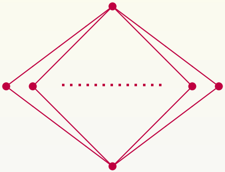
Families of Subsets with a Forbidden Subposet



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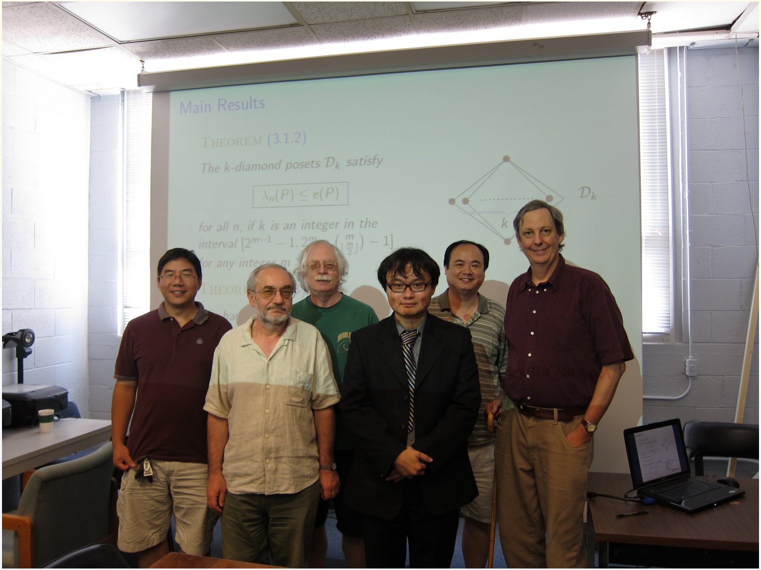
Workshop on Graph Theory and Combinatorics
Symposium for Young Combinatorialists
NSYSU Kaohsiung, August 10, 2012

UNIVERSITY OF SOUTH CAROLINA

Joint work with Wei-Tian Li, Academia Sinica



We celebrate the recent 60th birthday of Prof. Gerard Jennhwa Chang

For a poset P, we consider how large a family $\mathcal{F}$ of subsets of $[n] := \{1, \ldots, n\}$ we may have in the Boolean Lattice $\mathcal{B}_n : (2^{[n]}, \subseteq)$ containing no (weak) subposet P. We are interested in determining or estimating $\boxed{\mathrm{La}(n, P) := \max\{|\mathcal{F}| : \mathcal{F} \subseteq 2^{[n]}, P \not\subset \mathcal{F}\}.}$

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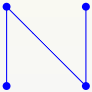
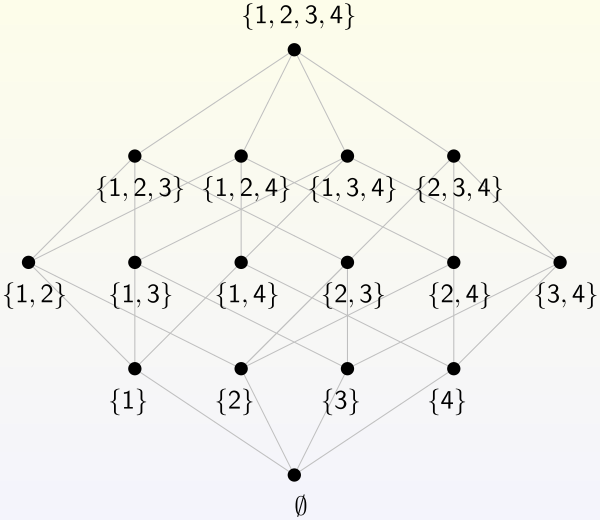
Example

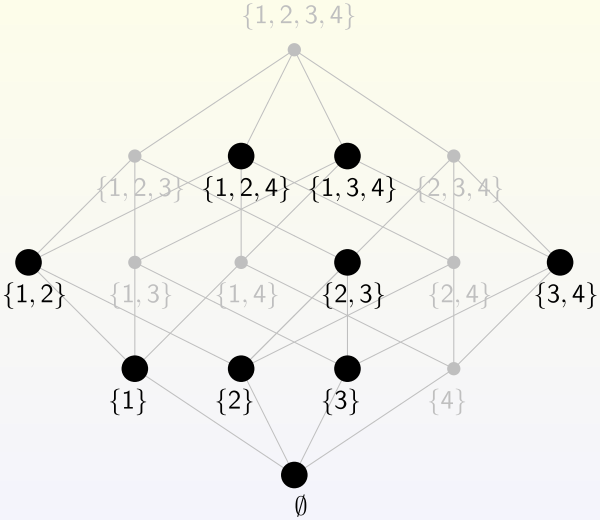


For the poset $P = \mathcal{N}$, $\mathcal{F} \not\supset \mathcal{N}$ means $\mathcal{F}$ contains no 4 subsets A, B, C, D such that $A \subset B$, $C \subset B$, $C \subset D$. Note that $A \subset C$ is allowed: The subposet does not have to be induced.

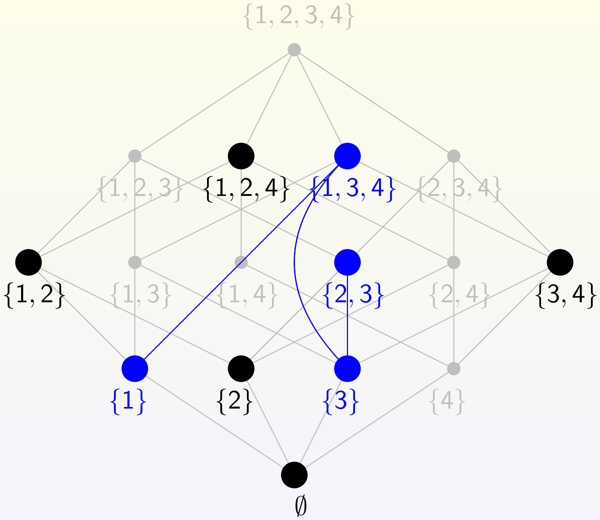
The Boolean Lattice $\mathcal{B}_4$



A Family of Subsets $\mathcal{F}$ in $\mathcal{B}_4$



$\mathcal{F}$ contains the poset $\mathcal{N}$



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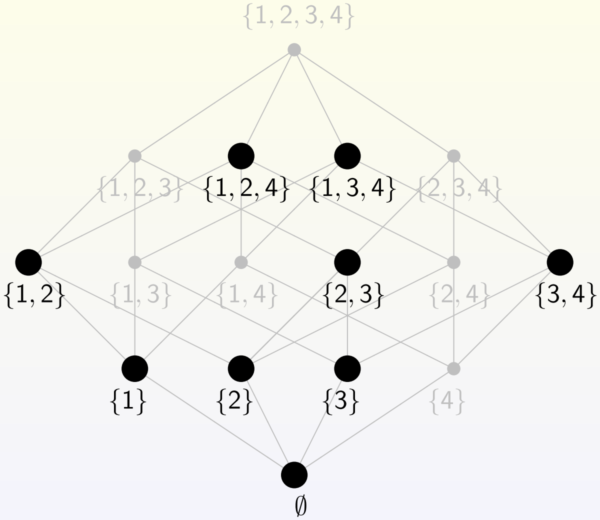
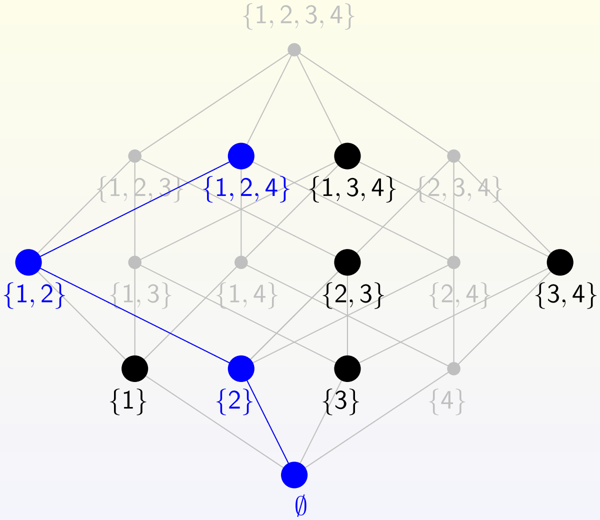
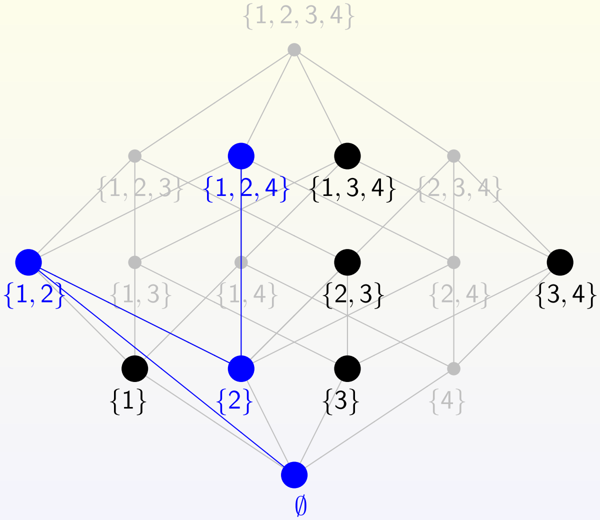


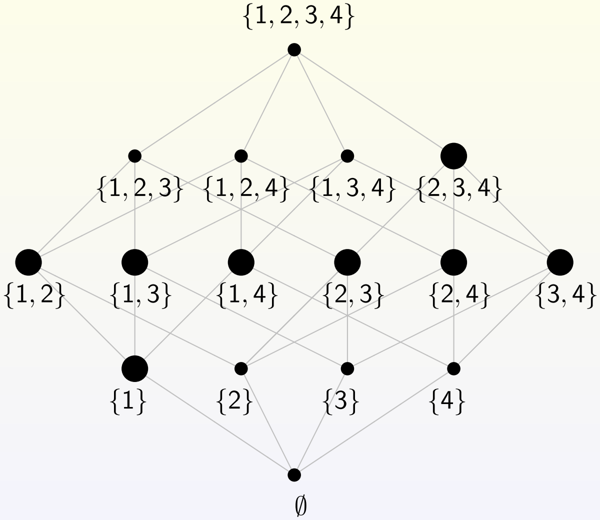
$\mathcal{F}$ Contains a 4-Chain $\mathcal{P}_4$



Hence, $\mathcal{F}$ Contains Another $\mathcal{N}$



A Large $\mathcal{N}$-free Family in $\mathcal{B}_4$



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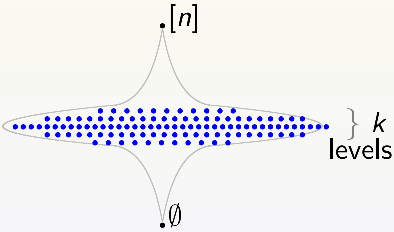
For many posets, $\mathrm{La}(n,P)$ is exactly equal to the sum of middle k binomial coefficients, denoted by $\Sigma(n,k)$.

Moreover, the largest families may be $\mathcal{B}(n,k)$, the families of subsets of middle k sizes.



Foundational results: Let $\mathcal{P}_k$ denote the k-element chain (path poset).

Theorem (Sperner, 1928)

For all n,

$$\mathrm{La}(n, \mathcal{P}_2) = \binom{n}{\lfloor \frac{n}{2} \rfloor},$$

and the extremal families are $\mathcal{B}(n, 1)$.

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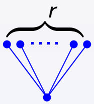
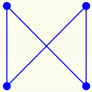
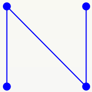
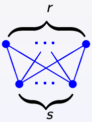
Theorem (Erdős, 1945)

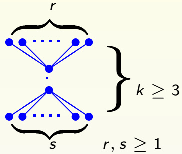
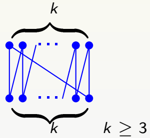
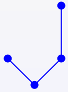
For general k and n,

$$\mathrm{La}(n, \mathcal{P}_k) = \Sigma(n, k-1),$$

and the extremal families are $\mathcal{B}(n, k-1)$.

Excluded subposet P	$\mathrm{La}(n, P)$	
$\mathcal{P}_2$ 	$\binom{n}{\lfloor \frac{n}{2} \rfloor}$	[Sperner, 1928]
Path $\mathcal{P}_k$, $k \geq 2$ 	$\Sigma(n, k-1)$ $\sim (k-1)\binom{n}{\lfloor \frac{n}{2} \rfloor}$	[P. Erdős, 1945]
r-fork $\mathcal{V}_r$ 	$\sim \binom{n}{\lfloor \frac{n}{2} \rfloor}$	[Katona-Tarján, 1981] [DeBonis-Katona 2007]

Excluded subposet P	$\mathrm{La}(n,P)$	
Butterfly B 	$\Sigma(n,2)$ $\sim 2\binom{n}{\lfloor \frac{n}{2} \rfloor}$	[DeBonis-Katona-Swanepoel, 2005]
$\mathcal{N}$ 	$\sim \binom{n}{\lfloor \frac{n}{2} \rfloor}$	[G.-Katona, 2008]
$\mathcal{K}_{r,s}(r,s \geq 2)$  $r,s \geq 2$	$\sim 2\binom{n}{\lfloor \frac{n}{2} \rfloor}$	[De Bonis-Katona, 2007]

Excluded subposet P	$\mathrm{La}(n, P)$	
Batons, $\mathcal{P}_k(s, t)$ 	$\sim (k-1)\binom{n}{\lfloor \frac{n}{2} \rfloor}$	[G.-Lu, 2009]
Crowns $\mathcal{O}_{2k}$ 	k even: $\sim \binom{n}{\lfloor \frac{n}{2} \rfloor}$ k odd: $\leq (1 + \frac{1}{\sqrt{2}})\binom{n}{\lfloor \frac{n}{2} \rfloor}$	[G.-Lu, 2009]
$\mathcal{J}$ 	$\Sigma(n, 2)$ $\sim 2\binom{n}{\lfloor \frac{n}{2} \rfloor}$	[Li, 2009]

Asymptotic behavior of $\mathrm{La}(n, P)$

Definition

$\pi(P) := \lim_{n\to\infty} \frac{\mathrm{La}(n,P)}{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$.

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Conjecture (G.-Lu, 2009)

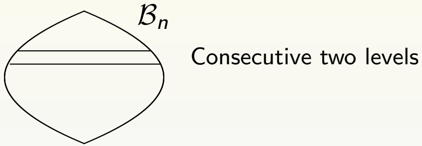
For all P, $\pi(P) = e(P)$, where

Definition

$e(P) :=$ max m *such that for all* n, $P \not\subset \mathcal{B}(n, m)$.

Example: Butterfly B

For all n, $\mathcal{B}(n,2) \not\supset \bowtie \Rightarrow e(\bowtie) = 2$,



while $\mathrm{La}(n, \bowtie) = \Sigma(n,2) \Rightarrow \pi(\bowtie) = 2$.

$\pi(P)$ and Height

Definition

The height $h(P)$ is the maximum size of any chain in P.

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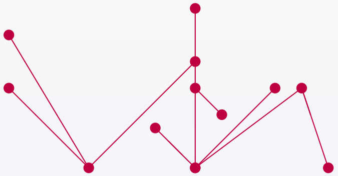
Theorem (Bukh, 2010; cf. G.-L.-Lu 2011)

Let T be a poset such that the Hasse diagram is a tree. Then

$$\pi(T) = e(T) = h(T) - 1.$$



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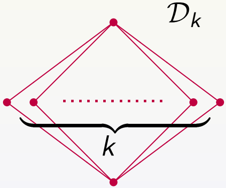
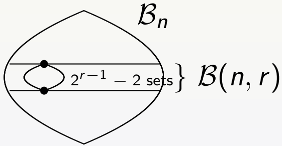
What about taller posets P?

For P of height 3 $\pi(P)$ cannot be bounded:

Example (Jiang, Lu) k-diamond poset $\mathcal{D}_k$



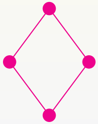


$\mathcal{B}(n, r) \not\supset \mathcal{D}_k$ for $k = 2^{r-1} - 1$, so $\pi(\mathcal{D}_k) \geq r$ if it exists.

On the Diamond $\mathcal{D}_2$

Problem

Despite considerable effort it remains open to determine the value $\pi(\mathcal{D}_2)$ or even to show it exists!



Easy bounds:

$\Sigma(n, 2) \le \mathrm{La}(n, \mathcal{D}_2) \le \Sigma(n, 3)$

$\Rightarrow 2 \le \pi(\mathcal{D}_2) \le 3$

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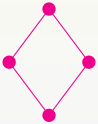

Easy bounds:
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The conjectured value of $\pi(\mathcal{D}_2)$ is its lower bound, $e(\mathcal{D}_2) = 2$.

The D_2 Diamond Theorem

Theorem (G.-L.-Lu, 2011)

As $n \to \infty$,

$$\Sigma(n,2) \le \mathrm{La}(n,\mathcal{D}_2) \le \left(2\frac{3}{11} + o_n(1)\right)\binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

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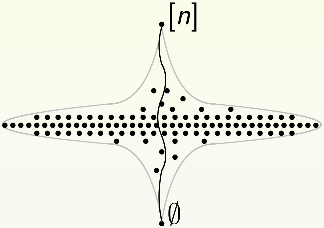
We prove this and most of our other results by considering, for a P-free family $\mathcal{F}$ of subsets of $[n]$, the average number of times a random full (maximal) chain in the Boolean lattice $\mathcal{B}_n$ meets $\mathcal{F}$, called the *Lubell function*.

Lubell Function

A *full chain* $\mathcal{C}$ in $\mathcal{B}_n$ is a collection of $n+1$ subsets as follows:

$$\emptyset \subset \{a_1\} \subset \cdots \subset \{a_1, \ldots, a_n\}.$$



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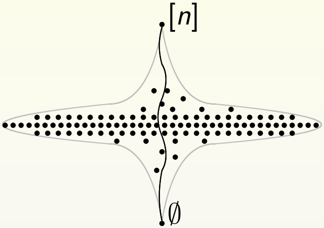

Definitions

Let $\mathcal{C} = \mathcal{C}_n$ be the set of full chains in $\mathcal{B}_n$.

For $\mathcal{F} \subset 2^{[n]}$, the height $h(\mathcal{F}) := \max_{C \in \mathcal{C}} |\mathcal{F} \cap C|$.

The Lubell function $\bar{h}(\mathcal{F}) := \mathrm{ave}_{C \in \mathcal{C}} |\mathcal{F} \cap C|$.

Lubell Function

Lemma (G.-L.-Lu, 2011)

Let $\mathcal{F}$ be a collection of subsets of $[n]$.

1. We have

$$\bar{h}(\mathcal{F}) = \sum_{A \in \mathcal{F}} \frac{1}{\binom{n}{|A|}}.$$

2. If $\bar{h}(\mathcal{F}) \leq m$, for some real number $m > 0$, then

$$|\mathcal{F}| \leq m \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

It means that the Lubell function provides an upper bound on $|\mathcal{F}|/\binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Lubell Function

Lemma

(ctd.) Let $\mathcal{F}$ *be a collection of subsets of* $[n]$.

3. If $\bar{h}(\mathcal{F}) \leq m$, *for some* **integer** $m > 0$, *then*

$$|\mathcal{F}| \leq \Sigma(n, m),$$

and equality holds if and only if

(1) $\mathcal{F} = \mathcal{B}(n, m)$ *when* $n + m$ *is* **odd**, *or*

(2) $\mathcal{F} = \mathcal{B}(n, m-1)$ *together with any* $\binom{n}{(n+m)/2}$ *subsets of sizes* $(n \pm m)/2$ *when* $n + m$ *is* **even**.

Lubell Function

Let $\lambda_n(P)$ be $\max \bar{h}(\mathcal{F})$ over all P-free families $\mathcal{F} \subset 2^{[n]}$. Then we have

$$\Sigma(n, e(P)) \leq \mathrm{La}(n, P) \leq \lambda_n(P)\binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

We study $\lambda_n(P)$ and use it to investigate the $\pi(P) = e(P)$ conjecture for many posets.

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Asymptotics: Recall the limit $\pi(P) := \lim_{n\to\infty} \frac{\mathrm{La}(n,P)}{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$. Let $\lambda(P) := \lim_{n\to\infty} \lambda_n(P)$.

$$\boxed{e(P) \leq \pi(P) \leq \lambda(P),}$$

if both limits exist.

Successive upper bounds on $\pi(\mathcal{D}_2)$:

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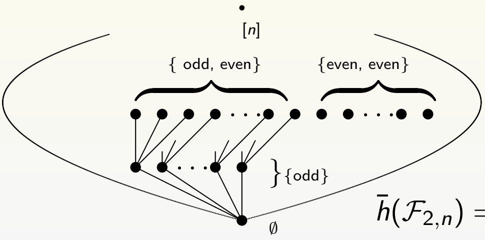
2.273 [G.-L.-Lu, 2011]
2.25 [Kramer-Martin-Young, 2012]

How well can this Lubell function method do? Consider this diamond-free family:

Ex: $\mathcal{F}_{2,n}$



$$\bar{h}(\mathcal{F}_{2,n}) = 1 + \frac{\lceil \frac{n}{2} \rceil}{n} + \frac{\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil + \binom{\lfloor \frac{n}{2} \rfloor}{2}}{\binom{n}{2}}$$

For $n > 1$, $\bar{h}(\mathcal{F}_{2,n}) > 2.25$.

What we then see is there are families of subsets with Lubell function values $\to 2.25$ as $n \to \infty$. On the other hand, G.-L.-Lu proved that the values λ_n are nonincreasing for $n \geq 2$. Hence, $\lambda(\mathcal{D}_2)$ exists, and is at least 2.25, which is a barrier for this approach to showing $\pi(\mathcal{D}_2) = 2$.

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Problem

Is $\bar{h}(\mathcal{F}) < 2 + \epsilon$ *if* $\mathcal{F} \not\supset \Diamond$ *such that* $||F| - \frac{n}{2}| < C\sqrt{n \log n}$ *for all* $F \in \mathcal{F}$*?*

Three level problem

To make things simpler, what if we restrict attention to D_2-free families in the middle three levels of the Boolean lattice B_n. We should get better upper bounds on $|\mathcal{F}|/\binom{n}{\lfloor \frac{n}{2} \rfloor}$:

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This is a restricted case of the problem of finding a largest set of vertices in the hypercube graph with no C_4.

Lubell-bounded Posets

For many posets we can use the Lubell function to completely determine $\mathrm{La}(n, P)$ and the extremal families.

Proposition

For a poset P satisfying $\boxed{\lambda_n(P) \leq e(P) \text{ for all } n}$, *we have*

$$\mathrm{La}(n, P) = \Sigma(n, e(P)) \text{ for all } n.$$

If $\mathcal{F}$ is a P-free family of the largest size, then

$$\mathcal{F} = \mathcal{B}(n, e(P)).$$

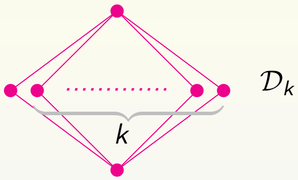
We say posets that satisfy the inequality above are *uniformly L-bounded*.

The k-Diamond Theorem

Theorem (G.-L.-Lu, 2012)
The k-diamond posets $\mathcal{D}_k$ satisfy

$$\lambda_n(P) \leq e(P)$$

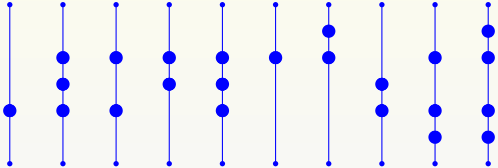
for all n, if k is an integer in the interval $[2^{m-1}-1, 2^m - \binom{m}{\lfloor \frac{m}{2} \rfloor} - 1]$ for any integer $m \geq 2$.



This means the posets $\mathcal{D}_k$ are uniformly L-bounded for $k = 1, 3, 4, 7, 8, 9, \ldots$. Consequently, for most values of k, D_k satisfies the $\pi = e$ conjecture, and, moreover, we know the largest D_k-families for all values of n.

Proof Sketch: The Partition Method

The Lubell function $\bar{h}(\mathcal{F})$ is equal to the average number of times a full chain intersects the family $\mathcal{F}$.



One of the key ideas (due to Li) involves splitting up the collection $\mathcal{C}_n$ of full chains into blocks that have a nice property, and computing the average on each block. Then $\bar{h}(\mathcal{F})$ is at most the maximum of those averages.

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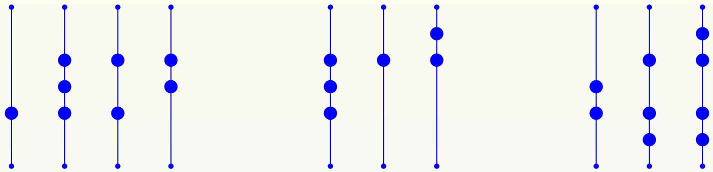
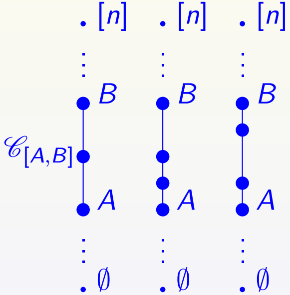


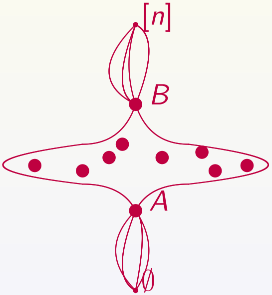
One of the key ideas (due to Li) involves splitting up the collection $\mathcal{C}_n$ of full chains into blocks that have a nice property, and computing the average on each block. Then $\bar{h}(\mathcal{F})$ is at most the maximum of those averages.

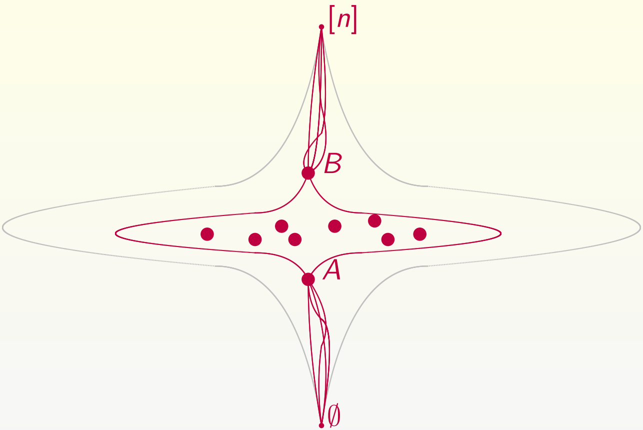
Proof Sketch: The k-Diamond Theorem

Min-Max Partition

The block $\mathscr{C}_{[A,B]}$ consists of full chains with $\min \mathcal{F} \cap \mathcal{C} = A$ and $\max \mathcal{F} \cap \mathcal{C} = B$.





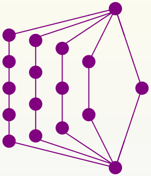


Compute $\mathrm{ave}_{\mathcal{C}\in\mathscr{C}_{[A,B]}}|\mathcal{F}\cap\mathcal{C}|$ for each block $\mathscr{C}_{[A,B]}$. If say we forbid $\mathcal{D}_3$, there are at most two points between A and B, and the largest average value $|\mathcal{F}\cap\mathcal{C}|$ is when we get a diamond $\mathcal{D}_2$ for $[A,B]$, which is $3=e(\mathcal{D}_3)$. □

The Harp Theorem

Theorem

The harp posets $\mathcal{H}(\ell_1, ..., \ell_k)$ are uniformly L-bounded, if $\ell_1 > \cdots > \ell_k \geq 3$.



$\mathcal{H}(7, 6, 5, 4, 3)$

Hence, harps with distinct path lengths are uniformly L-bounded and satisfy the $\pi = e$ conjecture.

More on the Lubell Function

Recall that $e(P) \leq \pi(P) \leq \lambda(P)$ when the limits $\pi(P)$ and $\lambda(P)$ both exist. For a uniformly L-bounded poset P, $e(P) = \pi(P) = \lambda(P)$.

More on the Lubell Function

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Many posets of interest are NOT uniformly L-bounded, including $\mathcal{V}_2$, $\mathcal{D}_2$, and the butterfly B.

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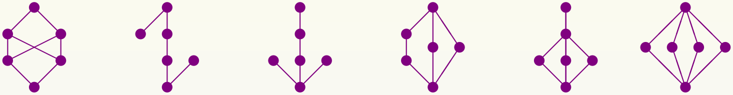
$e(P) = \pi(P) = \lambda(P)$.
Many posets of interest are NOT uniformly L-bounded, including $\mathcal{V}_2$, $\mathcal{D}_2$, and the butterfly B.
Still, it can be proven that $\lambda(P)$ exists whenever P is any diamond $\mathcal{D}_k$ or a harp $\mathcal{H}(\ell_1, \ldots, \ell_k)$.

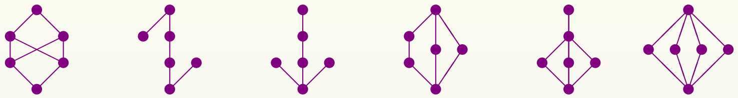
More on the Lubell Function

More uniformly L-bounded posets



More on the Lubell Function

More uniformly L-bounded posets



To prove $\pi(P) = e(P)$, it is actually enough to bound $\bar{h}(\mathcal{F})$ for families $\mathcal{F}$ of subsets of sizes "near the middle." We introduce weaker conditions of the Lubell function to show more posets P satisfy the conjecture.

More on the Lubell Function

Definition

For integer $m \geq 0$ *we say poset* P *is m-L-bounded if for all* n*,* $\bar{h}(|F) \leq n$ *for all families* $\mathcal{F}$ *of subsets* $A \subseteq [n]$ *such that* $m \leq |A| \leq n - m$*.*

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1-L-bounded is called centrally L-bounded.

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1-L-bounded is called centrally L-bounded.

Definition

We say poset P is L-bounded if it is m-L-bounded for some m.

0-L-bounded $\subseteq$ 1-L-bounded $\subseteq$ 2-L-bounded $\cdots \subseteq$ L-bounded

Lemma

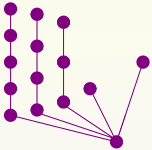
If P is L-bounded, then $\pi(P) = e(P)$.

The Fan Theorem

Theorem

Let P be the fan poset (wedge of paths) $\mathcal{V}(\ell_1, ..., \ell_k)$, $\ell_1 \geq \cdots \geq \ell_k$.

- *a) If $k = 1$, or if $\ell_1 > \ell_2 + 1$, then P is uniformly L-bounded.*
- *b) If $\ell_1 > \cdots > \ell_k$, then P is centrally L-bounded.*
- *c) If $\ell_1 > \ell_2$, then P is L-bounded.*
- *d) If $\ell_1 = \ell_2$, then P is NOT L-bounded, but it is "lower L-bounded".*



$\mathcal{V}(6, 5, 4, 2, 2)$

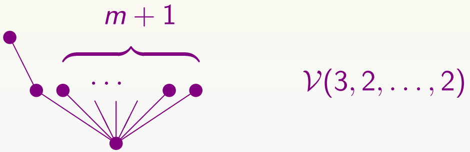
Hence, all fans satisfy the $\pi = e$ conjecture.

Fans and L-bounded Properties

Theorem

For $m \geq 1$ let P be the fan poset (wedge of paths) $\mathcal{V}(3, 2, \ldots, 2)$ with $m + 1 \geq 2$'s.

Then P is m-L-bounded, but not $(m-1)$-L-bounded.



Hence, for each property m-L-bounded, there is a poset with that property (and no stronger ones).

Constructions

Definition

The ordinal sum *of disjoint posets* P_1, P_2*, denoted* $P_1 \oplus P_2$*, is the poset consisting on* $P_1 \cup P_2$*, ordered by* $x \leq y$ *if* $x \in P_1$ *and* $y \in P_2$*, or if* x, y *are in the same* P_i *with* $x \leq y$*. The boldface number* $\mathbf{k}$ *is the* k*-element antichain.*

For example, the k-diamond poset $\mathcal{D}_k$ is the same as $\mathbf{1} \oplus \mathbf{k} \oplus \mathbf{1}$.

Theorem

For any centrally L-bounded poset P*, both* $\mathbf{1} \oplus P$ *and* $P \oplus \mathbf{1}$ *are centrally L-bounded. Furthermore,* $\mathbf{1} \oplus P \oplus \mathbf{1}$ *is uniformly L-bounded.*

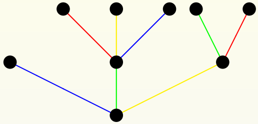
Constructions

Definition

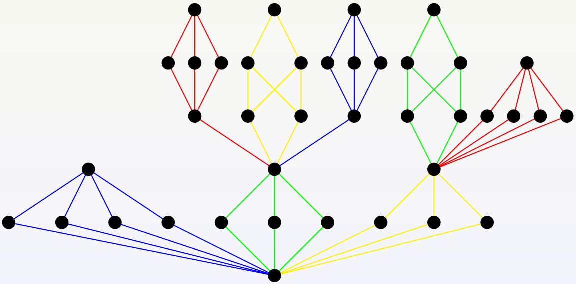
Suppose posets $P_1, \ldots, P_k$ are uniformly L-bounded with 0 and 1. A blow-up of a rooted tree T on k edges has each edge replaced by a P_i.

Constructions

Theorem (Li, 2011)
If P is a blow-up of a rooted tree T, then $\pi(P) = e(P)$. If the tree is a path, then P is uniformly L-bounded.



A blow-up of the rooted tree above:



Future Research

Conjecture (G.-Lu, 2009)

For any finite poset, $\pi(P)$ exists and is $e(P)$.

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Determine for the crown $\mathcal{O}_6$ whether $\pi(\mathcal{O}_6)$ exists and equals 1. The current best upper bound is $1.707\ldots$. Lu believes we can solve $\mathcal{O}_{14}$.

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Problem

Prove that $\lambda(P)$ exists for general P.

Problem

We know $\lambda(\mathcal{D}_2)$ exists. Prove it is 2.25.

Future Research

Problem

Provide insight into why

- $\mathrm{La}(n, P)$ *behaves very nicely for some posets, equalling* $\Sigma(n, e(P))$ *for all* $n \geq n_o$ *(such as the butterfly* B *and the diamonds* $\mathcal{D}_k$ *for most values of* k*);*
- *Is more complicated, but behaves well asymptotically (such as* $\mathcal{V}_2$*); or*
- *Continues to resist asymptotic determination (such as* $\mathcal{D}_2$ *and* $\mathcal{O}_6$*).*

At dinner this week. Gerard is the young chap on the right.


Thank You

Foundational results: Let $\mathcal{P}_k$ denote the k-element chain (path poset).

Theorem (Sperner, 1928)

For all n,

$$\mathrm{La}(n, \mathcal{P}_2) = \binom{n}{\lfloor \frac{n}{2} \rfloor},$$

and the extremal families are $\mathcal{B}(n, 1)$.

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Theorem (Erdős, 1945)

For general k and n,

$$\mathrm{La}(n, \mathcal{P}_k) = \Sigma(n, k-1),$$

and the extremal families are $\mathcal{B}(n, k-1)$.

Foundational results: Let $\mathcal{V}_r$ denote the poset of r elements above a single element.

Theorem (Katona-Tarján, 1981)

As $n \to \infty$,

$$\left(1 + \frac{1}{n} + \Omega\left(\frac{1}{n^2}\right)\right)\binom{n}{\lfloor \frac{n}{2} \rfloor} \leq \mathrm{La}(n, \mathcal{V}_2) \leq \left(1 + \frac{2}{n}\right)\binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

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Theorem (Thanh 1998, DeBonis-Katona, 2007)

For general r, as $n \to \infty$,

$$\left(1 + \frac{r-1}{n} + \Omega\left(\frac{1}{n^2}\right)\right) \le \frac{\mathrm{La}(n, \mathcal{V}_r)}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} \le \left(1 + 2\frac{r-1}{n} + O\left(\frac{1}{n^2}\right)\right).$$

More results for small posets: Let B denote the Butterfly poset with two elements each above two other elements. Let $\mathcal{N}$ denote the four-element poset shaped like an N.

Theorem (DeBonis-Katona-Swanepoel, 2005)

For all $n \geq 3$

$$\mathrm{La}(n, B) = \Sigma(n, 2),$$

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Easy Upper Bound for $\mathcal{D}_2$

Let

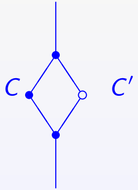
$$d_n := \max_{\substack{\mathcal{F} \subset 2^{[n]} \\ \mathcal{F} \not\supset \Diamond}} \bar{h}(\mathcal{F})$$

Proposition

For all n, $d_n \leq 2.5$. Hence, $\pi(\Diamond) \leq 2.5$.

Proof.

Suppose $\mathcal{F} \not\supseteq \Diamond$.
Let $\gamma_i := Pr(|\mathcal{F} \cap C| = i)$. $\bar{h}(\mathcal{F}) = \mathrm{E}(|\mathcal{F} \cap C|) = \sum_{i=1}^{3} i\gamma_i$.
One shows easily that $\gamma_3 \leq \gamma_2$. □



Improved Bound for $\mathcal{D}_2$

Theorem

$\pi(\Diamond) < 2.3$

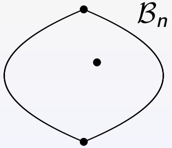
Lemma

For $n \geq 3$, $d_n \leq d_{n-1}$.

n	2	3	4	5	6	7
d_n	$2\frac{1}{2}$	$2\frac{1}{3}$	$2\frac{1}{3}$	2.3	2.3	< 2.3

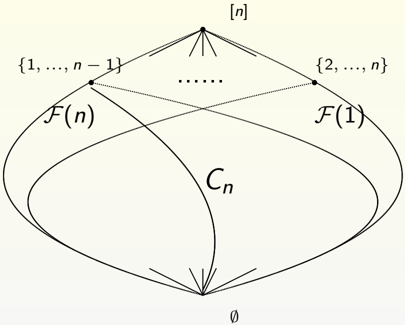
Proof.

Let $\mathcal{F}$ achieve d_n. If $\emptyset$, $[n] \in \mathcal{F}$, then $\bar{h}(\mathcal{F}) \leq 2 + \frac{1}{n} \leq d_n$.



Else we may assume $[n] \notin \mathcal{F}$.

$$\begin{aligned} d_n &= \bar{h}(\mathcal{F}) = \mathrm{E}(|\mathcal{F} \cap C|) \\ &\le \frac{\sum_{i=1}^{n} \mathrm{E}(|\mathcal{F}(i) \cap C_i|)}{n} \\ &\le d_{n-1} \end{aligned}$$



where $\mathcal{F}(i) := \{F \in \mathcal{F} | i \notin \mathcal{F}\}$ and C_i is a random full chain of subset of $[n] - \{i\}$. □

Forbidding Induced Subposets

Less is known for this problem:

Definition

We say P is an induced subposet of Q, written $P \subset^* Q$ if there exists an injection $f : P \to Q$ such that for all $x, y \in P$, $x \leq y$ iff $f(x) \leq f(y)$. We define $\mathrm{La}^*(n, P)$ to be the largest size of a family of subsets of $[n]$ that contains no induced subposet P.

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Theorem (Carroll-Katona, 2008)

As $n \to \infty$,

$$\left(1 + \frac{1}{n} + \Omega\left(\frac{1}{n^2}\right)\right)\binom{n}{\lfloor \frac{n}{2} \rfloor} \leq \mathrm{La}^*(n, \mathcal{V}_2) \leq \left(1 + \frac{2}{n} + O\left(\frac{1}{n^2}\right)\right)\binom{n}{\lfloor \frac{n}{2} \rfloor}$$

Forbidding Induced Subposets

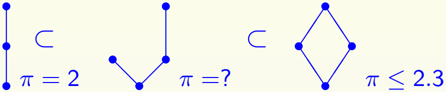
Extending Bukh's Forbidden Tree Theorem:

Theorem (Boehnlein-Jiang, 2011)

For every tree poset T,

$$\mathrm{La}^*(n, T) \sim (h(T) - 1)\binom{n}{\lfloor \frac{n}{2} \rfloor}, \mathrm{as} n \to \infty.$$

What about $\mathcal{J}$? Let us use the Lubell function.



Theorem (Li)

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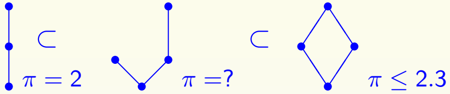

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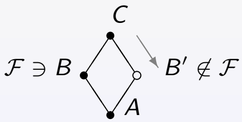
Proof.

Let $\mathcal{F} \subset 2^{[n]}$ achieve $\mathrm{La}(n, \mathcal{J})$. Then $\bar{h}(\mathcal{F}) \leq 3$. If $\mathcal{F}$ contains some $\mathcal{P}_3$, make a swap:



Then $\mathcal{F}' := \mathcal{F} - \{C\} + \{B'\}$.

- contains no $\mathcal{J}$
- $|\mathcal{F}'| = |\mathcal{F}|$
- $|\mathcal{F}'|$ contains fewer $\mathcal{P}_3's$

Iterate until we get $\mathcal{J}$-free $\tilde{\mathcal{F}}$ of height 2, so

$$|\mathcal{F}| = |\tilde{\mathcal{F}}| \leq \mathrm{La}(n, \mathcal{P}_3).$$

$\square$

The Union-free Family Theorem

A related problem

Theorem (Kleitman, 1965)

Let $\mathcal{F}$ be a collection of subsets of $[2n]$, that contains no two sets and their union. Then

$$|\mathcal{F}| \leq \binom{2n}{n}(1 + O(n^{-1/2})).$$

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In connection with this, he proposes to investigate two-level "triangle-free" families of subsets of $[2n]$.

Triangle-free Families

Let $k \geq 1$. Consider a family $\mathcal{F}$ of subsets of $[2n]$ such that every $A \in \mathcal{F}$ has size n or $n - k$. Further, suppose that there are no three sets $A_1, A_2, B \in \mathcal{F}$ with $|A_1| = |A_2| = n - k$, $|B| = n$, $A_1, A_2 \subset B$, and A_1, A_2 are at Hamming distance $2k$. This forbidden configuration we call a *triangle*.

Note that it means $A_1 \cup A_2 = B$.

Kleitman asked for a good upper bound on triangle-free $\mathcal{F}$ for $k = 2$ and for general k. Trivially, $\binom{2n}{n} \leq |\mathcal{F}| \leq 2\binom{2n}{n}$.

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Proposition (G.-Li)

For triangle-free $\mathcal{F}$,

$$|\mathcal{F}| \leq \binom{2n}{n}(1 + (k/n)).$$

This can be proven with the Lubell function.

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Kleitman believes it is possible to remove the factor k:

Conjecture

For triangle-free families $\mathcal{F}$ for $k \geq 2$,

$$\boxed{|\mathcal{F}| \leq \binom{2n}{n}(1 + (1/n)).}$$