The Nine Dragon Tree Conjectures

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Abstract

Given a (not necessarily simple) graph G, the *fractional arboricity* of G is defined as

$$\gamma_f(G) = \max_{X \subseteq V(G)} \frac{|E(G[X])|}{|X|-1}.$$

By the well-known Nash-William Theorem, G decomposes into $\lceil \gamma_f(G) \rceil$ forests, i.e., $E(G)$ can be covered by the edge sets of $\lceil \gamma_f(G) \rceil$ forests. This result is sharp in the sense that $E(G)$ cannot be covered by fewer forests. However, by taking the ceiling of $\gamma_f(G)$, it seems that some information about the graph contained in the parameter $\gamma_f(G)$ got lost. For example, if $\gamma_f(G_1) = 2.01$ and $\gamma_f(G_2) = 2.99$, then what we can derive from Nash-William Theorem is that both G_1 and G_2 decompose into three forests. Is it possible that by using the fact that $\gamma_f(G_1)$ is considerably less than $\gamma_f(G_2)$, one can show that G_1 has a forest decomposition which is 'better' (or 'worse') than that of G_2? Intuitively, if $\gamma_f(G) = k + \epsilon$, where $0 < \epsilon$ is close to 0, then G can almost be covered by k forests. Is it possible that G be decomposed into k forests plus an ϵ-forest? The question is meaningful only if there is a proper definition of an ϵ-forest. In this talk, we propose two conjectures concerning the decomposition of graphs.

On the top of the Longevity Mountain in Kaohsiung, Taiwan, there is a giant banyan tree, called the 'Nine Dragon Tree'. It resembles a number of dragons resting on the top of the mountain. Many graph theorists visited this tree and are amazed at the fact that this tree is far from being acyclic. We would like to associate the conjectures below with this beautiful tree, whose fractional arboricity remains a mystery.

A graph G is called (k, d)-coverable (respectively, strongly (k, d)-coverable) if G decomposes into k forests plus a graph of maximum degree at most d (respectively, $k + 1$ forests with one of them having maximum degree at most d).

Conjecture 1 *[The Weak NDT (Nine Dragon Tree) Conjecture] For any positive integer k and real number $0 \le \epsilon < 1$, there is a constant d such that every graph G with $\gamma_f(G) \le k + \epsilon$ is (k, d)-coverable (respectively, strongly (k, d)-coverable).*

If the conjecture is true, then one naturally wonder that for given k and ϵ, what is the smallest d for which the statement is true.

Conjecture 2 *[The Strong NDT (Nine Dragon Tree) Conjecture] Suppose G is a graph and k, d are positive integers. If $\gamma_f(G) \le k + \frac{d}{k+1+d}$ then G is (k, d)-coverable (respectively, strongly (k, d)-coverable).*

We prove two special cases of the Strong NDT Conjecture: If G is a graph with fractional arboricity at most $\frac{4}{3}$, then G decomposes into a forest and a matching. If G is a graph with fractional arboricity at most $\frac{3}{2}$, then G decomposes into a forest and

a linear forest. In particular, every planar graph of girth at least 8 decomposes into a forest and a matching, and every planar graph of girth at least 6 decomposes into a forest and a linear forest. This improves earlier results concerning decomposition of planar graphs. We also show that the bound in the conjecture above is sharp, i.e., for any $\epsilon > 0$, there is a graph G with $\gamma_f(G) < k + \frac{d}{k+d+1} + \epsilon$, and yet G cannot be decomposed into $k+1$ forests, with one of them having maximum degree at most d.

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