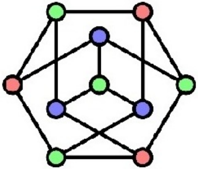
THE FORBIDDEN SUBPOSET PROBLEMS AND TURÁN PROBLEMS



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2012 Symposium for Young Combiantorialists

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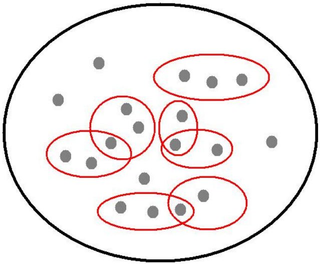
Introduction

Consider a family of subsets of $[n] := \{1, 2, \ldots, n\}$ such that $A \not\subset B$ is required for any distinct members A and B of this family. Such a family is said to be **inclusion-free**.



Question: What is the maximum size of such a family?

Theorem (Sperner, 1928)

Let $\mathcal{F}$ be an inclusion-free family of subsets of $[n]$. Then

$$|\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

The upper bound is achieved by taking all sets of size $\lfloor \frac{n}{2} \rfloor$.

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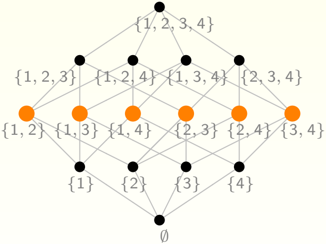


Forbidden Subposet Problems

A *poset (partially ordered set)* $P = (P, \leq)$ is a set P with a binary partial order relation $\leq$ satisfying

1. For all $x \in P$, $x \leq x$. (reflexivity)
2. If $x \leq y$ and $y \leq x$, then $x = y$. (antisymmetry)
3. If $x \leq y$ and $y \leq z$, then $x \leq z$. (transitivity)

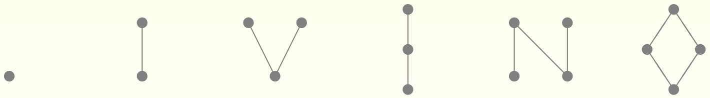


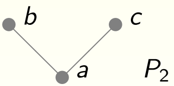
Figure: The Hasse diagrams of some small posets.

The *Boolean lattice* $\mathcal{B}_n = (2^{[n]}, \subseteq)$ is the poset consisting of the power set of $[n]$ and the inclusion relation as the partial order.

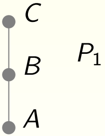
A poset $P_1 = (P_1, \leq_1)$ contains another poset $P_2 = (P_2, \leq_2)$ as a *(weak) subposet* if there exists an injection f from P_2 to P_1, which preserves the order, that is $f(a) \leq_1 f(b)$ whenever $a \leq_2 b$.

Example:

$P_2 = (\{a, b, c\}, \{(a, b), (a, c)\})$



$$f\colon \begin{array}{l} c \longmapsto C \\ b \longmapsto B \\ a \longmapsto A \end{array}$$



$P_1 = (\{A, B, C\}, \{(A, B), (B, C), (A, C)\})$

A family $\mathcal{F} = (\mathcal{F}, \subseteq)$ of subsets of $[n]$ is said to be P-free, if it does not contain $P = (P, \leq)$ as a subposet.

Let $\mathrm{La}(n, P)$ be the largest size of a P-free family of subsets of $[n]$.

There are many results on $\mathrm{La}(n, P)$ for various posets P, mostly obtained by G. O. H. Katona and his collaborators.

The weight of a P-free family

Upper Bound of $\mathrm{La}(n, P)$

The *Lubell function* of a family $\mathcal{F}$ of subsets of $[n]$ is

$$\boxed{\bar{h}_n(\mathcal{F}) = \sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}},}$$

which is a weighted sum of the sets in the family.

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THEOREM (Yamamoto, 1954; Mashalkin, 1963; Bollobás, 1965; Lubell, 1966)

For any antichain $\mathcal{F}$ os subsets of $[n]$,

$$\sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} \leq 1 \quad (\textit{LYM-inequality}).$$

Proposition

For any (P-free) family $\mathcal{F}$ of subsets of $[n]$, if $\bar{h}_n(\mathcal{F}) \leq k$, then

$$|\mathcal{F}| \leq k \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

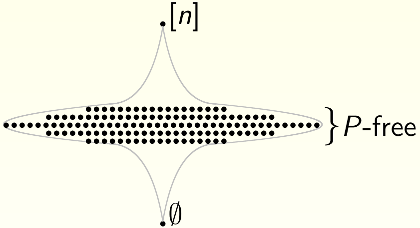
Define

$$\lambda_n(P) := \max_{\mathcal{F}:P\text{-free}} \bar{h}_n(\mathcal{F}).$$

The value of $\lambda_n(P)$ gives an upper bound of $\mathrm{La}(n, P)$.

Lower Bound of $\mathrm{La}(n, P)$

Take as many levels in the middle of $\mathcal{B}_n$ as possible until the family will contain P as a subposet when taking one more level.



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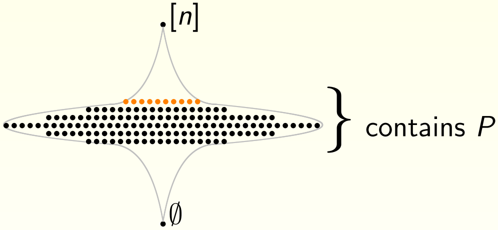

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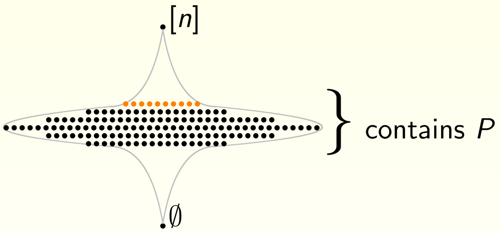

$e(P)$: the largest integer k such that the family consisting of the middle k levels of $\mathcal{B}_n$ is P-free for any n.

Observation: If $\lim_{n\to\infty} \lambda_n(P) = e(P)$, then

$$\mathrm{La}(n, P) \sim e(P)\binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Define $\lambda(P) = \lim_{n\to\infty} \lambda_n(P)$. Note that $\lambda(P) \geq e(P)$.

There exist posets satisfying $\lambda(P) = e(P)$ but also many posets satisfy $\lambda(P) > e(P)$.

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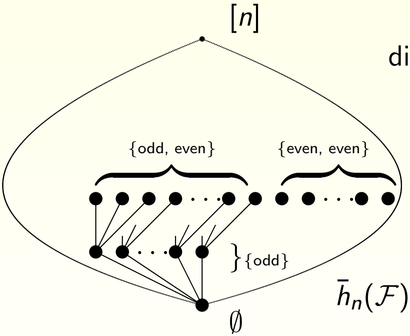
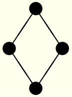
The smallest poset P for which $\mathrm{La}(n, P)$ is not clearly understood is the *diamond poset* $\mathcal{D}_2$. It does not satisfy $\lambda(P) = e(P)$.

CONJECTURE (Griggs, L., and Lu, 2012)

For the diamond poset $\mathcal{D}_2$, $\lambda_n(\mathcal{D}_2) = 2 + \frac{\lfloor \frac{n^2}{4} \rfloor}{n(n-1)}$.



diamond $\mathcal{D}_2$:



$$\bar{h}_n(\mathcal{F}) = 1 + \frac{\lceil \frac{n}{2} \rceil}{n} + \frac{\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil + \binom{\lfloor \frac{n}{2} \rfloor}{2}}{\binom{n}{2}}$$

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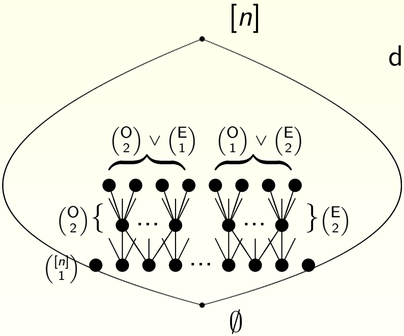

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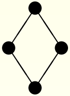


THEOREM (Kramer, Martin, and Young, preprint)

If a $\mathcal{D}_2$-free family $\mathcal{F}$ contains $\emptyset$, then

$$\bar{h}_n(\mathcal{F}) \leq 2.25 + o_n(1).$$

THEOREM (Kramer, Martin, and Young, preprint)

$$\mathrm{La}(n, \mathcal{D}_2) \leq 2.25 + o_n(1).$$

Question: Does $\bar{h}_n(\mathcal{F}) > 2.25 + \varepsilon$ for some $\mathcal{F}$ with $\emptyset \notin \mathcal{F}$?

A sketch of the proof:

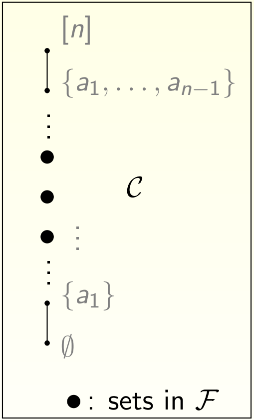


Figure: A full chain $\mathcal{C}$.

A *full chain* $\mathcal{C}$ in $\mathcal{B}_n$ is a family of subsets as follows.

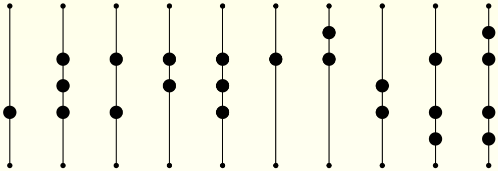
$$\emptyset \subset \{a_1\} \subset \{a_1, a_2\} \cdots \subset [n]$$

Associate a set $F \in \mathcal{F}$ to a full chain $\mathcal{C}$ if $F \in \mathcal{C}$.

By counting the number of pairs $(F, \mathcal{C})$ in two different ways, we have

$$\bar{h}_n(\mathcal{F}) = \sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} = \sum_{\mathcal{C}\text{:full chain}} \frac{|\mathcal{C} \cap \mathcal{F}|}{n!}$$

The Lubell function of $\bar{h}_n(\mathcal{F})$ is equal to the average number of times that the full chains intersect the family $\mathcal{F}$.



(1) Partition the set of full chains into blocks.
(2) Compute the average of $|\mathcal{C} \cap \mathcal{F}|$ for full chains in each block.
(3) $\bar{h}_n(\mathcal{F})$ is bounded by the maximum of those averages.

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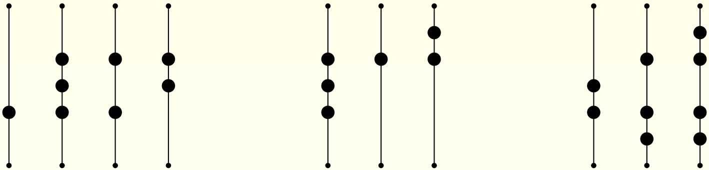

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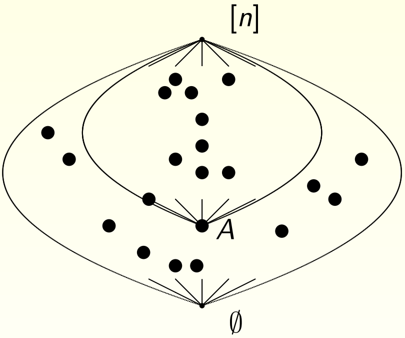
Let $\mathcal{C}_A$ be a block of full chains $\mathcal{C}$ with $\boxed{\min \mathcal{C} \cap \mathcal{F} = A}$. Then

$$\sum_{\mathcal{C} \in \mathcal{C}_A} \frac{|\mathcal{C} \cap \mathcal{F}|}{|\mathcal{C}_A|} = \bar{h}_m(\mathcal{F}'),$$

where $\mathcal{F}'$ is some $\mathcal{D}_2$-free family as $\mathcal{F}$ is $\mathcal{D}_2$-free, and $m \leq n$.
Moreover $\emptyset \in \mathcal{F}'$.



□

Sizes versus weights

Define

$$\pi(P) = \lim_{n\to\infty} \frac{\mathrm{La}(n, P)}{\binom{n}{\lfloor \frac{n}{2} \rfloor}}.$$

CONJECTURE (Griggs and Lu, 2009)

For any finite poset P, $\pi(P)$ is an integer.

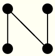
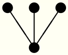
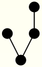
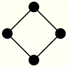
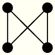
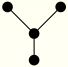
P							
$e(P)$	1	1	2	2	2	2	3
$\pi(P)$	1	2	2	**?**	2	2	3
$\lambda(P)$	**2**	**2.25**	**2.25**	$<$ **2.273**	**3**	**3**	**3**

Table: $e(P)$, $\pi(P)$, and $\lambda(P)$ for $|P| = 4$.

Turán's Problems

Question: What is the largest number of edges of a triangle-free graph on n vertices?

THEOREM (Mantel, 1907)

The balanced complete bipartite graph $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$ is the only triangle-free graph that contains most edges.

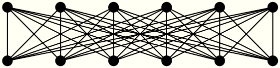


Figure: $K_{6,6}$

Theorem (Turán, 1941)

Let G be a K_r-free graph with n vertices. Then

$$E(G) \leq |E(T_{r-1}(n))|,$$

where $T_{r-1}(n)$is the balanced $(r-1)$-partite graph with n vertices, with equality if and only if G is isomorphic to $T_{r-1}(n)$.

Theorem (Turán, 1941)

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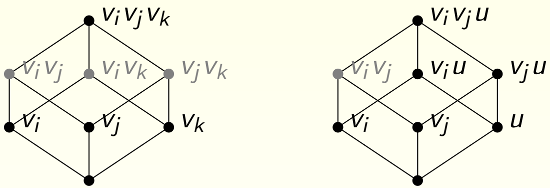
Turán density:

$$\pi(H) := \lim_{n\to\infty} \max_{G:H\text{-free}} \frac{|E(G)|}{\binom{n}{2}}.$$

We have $\pi(K_r) = 1 - \frac{1}{r}$ for any complete graph K_r.

Consider a family $\mathcal{F}$ consisting of $\emptyset$, $\binom{[n]}{1}$, $\binom{[n]}{3}$, and the edge set of $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$. It is $\mathcal{D}_6$-free



Hence $\lambda_n(\mathcal{D}_6) \geq 3\frac{1}{2}$. On the other hand, we have $\lambda_n(\mathcal{D}_6) \leq 3\frac{2}{3}$.

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Open problems

Problem

Does $\lambda(P)$ exist for any poset P?

Problem

What posets satisfy $e(P) = \lambda(P)$(hence $\pi(P)$ as well)?

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Is $\lambda(\mathcal{D}_6) = 3\frac{1}{2}$? If this is true, it implies Mantel's Theorem.

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Thank you!!