On the distance matrices of the CP graphs

Jephian C.-H. Lin

Department of Applied Mathematics, National Sun Yat-sen University

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Distance matrix

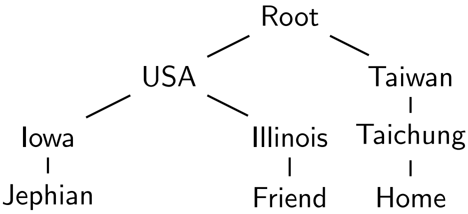
- Let G be a connected simple graph on vertex set $V = \{1, \ldots, n\}$.
- The distance $d_G(i, j)$ between two vertices i, j on G is the length of the shortest path.
- The distance matrix of G is an $n \times n$ matrix

$$\mathcal{D} = \left[d_G(i, j)\right].$$

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 2 & 3 \\ 2 & 1 & 0 & 1 & 2 \\ 3 & 2 & 1 & 0 & 1 \\ 4 & 3 & 2 & 1 & 0 \end{bmatrix}$$

Motivation: Pierce's loop switching scheme

- How two build a phone call between two persons?
  - Root-USA-Iowa-Jephian
  - Root-USA-Illinois-Friend
  - Root-Taiwan-Taichung-Home



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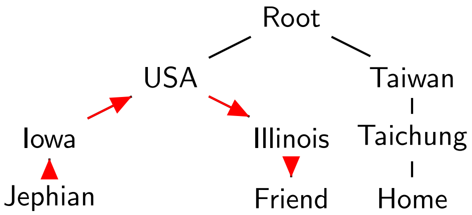

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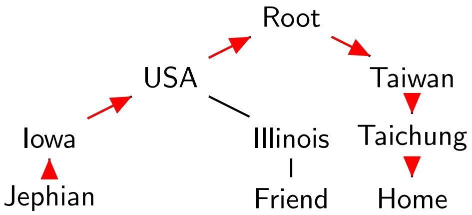
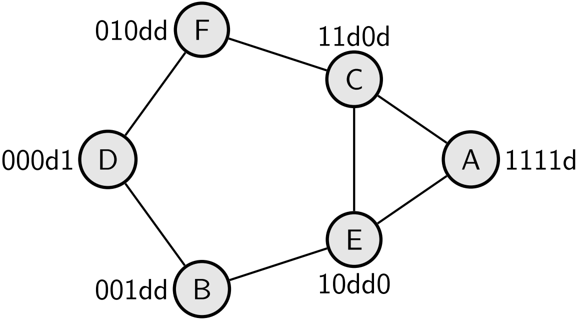


Graham and Pollak's model

- A model works for all graphs, not limited to trees.
- Each vertex is assigned with an address, and the distance between two vertices is the Hamming distance of the address.
- Find the neighbor that decrease the Hamming distance.



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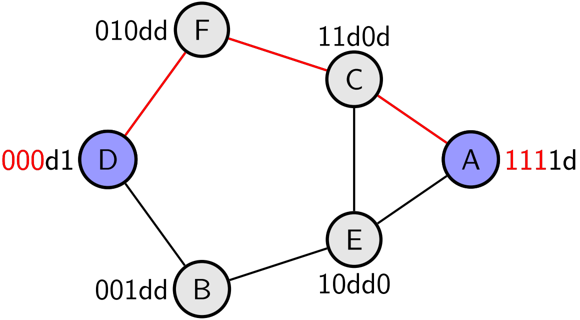

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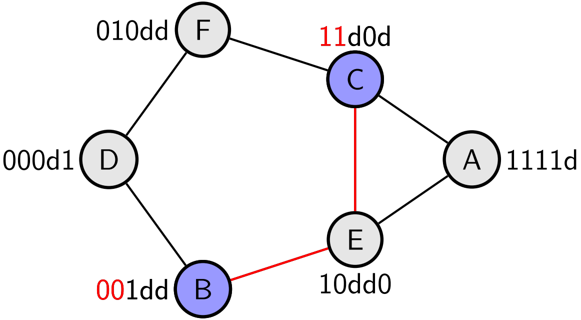

Matrix representation of each digit

- Consider the k-th digit of each vertex. Let α_k be the ones; let β_k be the zeros. Let B_k be the adjacency matrix of the complete bipartite graph between α_k and β_k.
- The i,j-entry of B_k indicate the contribution of the k-th digit to the Hamming distance.

A	1111d
B	001dd
C	11d0d
D	000d1
E	10dd0
F	010dd

$$B_1 = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

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$$B_3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Equivalent definitions

Let G be a graph and $\mathcal{D}$ its distance matrix. The following questions are equivalent.

Q1: Find an addressing scheme of length t such that the Hamming distance of the strings is the distance of the vertices.

Q2: Find $B_1, \ldots, B_t$ such that $\sum_{k=1}^{t} B_k = \mathcal{D}$, where each B_k is the adjacency matrix of a complete bipartite graph.

Length of the address

Theorem (Graham and Pollak 1971)

Let G be a graph and $\mathcal{D}$ its distance matrix. Then such an address always exist and its length is at least

$$\max\{n_-, n_+\},$$

where n_-, n_+ are the negative and positive inertia.

Corollary (Graham and Pollak 1971)

When G is a complete graph or a tree, then the minimum length of the address is $|V(G)| - 1$.

Length of the address

Conjecture (Graham and Pollak 1971)

For any graph on n vertices, the address can be chosen with length at most $n-1$.

Theorem (Winkler 1983)

The squashed cube conjecture is true.

How to compute the inertia?

- Let A be a matrix. The k-th leading minor D_k of A is the determinant of the submatrix on the first k rows/columns.
- Suppose $D_1, \dots, D_n$ are the leading minors with $D_n \neq 0$. Jones showed that there are no two consecutive zeros.

Theorem (Jones 1950)

Let A be a nonsingular symmetric $n \times n$ matrix with principal leading minors $D_1, \dots, D_n$. Then n_- is the number of sign changes in the sequence $1, D_1, \dots, D_n$, ignoring the zeros in the sequence.

Trees and complete graphs

Theorem (Graham and Pollak 1971)

For every tree T on n vertices,

$$\det_{\mathcal{D}}(T) = (-1)^{n-1}(n-1)2^{n-2}.$$

Proposition

Let K_n be the complete graph on n vertices. Then

$$\det_{\mathcal{D}}(K_n) = (-1)^{n-1}(n-1).$$

What other graphs whose distance determinant only depends on the order?

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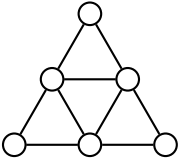
What other graphs whose distance determinant only depends on the order?

The k-tree

Start with K_k with vertex labeled as $1, \ldots, k$. Then for $j = k+1, \ldots, n$, add a new vertex j inductively such that

- j joins with a k-clique.

2-tree



The linear k-tree

Start with K_k with vertex labeled as $1, \ldots, k$. Then for $j = k+1, \ldots, n$, add a new vertex j inductively such that

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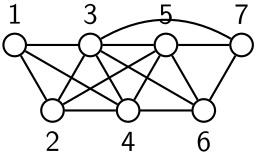
- j joins with the last vertex $j-1$.

linear 3-tree



The backward degrees are $0, 1, \ldots, k-1, k, \ldots, k$.

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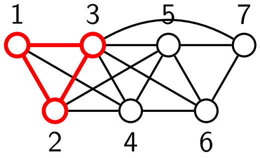

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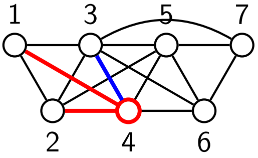

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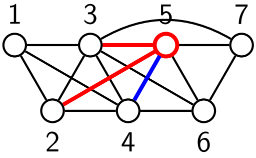

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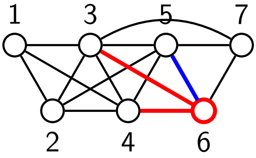

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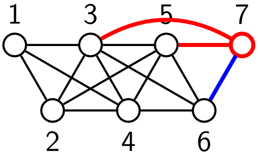


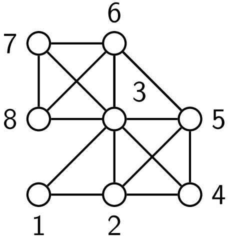
The backward degrees are $0, 1, \ldots, k-1, k, \ldots, k$.

The CP graph

Let $s = q_1, \ldots, q_n$ be a given backward degree sequence. Start with K_2 with vertex labeled as $1, 2$. Then for $j = 3, \ldots, n$, add a new vertex j inductively such that

- j joins with a q_j-clique.
- j joins with the last $q_j - 1$ vertex.

$$s = 0, 1, 2, 2, 3, 2, 2, 3$$



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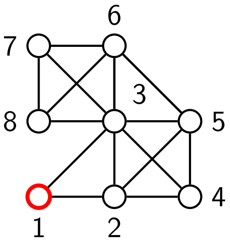

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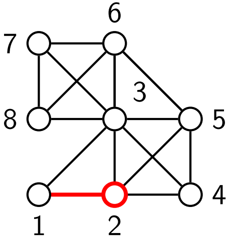

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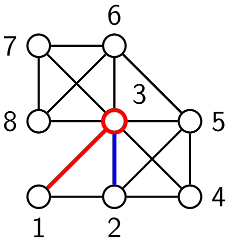

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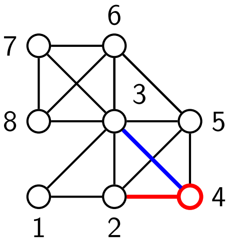

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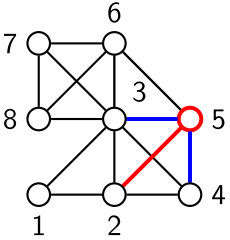

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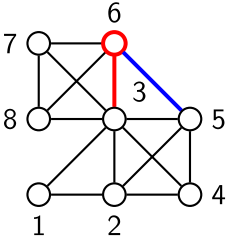

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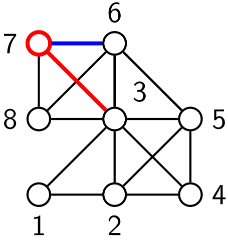

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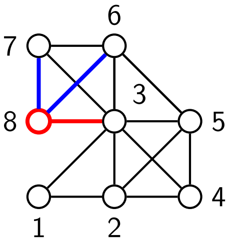
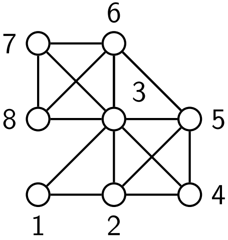


Special case: 2-clique path

- 2-clique path: combining cliques of sizes $p_1, \dots, p_m$ by distinct edges
- Let $[a] = \{1, \dots, a\}$ and $[a, b] = [a, a+1, \dots, b]$.
- Then $s = 0, 1, [2, p_1 - 1], [2, p_2 - 1], \dots, [2, p_m - 1]$.
- Denoted as $s = 2 : p_1, \dots, p_m$.

s	
$0, 1$	
2	$p_1 = 3$
$2, 3$	$p_2 = 4$
2	$p_3 = 3$
$2, 3$	$p_4 = 4$



Fixed b_j and flexible a_j

Let $s = q_1, \ldots, q_n$. In a CP graph of s, each vertex j has q_j backward neighbors:

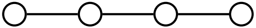
- $q_j - 1$ fixed neighbors, and
- 1 flexible neighbor.

That is,

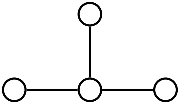
$$N(j) \cap [j-1] = \{a_j\} \,\dot{\cup}\, \{b_j, j-1\},$$

where $b_j = j - q_j + 1$ is fixed and a_j may vary.

The family $\mathcal{CP}_s$ includes all CP graphs build from the sequence s.



$\det_{\mathcal{D}}(P_4) = -12$



$\det_{\mathcal{D}}(K_{1,3}) = -12$

Theorem (Graham and Pollak 1971)

For every tree T on n vertices,

$$\det_{\mathcal{D}}(T) = (-1)^{n-1}(n-1)2^{n-2}.$$

History

- Graham and Pollak 1971: $\det_{\mathcal{D}}(T)$ of a tree T only depends on n. [Yan and Yeh gave a simpler proof in 2006.]
- Graham, Hoffman, and Hosoya 1977: $\det_{\mathcal{D}}(G)$ only depends on its blocks, but not how blocks attached together.
- Bapat, Kirkland, and Neumann: weighted distance matrix of a tree.
- Bapat, Lal, and Pati; Yan and Yeh: q-analog and the q-exponential distance matrix of a tree.

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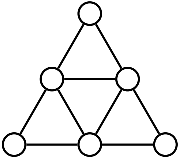
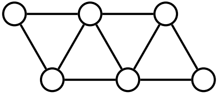
How about graphs without a cut vertex?

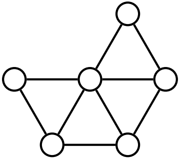
How about k-trees?



$\det_{\mathcal{D}}(G_1) = -8$

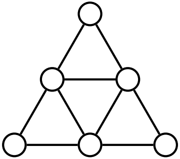


$\det_{\mathcal{D}}(G_2) = -9$



$\det_{\mathcal{D}}(G_3) = -9$

How about k-trees?



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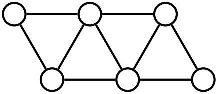

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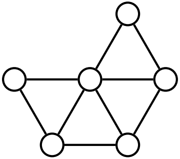
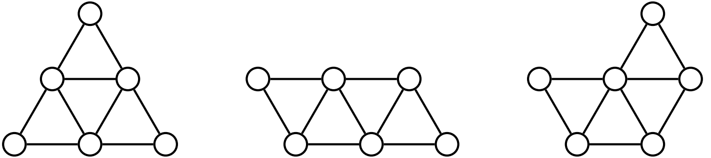


$\det_{\mathcal{D}}(G_3) = -9$

Linear 2-trees seems promising.

How about k-trees?



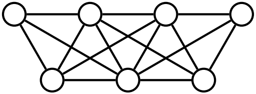
$\det_{\mathcal{D}}(G_1) = -8$ $\det_{\mathcal{D}}(G_2) = -9$ $\det_{\mathcal{D}}(G_3) = -9$

Theorem (Cheng and L 2018+)

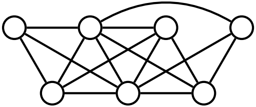
For every linear 2-tree G on n vertices,

$$\det_{\mathcal{D}}(G) = (-1)^{n-1}\left(1+\left\lfloor\frac{n-2}{2}\right\rfloor\right)\left(1+\left\lceil\frac{n-2}{2}\right\rceil\right).$$

How about linear k-tree?



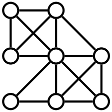
$\det_{\mathcal{D}}(G_1) = 4$



$\det_{\mathcal{D}}(G_2) = 6$

2-clique paths

Given $p_1, \dots, p_m \geq 3$, a 2-clique path is obtained from a sequence of complete graphs $K_{p_1}, \dots, K_{p_m}$ by gluing an edge of K_{p_i} to an edge of $K_{p_{i+1}}$, $i = 1, \dots, m$; an edge cannot be glued twice. The family $\mathcal{CP}_{2:p_1,\dots,p_m}$ collects all such graphs.



$G \in \mathcal{CP}_{2:3,4,3,4}$
$\det_{\mathcal{D}}(G) = (1+1+1)(1+2+2) = 15$

Theorem (Cheng and L 2018+)

For every graph $G \in \mathcal{CP}_{2:p_1,\dots,p_m}$ on n vertices,

$$\det_{\mathcal{D}}(G) = (-1)^{n-1}\left(1 + \sum_{k \text{ odd}} (p_k - 2)\right)\left(1 + \sum_{k \text{ even}} (p_k - 2)\right).$$

The CP graphs

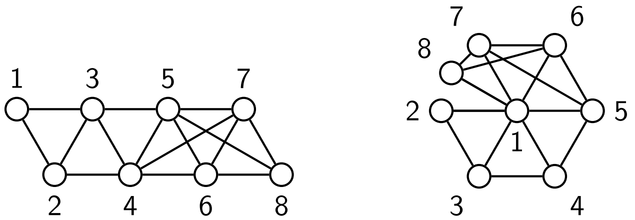
Let $s = q_1, \ldots, q_n$.

- Vertex k has q_k backward neighbors: $q_k - 1$ fixed and 1 flexible.

$$N(j) \cap [j-1] = \{a_j\} \,\dot\cup\, \{b_j, j-1\},$$

where $b_j = j - q_j + 1$ is fixed and a_j may vary.

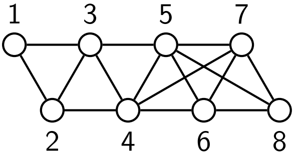
- Examples of $\mathcal{CP}_{0,1,2,2,2,2,3,3}$:



Reducing matrix

- The reducing matrix E of a CP graph is an $n \times n$ matrix whose k-th column is

$$\begin{cases} \mathbf{e}_k & \text{if } k \in \{1,2\}, \\ \mathbf{e}_k - \mathbf{e}_{a_k} - \mathbf{e}_{k-1} + \mathbf{e}_{a_{k-1}} & \text{if } k \geq 3. \end{cases}$$



$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Theorem (Cheng and L 2018+)

Let s be a sequence of backward degrees. For any $G \in \mathcal{CP}_s$ with the distance matrix $\mathcal{D}$ and the reducing matrix E, the matrix

$$E^\top \mathcal{D} E$$

only depends on s.

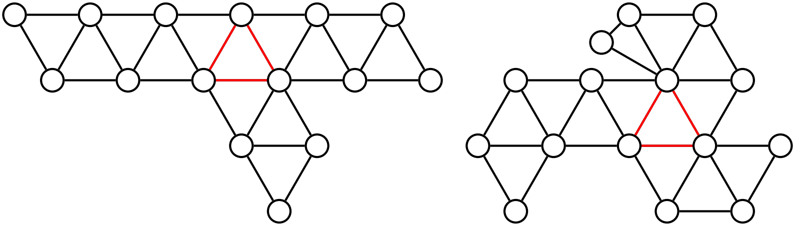
- Note that E is an upper triangular matrix with every diagonal entry equal to 1.

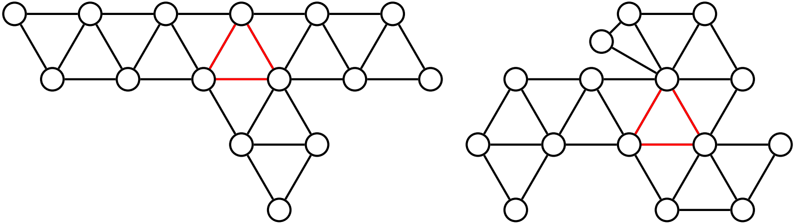
Corollary (Cheng and L 2018+)

Let s be a sequence of backward degrees. Then

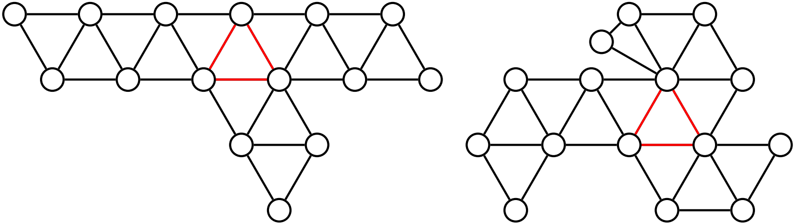
$$\det_{\mathcal{D}}(G) \text{ and } \mathrm{inertia}_{\mathcal{D}}(G)$$

are independent of the choice of $G \in \mathcal{CP}_s$.





$$\det_{\mathcal{D}}(G_1) = \det_{\mathcal{D}}(G_2) = 56$$



$$\det_{\mathcal{D}}(G_1) = \det_{\mathcal{D}}(G_2) = 56$$

Thank you!

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