Zero forcing and its applications

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System of linear equations

$$\begin{aligned} 2x &+3y &-z &= 4 \\ x &-y &+2z &= 3 \\ -3x &+2y &+z &= 2 \end{aligned}$$

Hard to know if the solution exists, or if the solution is unique.

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Main philosophy

$$2x + y + 3z = 7$$

$$x = 1, y = 2 \implies z = 1$$

In a linear equation, if all but one variable are known, then this remaining variable is also known.

$$2x + y + 3z = 0$$

$$x = 0, y = 0 \implies z = 0$$

In a homogeneous linear equation, if all but one variable are zero, then this remaining variable is also zero.

Hidden triangle in a system

$$\begin{array}{rcccccl}
1. & x & & +z & & +u & = 0 \\
2. & & y & +z & & & = 0 \\
3. & x & +y & +z & +w & +u & = 0 \\
4. & & & z & +w & & = 0 \\
5. & x & & +z & & +u & = 0
\end{array}$$

Given information: $x = y = 0$. Then

$$2. \implies z = 0,$$

$$4. \implies w = 0,$$

$$3. \implies u = 0.$$

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$$\begin{array}{lllllll} 2. & & y & +z & +0 & +0 & =0 \\ 4. & & & z & +w & +0 & =0 \\ 3. & x & +y & +z & +w & +u & =0 \\ 1. & x & & +z & & +u & =0 \\ 5. & x & & +z & & +u & =0 \end{array}$$

Given information: $x = y = 0$. Then

$$\begin{aligned} 2. &\implies z = 0, \\ 4. &\implies w = 0, \\ 3. &\implies u = 0. \end{aligned}$$

As long as the red terms has nonzero coefficients and the orange terms are zero, the same argument always works.

Application to algebra

Find the inverse of a formal power series.

$$\begin{array}{rcccccc} & 1 & +2x & +3x^2 & +4x^3 & +5x^4 & +\cdots \\ \times) & b_0 & +b_1x & +b_2x^2 & +b_3x^3 & +b_4x^4 & +\cdots \\ \hline & 1 & & & & & \end{array}$$

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$$\begin{array}{llllll} 1b_0 & & & & & = 1 \\ 2b_0 & +1b_1 & & & & = 0 \\ 3b_0 & +2b_1 & +1b_2 & & & = 0 \\ 4b_0 & +3b_1 & +2b_2 & +1b_3 & & = 0 \\ & & \vdots & & & \end{array}$$

Application to algebra

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\hline
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\end{array}
$$

$$
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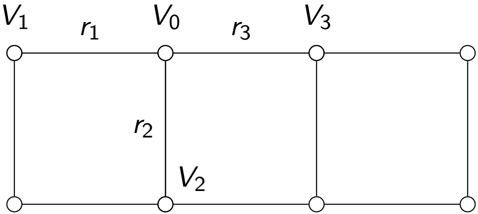
A formal power series has an inverse if and only if the constant term is nonzero.

Electronic circuit

$$\frac{1}{r_1}(V_0 - V_1) + \frac{1}{r_2}(V_0 - V_2) = \frac{1}{r_3}(V_3 - V_0) + \epsilon V_0$$



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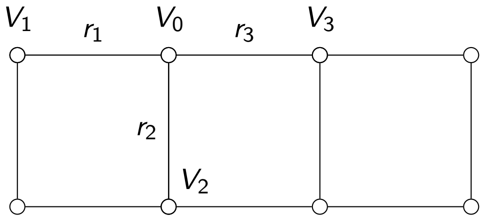
$$\frac{1}{r_1}V_1 + \frac{1}{r_2}V_2 + \frac{1}{r_3}V_3 + (\epsilon - \frac{1}{r_1} - \frac{1}{r_2} - \frac{1}{r_3})V_0 = 0$$



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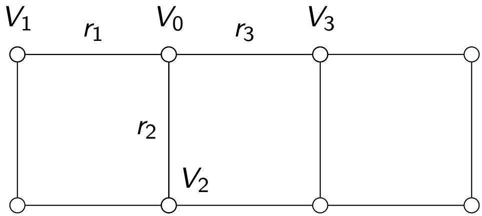
$$a_1 V_1 + a_2 V_2 + a_3 V_3 + a_0 V_0 = 0$$

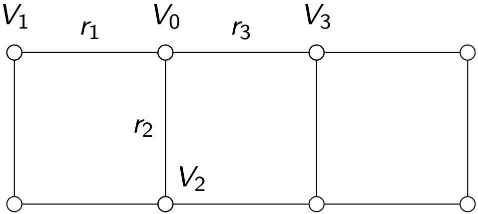
nonzero — zero or nonzero



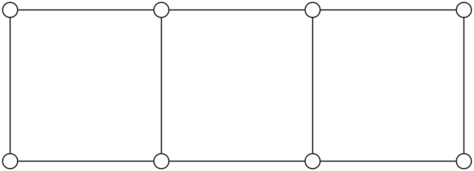
Electronic circuit

$$\underbrace{a_1 V_1 + a_2 V_2 + a_3 V_3}_{\text{nonzero}} + \underbrace{a_0 V_0}_{\text{zero or nonzero}} = 0$$

The conservation law leads to a linear equation on each node; itself and its neighbours represent the variables.



How many sensors required to monitor the voltages?



- Each vertex represents a linear equation; variables are itself and its neighbors (closed neighbourhood).
- If in a closed neighbourhood, all but one voltages are known, then this remaining one are also known.

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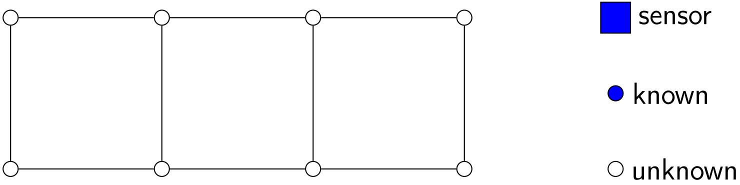

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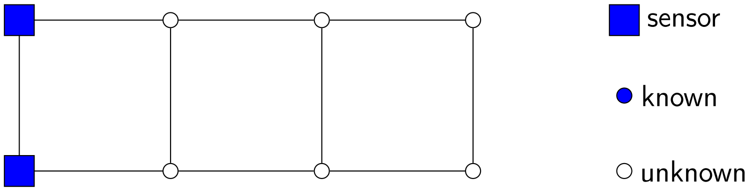

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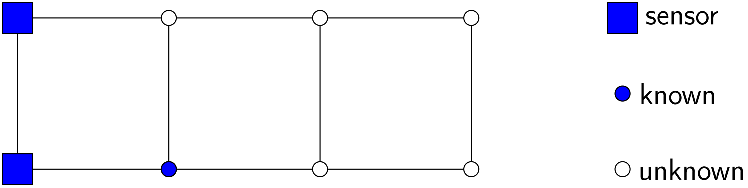

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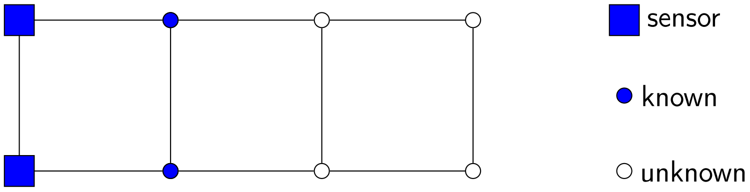

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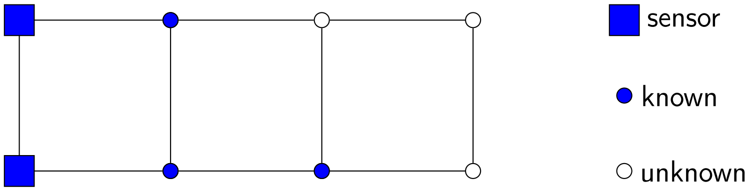

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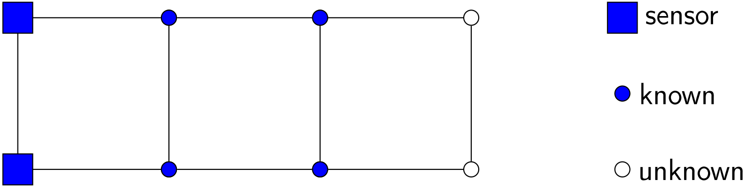

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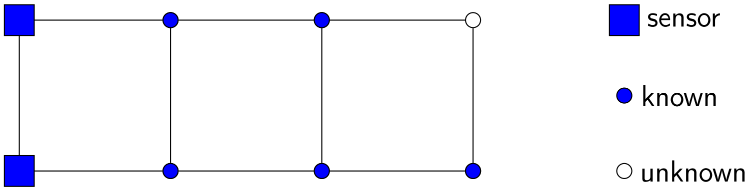

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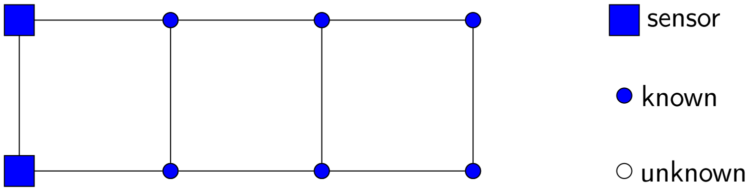

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Model by graphs and matrices

- A electronic circuit can be represented by a graph; each vertex represents a node, and each edge represents a connection.
- The linear equations can be recorded into a matrix; each row represents a equation, and each column represents an unknown voltage.
- This is a symmetric matrix where rows and columns are both indexed by the vertices.

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Let G be a simple graph on n vertices. The family $\mathcal{S}(G)$ consists of all $n \times n$ real symmetric matrix $M = \left[M_{i,j}\right]$ with

$$\begin{cases} M_{i,j} = 0 & \text{if } i \neq j \text{ and } \{i,j\} \text{ is not an edge,} \\ M_{i,j} \neq 0 & \text{if } i \neq j \text{ and } \{i,j\} \text{ is an edge,} \\ M_{i,j} \in \mathbb{R} & \text{if } i = j. \end{cases}$$

$$\mathcal{S}(\circ\!-\!\circ\!-\!\circ) \ni \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0.1 & 0 \\ 0.1 & 1 & \pi \\ 0 & \pi & 0 \end{bmatrix}, \cdots$$

Zero forcing

Zero forcing process:

- Start with a given set of blue vertices (sensors).
- If for some x, the closed neighbourhood $N_G[x]$ are all blue except for one vertex y and $y \neq x$, then y turns blue.

An initial blue set that can make the whole graph blue is called a zero forcing set. The zero forcing number $Z(G)$ of a graph G is the minimum size of a zero forcing set.

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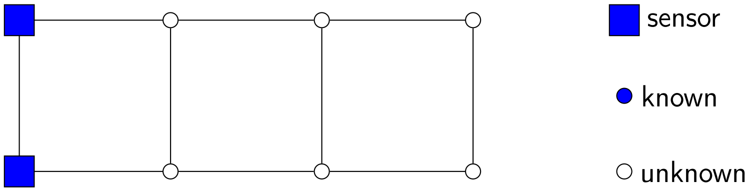

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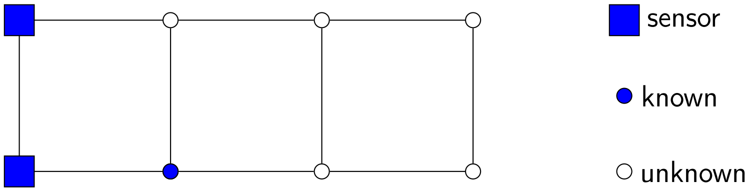

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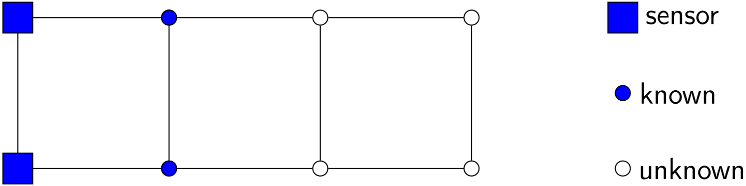

How to deploy the sensors?

Any zero forcing set is a good deployment of sensors that can monitor the whole graph.

The zero forcing number is the minimum number of sensors required.

Zero forcing sets suggest a good deployment before knowing the details of the network.

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Proposition (Kenter and L 2018)

Let G be a graph on the vertex set V. The following are equivalent:

1. *B is a zero forcing set.*
2. *For any $A \in \mathcal{S}(G)$, the columns corresponding to $V \setminus B$ hides a lower triangular matrix.*
3. *For any $A \in \mathcal{S}(G)$, the columns corresponding to $V \setminus B$ are linearly independent.*

Theorem (AIM Work Group 2008)

Let G be a graph on n vertices. For any matrix $A \in \mathcal{S}(G)$, $n - Z(G) \leq \text{rank}(A)$.

More zero forcing

- Same argument works for non-symmetric matrices.
- When more information are known on the matrices, the design of the zero forcing process can be improved.
- For example, nonnegative matrices, zero diagonal entries, or nonzero diagonal entries.
- They all follow the same philosophy.
- Zero forcing is related to the minimum rank problem (Math), quantum control (Physics), building logic circuit (Physics), the graph searching problem (ComS).

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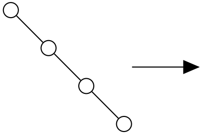
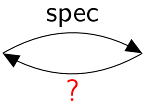
Inverse eigenvalue problem of a graph (IEP-G)

Let G be a graph. Define $\mathcal{S}(G)$ as the family of all real symmetric matrices $A = \left[a_{ij}\right]$ such that

$$a_{ij} \begin{cases} \neq 0 & \text{if } ij \in E(G), i \neq j; \\ = 0 & \text{if } ij \notin E(G), i \neq j; \\ \in \mathbb{R} & \text{if } i = j. \end{cases}$$



$$\begin{bmatrix} ? & * & 0 & 0 \\ * & ? & * & 0 \\ 0 & * & ? & * \\ 0 & 0 & * & ? \end{bmatrix}$$



$\{1, 2, 3, 4\}$

IEP-G: What are the possible spectra of a matrix in $\mathcal{S}(G)$?

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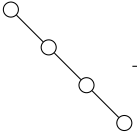

$$\longrightarrow \begin{bmatrix} ? & * & 0 & 0 \\ * & ? & * & 0 \\ 0 & * & ? & * \\ 0 & 0 & * & ? \end{bmatrix} \underset{?}{\overset{\text{spec}}{\rightleftarrows}} \{1, 2, 3, 4\}$$

IEP-G: What are the possible spectra of a matrix in $\mathcal{S}(G)$?

Supergraph Lemma

Lemma (BFHHLS 2017)

Let H be a spanning subgraph of G. If $A \in \mathcal{S}(H)$ has the strong spectral property (SSP), then there is a matrix $B \in \mathcal{S}(G)$ such that

- $\text{spec}(A) = \text{spec}(B)$,
- *B has the SSP, and*
- *$\|B - A\|$ can be chosen arbitrarily small.*

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} \sim 1 & \epsilon & 0 & 0 \\ \epsilon & \sim 2 & \epsilon & 0 \\ 0 & \epsilon & \sim 3 & \epsilon \\ 0 & 0 & \epsilon & \sim 4 \end{bmatrix}$$

SSP will be defined later

Entrywise product $\circ$

$$A \circ X = O$$

$$\Updownarrow$$

$$(X)_{ij} \neq 0 \text{ only when } (A)_{ij} = 0$$

$$I \circ X = O$$

$$\Updownarrow$$

X is zero on the diagonal

Let $A \in \mathcal{S}(G)$. Then

$$A \circ X = O \text{ and } I \circ X = O$$

$$\Updownarrow$$

$$(X)_{ij} \neq 0 \text{ only when } ij \notin E(G)$$

Strong spectral property (SSP)

Definition

A matrix A has the strong spectral property (SSP) if $X = O$ is the only real symmetric matrix that satisfies the following matrix equations:

- $A \circ X = O, I \circ X = O$,
- $AX - XA = O$.

Examples of matrices with the SSP:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Here we use the notation $[A, X]$ for $AX - XA$.

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Example of $A \in \mathcal{S}(P_4)$

Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ and } X = \begin{bmatrix} 0 & 0 & x & y \\ 0 & 0 & 0 & z \\ x & 0 & 0 & 0 \\ y & z & 0 & 0 \end{bmatrix}.$$

Then

$$[A, X] = \begin{bmatrix} 0 & -x & -y & -x+z \\ x & 0 & x-z & y \\ y & -x+z & 0 & z \\ x-z & -y & -z & 0 \end{bmatrix} = O.$$

$$\implies x = 0, z = 0, y = 0 \implies X = O$$

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$A \in \mathcal{S}(P_4)$ always has the SSP

Let

$$A = \begin{bmatrix} d_1 & a_{12} & 0 & 0 \\ a_{12} & d_2 & a_{23} & 0 \\ 0 & a_{23} & d_3 & a_{34} \\ 0 & 0 & a_{34} & d_4 \end{bmatrix} \text{ and } X = \begin{bmatrix} 0 & 0 & x & y \\ 0 & 0 & 0 & z \\ x & 0 & 0 & 0 \\ y & z & 0 & 0 \end{bmatrix}.$$

Then $[A, X] =$

$$\begin{bmatrix} 0 & -a_{23}x & ?x - a_{34}y & ? \\ ? & 0 & ? & a_{12}y + ?z \\ ? & ? & 0 & a_{23}z \\ ? & ? & ? & 0 \end{bmatrix} = O.$$

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Example of $A \in \mathcal{S}(K_{1,3})$

Let

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \text{ and } X = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & x & y \\ 0 & x & 0 & z \\ 0 & y & z & 0 \end{bmatrix}.$$

Then $[A, X] =$

$$\begin{bmatrix} 0 & x+y & x+z & y+z \\ -x-y & 0 & 0 & 0 \\ -x-z & 0 & 0 & 0 \\ -y-z & 0 & 0 & 0 \end{bmatrix} = O \text{ implies } \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

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$A \in \mathcal{S}(K_{1,3})$ always has the SSP

Let

$$A = \begin{bmatrix} d_1 & a_{12} & a_{13} & a_{14} \\ a_{12} & d_2 & 0 & 0 \\ a_{13} & 0 & d_3 & 0 \\ a_{14} & 0 & 0 & d_4 \end{bmatrix} \text{ and } X = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & x & y \\ 0 & x & 0 & z \\ 0 & y & z & 0 \end{bmatrix}.$$

Then $[A, X] = \begin{bmatrix} 0 & a_{13}x + a_{14}y & a_{12}x + a_{14}z & a_{12}y + a_{13}z \\ ? & 0 & ? & ? \\ ? & ? & 0 & ? \\ ? & ? & ? & 0 \end{bmatrix} = O$

implies $\begin{bmatrix} 0 & a_{12} & a_{13} \\ a_{12} & 0 & a_{14} \\ a_{13} & a_{14} & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \implies \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \implies X = O$

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Verification of the SSP

- Let $A \in \mathcal{S}(G)$.
- Let $E_{ij} =$ 0, 1-matrix with two ones on ij and ji.
- Define $X = \sum_{ij \in E(\overline{G})} x_{ij} E_{ij}$.

$$AX - XA = \sum_{ij \in E(\overline{G})} x_{ij}(AE_{ij} - E_{ij}A) = O$$

Verification:

A has the SSP $\iff \{AE_{ij} - E_{ij}A\}_{ij \in E(\overline{G})}$ is linearly independent

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Verification:

$$A \text{ has the SSP } \iff \{AE_{ij} - E_{ij}A\}_{ij \in E(\overline{G})} \text{ is linearly independent}$$

Verification matrix

Let $\mathrm{vec}_{\mathrm{o}}(M)$ be the vector that records the off-diagonal entries of a skew-symmetric matrix M.

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ -1 & 0 & 4 & 5 \\ -2 & -4 & 0 & 6 \\ -3 & -5 & -6 & 0 \end{bmatrix} \xrightarrow{\mathrm{vec}_{\mathrm{o}}} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}$$

Definition

Let $A \in \mathcal{S}(G)$ and $p = |E(\overline{G})|$. The SSP verification matrix $\Psi_{\mathrm{S}}(A)$ of A is a $p \times \binom{n}{2}$ matrix whose rows are composed of $\mathrm{vec}_{\mathrm{o}}(AE_{ij} - E_{ij}A)$ for $ij \in E(\overline{G})$.

A has the SSP $\iff$ $\Psi_{\mathrm{S}}(A)$ has full row-rank.

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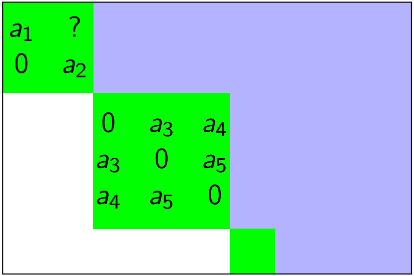
$$A \text{ has the SSP} \iff \Psi_\text{S}(A) \text{ has full row-rank.}$$

Key idea

The verification matrix *always* has full row-rank if the green parts are always invertible and the white part is zero.



Forcing process: general setting

Let G be a graph.

- Each edge on G is considered as "black".
- Each non-edge of G is initially white but can possibly be blue in the process.
- Color change rules will be defined later.

Theorem (L, Oblak, and Šmigoc 2020)

If starting with all white and ending with all non-edge blue, then every $A \in \mathcal{S}(G)$ has the SSP.

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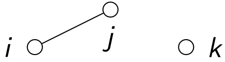
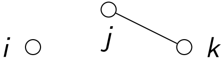
Forcing process: Rule 1

If

- ik is black or blue, and
- there is a unique black-white connection

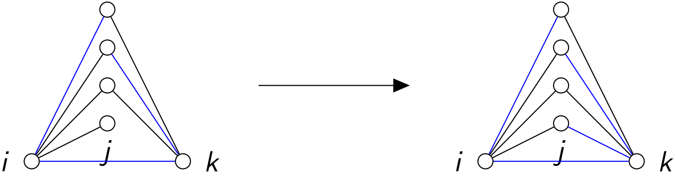


or



between i and k (say the former case)

then jk turns blue.

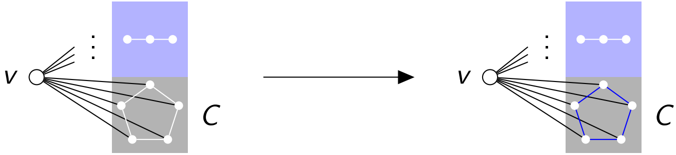


Forcing process: Rule 2

If

- $G[N(v)]$ contains a white odd cycle C as a component, and
- there are exactly two black-white connection between v and each vertex on C,

then the edges in $E(C)$ turn blue.

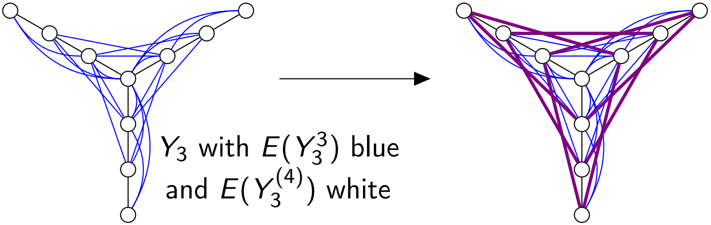


Forcing process: Rule 3

If

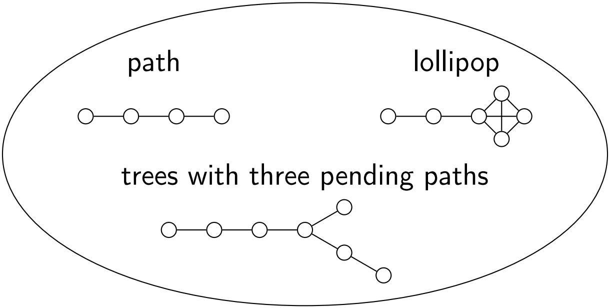
- G contains an induced subgraph Y_h,
- edges in $E(Y_h^h)$ are blue, edges in $E(Y_h^{(h+1)})$ are white, and
- there are exactly two black-white connections between the two endpoints of each edge in $E(Y_h^{(h)})$,

then the edges in $E(Y_h^{(h+1)})$ turn blue.

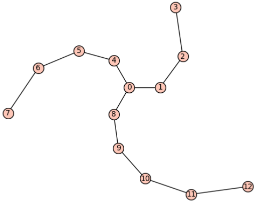


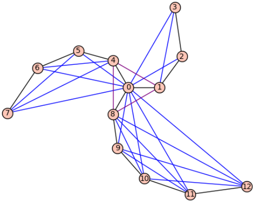
Graphs that guarantee the SSP

For the following graphs G, every $A \in \mathcal{S}(G)$ has the SSP.

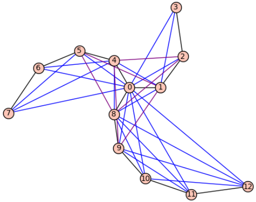


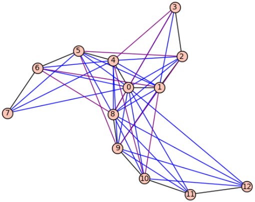
This includes all graphs with $q(G) = n - 1$.



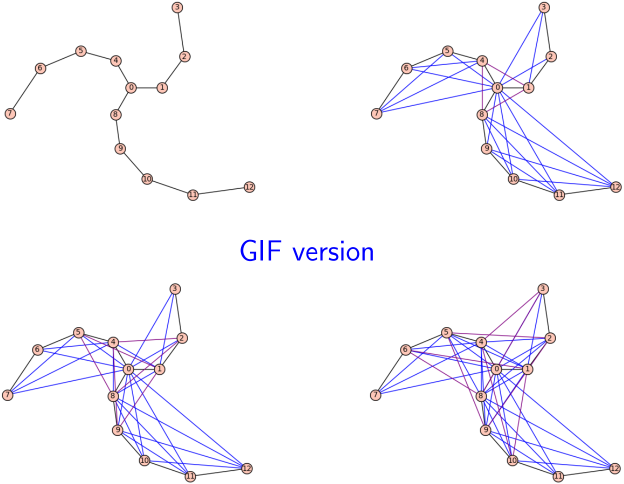


GIF version





Thanks!



GIF version

Thanks!

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