK]w FK??W F_?Hw



F_?Xw F?Cfw F?D~o F?D~w G?CaC?



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