Spectral Clustering: Theory and Practice

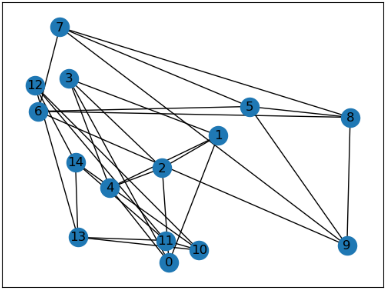
Jephian C.-H. Lin 林晉宏

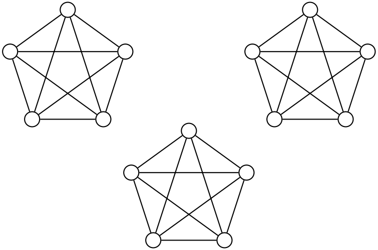
Department of Applied Mathematics, National Sun Yat-sen University

July 12, 2024

Colloquium at the University of Regina, Regina, SK

How to find the components?

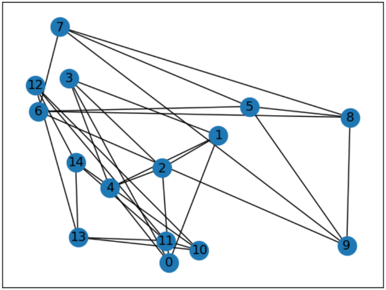


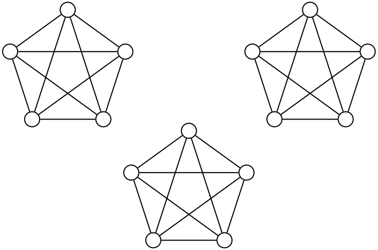


G_1

☺ Breadth-first search ☺ Laplacian matrix

How to find the components?

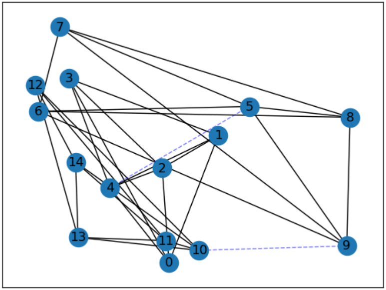


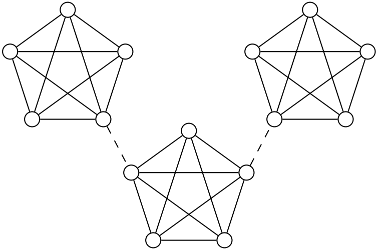


G_1

☺ Breadth-first search ☺ Laplacian matrix

How to find the clusters?

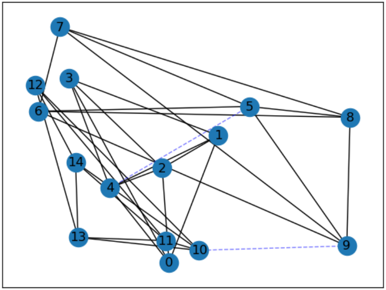


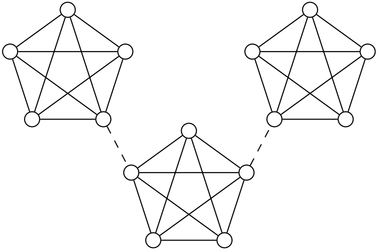


G_2

☹ Breadth-first search ☺ Laplacian matrix

How to find the clusters?





G_2

☹ Breadth-first search ☺ Laplacian matrix

Miroslav Fiedler
1926–2015

Known for

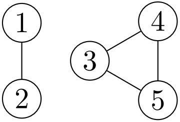
- algebraic connectivity,
- Fiedler vector,
- and more.

Have impact on

- graph partition,
- spectral clustering,
- image segmentation,
- and more.

Source: MacTutor `https://mathshistory.st-andrews.ac.uk/Biographies/Fiedler/`

Laplacian matrix



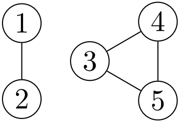
$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$

Definition

Let G be a graph on n vertices. The Laplacian matrix of G is the $n \times n$ matrix $L(G) = \left[\ell_{i,j}\right]$ such that

$$\ell_{i,j} = \begin{cases} -1 & \text{if } \{i,j\} \in E(G), \\ \deg_G(i) & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Laplacian matrix



$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$

Proposition

1. $L\mathbf{1} = \mathbf{0}$.
2. $\mathbf{x}^\top L\mathbf{x} = \sum_{\{i,j\}\in E}(x_i - x_j)^2$, *which means L is PSD.*
3. $L\mathbf{x} = \mathbf{0} \iff \mathbf{x}^\top L\mathbf{x} = 0$.

Example

For $G = K_2 \dot\cup K_3$,

$$\mathbf{x}^\top L\mathbf{x} = (x_1 - x_2)^2 + (x_3 - x_4)^2 + (x_4 - x_5)^2 + (x_3 - x_5)^2.$$

Count the number of components by the Laplacian matrix

Theorem (Fiedler 1973, Anderson and Morley 1971)

Let G be a graph and $L = L(G)$. Then $\mathrm{null}(L)$ is the number of components of G, and

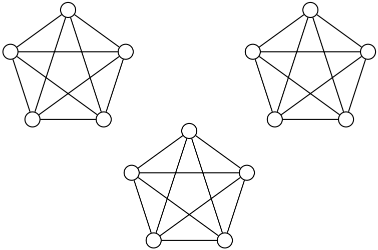
$$\ker(L) = \mathrm{span}\{\phi_{X_1}, \ldots, \phi_{X_k}\},$$

where $X_1, \ldots, X_k$ are the vertex sets of the components of G.

Example

For $G = K_2 \dot\cup K_3$,

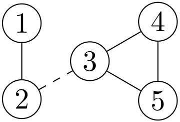
$$\mathrm{spec}(L) = \{0, 0, 2, 3, 3\} \text{ and } \ker(L) = \mathrm{span}\left\{\begin{bmatrix}1\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\\1\end{bmatrix}\right\}.$$



G_1

$$\mathrm{spec}(L) = \{0,0,0,5,\ldots\} \text{ and } \ker(L) = \mathrm{span}\left\{ \begin{bmatrix} \mathbf{1}_5 \\ \mathbf{0}_5 \\ \mathbf{0}_5 \end{bmatrix}, \begin{bmatrix} \mathbf{0}_5 \\ \mathbf{1}_5 \\ \mathbf{0}_5 \end{bmatrix}, \begin{bmatrix} \mathbf{0}_5 \\ \mathbf{0}_5 \\ \mathbf{1}_5 \end{bmatrix} \right\}.$$

Weighted Laplacian matrix



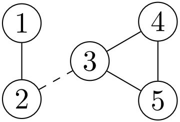
$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1.1 & -0.1 & 0 & 0 \\ 0 & -0.1 & 2.1 & -1 & -1 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$

Definition

Let G be a weighted graph on n vertices with weights $w_{i,j}$. The weighted Laplacian matrix of G is the $n \times n$ matrix $L(G) = \left[\ell_{i,j}\right]$ such that

$$\ell_{i,j} = \begin{cases} -w_{i,j} & \text{if } \{i,j\} \in E(G), \\ \sum_{k:k\sim i} w_{i,k} & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Weighted Laplacian matrix



$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1.1 & -0.1 & 0 & 0 \\ 0 & -0.1 & 2.1 & -1 & -1 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$

Proposition

1. $L\mathbf{1} = \mathbf{0}$.
2. $\mathbf{x}^\top L\mathbf{x} = \sum_{\{i,j\}\in E} w_{i,j}(x_i - x_j)^2$, *which means L is PSD.*
3. $L\mathbf{x} = \mathbf{0} \iff \mathbf{x}^\top L\mathbf{x} = 0$.

Example

For $G = K_2 \dot\cup K_3 + \{2,3\}$,

$$\mathbf{x}^\top L\mathbf{x} = (x_1 - x_2)^2 + (x_3 - x_4)^2 + (x_4 - x_5)^2 + (x_3 - x_5)^2 + 0.1(x_2 - x_3)^2.$$

Count the number of components by the Laplacian matrix

Theorem (Fiedler 1973, Anderson and Morley 1971)

Let G be a graph and $L = L(G)$. Then $\mathrm{null}(L)$ is the number of components of G, and

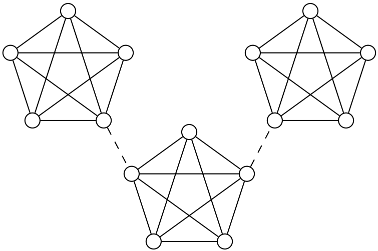
$$\ker(L) = \mathrm{span}\{\phi_{X_1}, \ldots, \phi_{X_k}\},$$

where $X_1, \ldots, X_k$ are the vertex sets of the components of G.

Example

For $G = K_2 \dot{\cup} K_3 + \{2, 3\}$,

$$\mathrm{spec}(L) = \{0, 0.08, 2.05, 3, 3.07\} \text{ and } \ker(L) = \mathrm{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right\}.$$



G_2

$$\mathrm{spec}(L) = \{0, 0.02, 0.06, 5, \ldots\} \text{ and } \ker(L) = \{\mathbf{1}\}.$$

Just a small perturbation: first few eigvals and eigvecs

$$\{0,0,0\} \rightarrow \{0,0.02,0.06\}$$

$$\begin{bmatrix} 0.45 & 0 & 0 \\ 0.45 & 0 & 0 \\ 0.45 & 0 & 0 \\ 0.45 & 0 & 0 \\ 0.45 & 0 & 0 \\ 0 & 0.45 & 0 \\ 0 & 0.45 & 0 \\ 0 & 0.45 & 0 \\ 0 & 0.45 & 0 \\ 0 & 0.45 & 0 \\ 0 & 0 & 0.45 \\ 0 & 0 & 0.45 \\ 0 & 0 & 0.45 \\ 0 & 0 & 0.45 \\ 0 & 0 & 0.45 \end{bmatrix} \rightarrow \begin{bmatrix} 0.26 & 0.32 & -0.18 \\ 0.26 & 0.32 & -0.18 \\ 0.26 & 0.32 & -0.18 \\ 0.26 & 0.32 & -0.18 \\ 0.26 & 0.31 & -0.17 \\ 0.26 & 0.01 & 0.36 \\ 0.26 & -0.00 & 0.37 \\ 0.26 & -0.00 & 0.37 \\ 0.26 & -0.00 & 0.37 \\ 0.26 & -0.01 & 0.36 \\ 0.26 & -0.31 & -0.17 \\ 0.26 & -0.32 & -0.18 \\ 0.26 & -0.32 & -0.18 \\ 0.26 & -0.32 & -0.18 \\ 0.26 & -0.32 & -0.18 \end{bmatrix}$$

Count the number of clusters by the Laplacian matrix

Theorem

Let G be a weighted graph and $L = L(G)$. Then the number of zeroish eigenvalues suggests the number of clusters of G, and vertices in the same cluster share similar values in each eigenvector.

Example

For $G = K_2 \dot\cup K_3 + \{2, 3\}$,

$$\operatorname{spec}(L) = \{0, 0.08, 2.05, 3, 3.07\} \text{ and } 0, 0.08 \to \begin{bmatrix} 0.45 \\ 0.45 \\ 0.45 \\ 0.45 \\ 0.45 \end{bmatrix}, \begin{bmatrix} 0.57 \\ 0.52 \\ 0.35 \\ 0.37 \\ 0.37 \end{bmatrix}$$

Spectral embedding algorithm

Algorithm

Input: a *weighted graph G* on n vertices and a *targeted dimension d*

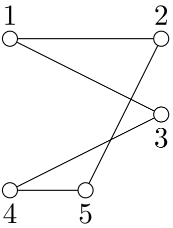
Output: an $n \times d$ *matrix Y*

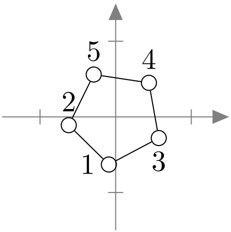
Steps:

1. $L \leftarrow L(G)$.
2. *Find the first d eigenvalues $\lambda_1 \leq \cdots \leq \lambda_d$ and the corresponding eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_d$.*
3. $Y \leftarrow$ *the matrix composed of columns $\mathbf{u}_1, \ldots, \mathbf{u}_d$.*
4. *Let $\mathbf{y}_1, \ldots, \mathbf{y}_n$ be the rows of Y. Define the embedding $f : V(G) \rightarrow \mathbb{R}^d$ by $i \mapsto \mathbf{y}_i$.*

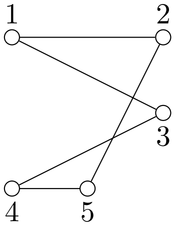
Remark

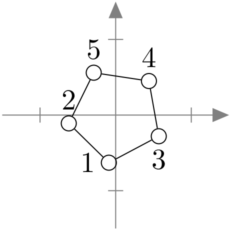
1. Since $\mathbf{u}_1 = \frac{1}{\sqrt{n}}\mathbf{1}$, people often take $\lambda_2 < \cdots < \lambda_{d+1}$ and their eigenvectors instead.
2. Main idea: *The embedding try to put adjacent vertices together—the stronger the weight, the closer they are.*





$$L = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 & -1 \\ -1 & 0 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & -1 & 0 & -1 & 2 \end{bmatrix} \to Y = \begin{bmatrix} -0.09 & -0.63 \\ -0.62 & -0.11 \\ 0.57 & -0.28 \\ 0.44 & 0.45 \\ -0.29 & 0.56 \end{bmatrix}$$





The spectral embedding algorithm occurs in

- graph drawing (Hall 1970, Koren 2005),
- graph partitioning (Pothen, Simon, and Liou 1990),
- graph ordering (Juvan and Mohar 1992),
- spectral clustering (Shi and Malik 2000),
- Laplacian eigenmap (Belkin and Niyogi 2003),
- and more.

How to draw a graph properly?

Problem

Given a weighted graph G on n vertices and a target dimension d, find an $n \times d$ matrix Y such that

$$\begin{aligned} \textbf{minimize} \quad & \operatorname{tr}(Y^\top LY) = \sum_{\{i,j\} \in E(G)} \|\mathbf{y}_i - \mathbf{y}_j\|^2 \\ \textbf{subject to} \quad & \mathbf{1}^\top Y = \mathbf{0}^\top \text{ and} \\ & Y^\top Y = I. \end{aligned}$$

Intuition:

- $\operatorname{tr}(T^\top LY)$: the potential energy of a spring-mass system.
- $\mathbf{1}^\top Y = \mathbf{0}^\top$: centered at the origin.
- $Y^\top Y = I$: normalized each coordinate.

Spectral embedding algorithm generates the answer!

How to draw a graph properly?

Problem

Given a weighted graph G *on* n *vertices and a target dimension* d*, find an* $n \times d$ *matrix* Y *such that*

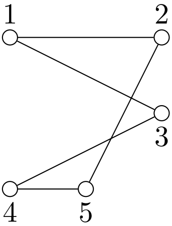
$$\begin{array}{rl} \textbf{minimize} & \operatorname{tr}(Y^\top LY) = \sum_{\{i,j\}\in E(G)} \|\mathbf{y}_i - \mathbf{y}_j\|^2 \\ \textbf{subject to} & \mathbf{1}^\top Y = \mathbf{0}^\top \textit{ and} \\ & Y^\top Y = I. \end{array}$$

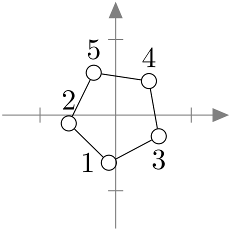
Intuition:

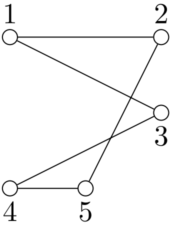
- $\operatorname{tr}(T^\top LY)$: the potential energy of a spring-mass system.
- $\mathbf{1}^\top Y = \mathbf{0}^\top$: centered at the origin.
- $Y^\top Y = I$: normalized each coordinate.

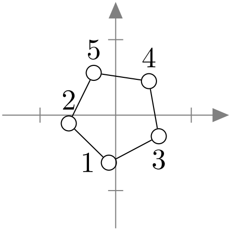
Spectral embedding algorithm generates the answer!

Some exmples


1
2
3
4
5


5
4
2
1
3





Thanks!

Kaohsiung

$\lambda_2^L = 1.6$

ILAS2025

4364962···

https://ilas2025.tw

References I

W. N. Anderson, Jr. and R. D. Morley.
Eigenvalues of the Laplacian of a graph.
University of Maryland Technical Report, TR-71-45, 1971.

M. Belkin and P. Niyogi.
Laplacian eigenmaps for dimensionality reduction and data representation.
Neural Computation, 15:1373–1396, 2003.

M. Fiedler.
Algebraic connectivity of graphs.
Czechoslovak Math. J., 23:298–305, 1973.

M. Fiedler.
Eigenvalues of acyclic matrices.
Czechoslovak Math. J., 25:607–618, 1975.

References II

M. Fiedler.
A property of eigenvectors of nonnegative symmetric matrices and its application to graph theory.
Czechoslovak Math. J., 25:619–633, 1975.

K. M. Hall.
An r-dimensional quadratic placement algorithm.
Management Science, 17:219–229, 1970.

M. Juvan and B. Mohar.
Optimal linear labelings and eigenvalues of graphs.
Discrete Appl. Math., 36:153–168, 1992.

Y. Koren.
Drawing graphs by eigenvectors: Theory and practice.
Computers & Mathematics with Applications, 49:1867–1888, 2005.

References III

A. Pothen, H. D. Simon, and K.-P. Liou.
Partitioning sparse matrices with eigenvectors of graphs.
SIAM J. Matrix Anal. Appl., 11:430–452, 1990.

J. Shi and J. Malik.
Normalized cuts and image segmentation.
IEEE Transactions on Pattern Analysis and Machine Intelligence, 22:888–905, 2000.