A Mathematical Model for Finding the Culprit Who Spreads Rumors

搜尋散佈謠言者的數學模型

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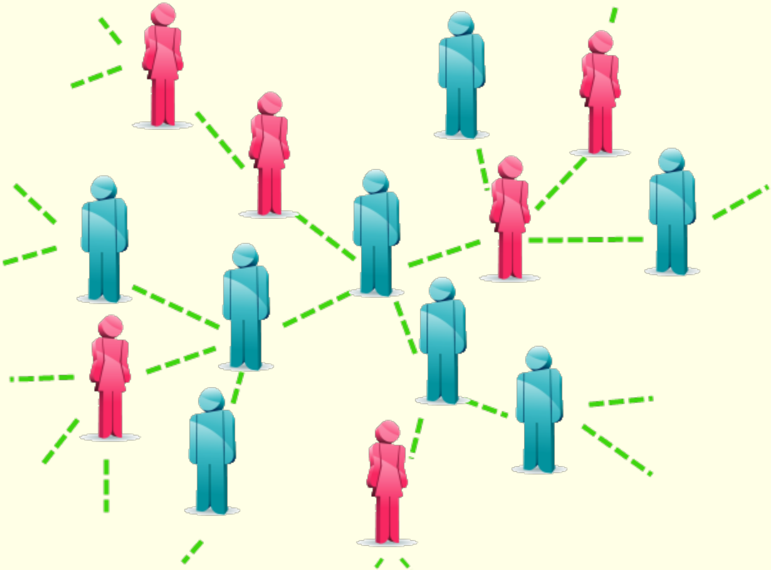
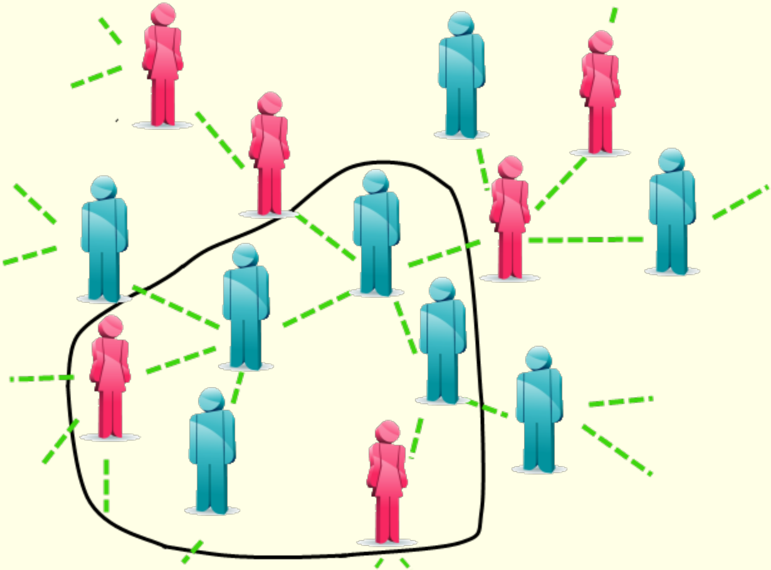
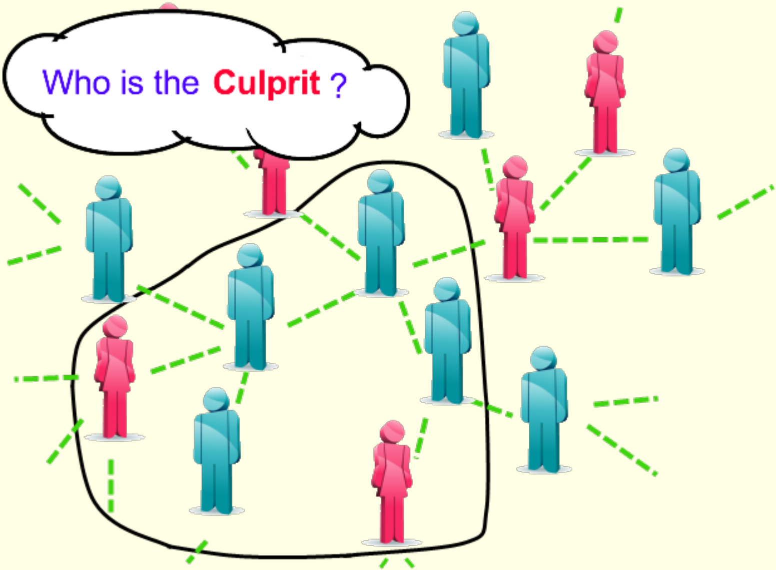
指導教授: 傅恆霖 教授

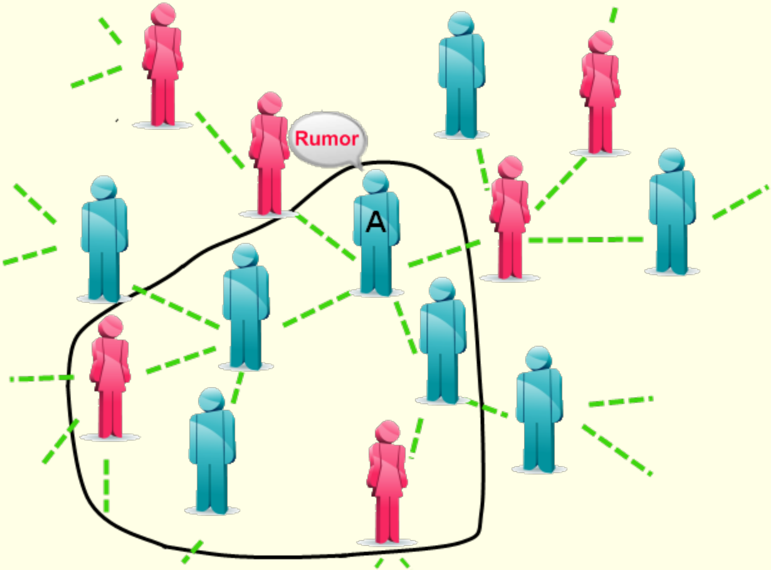
Department of Applied Mathematics
National Chiao Tung University

June 12, 2012






Who is the Culprit ?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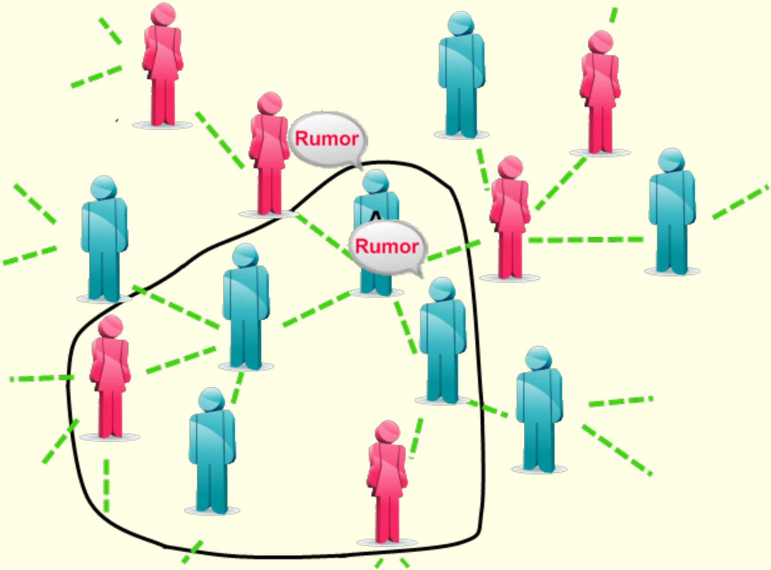
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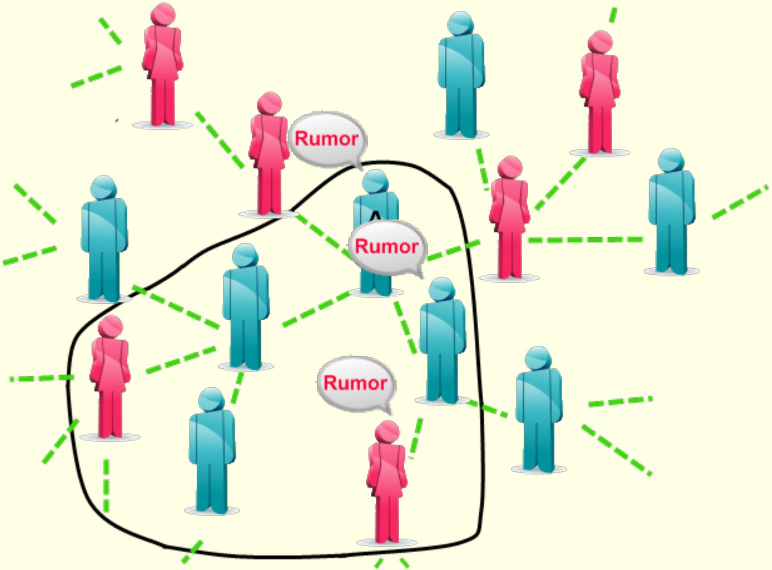
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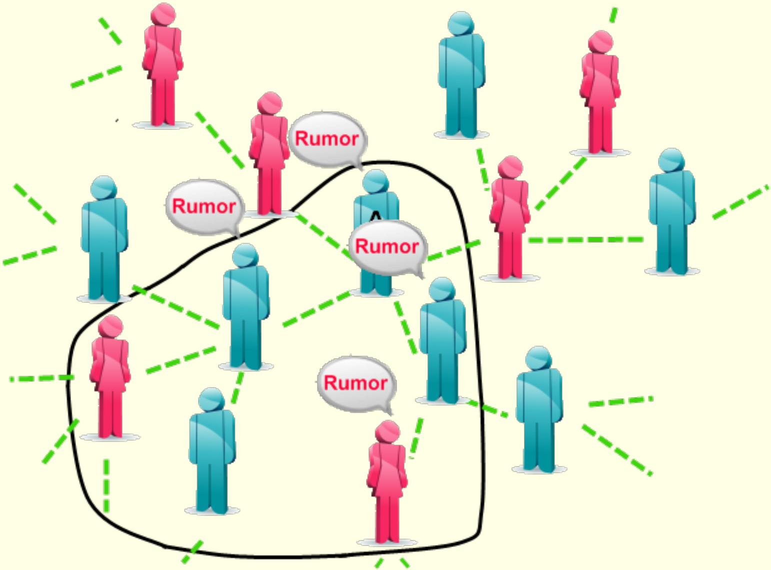
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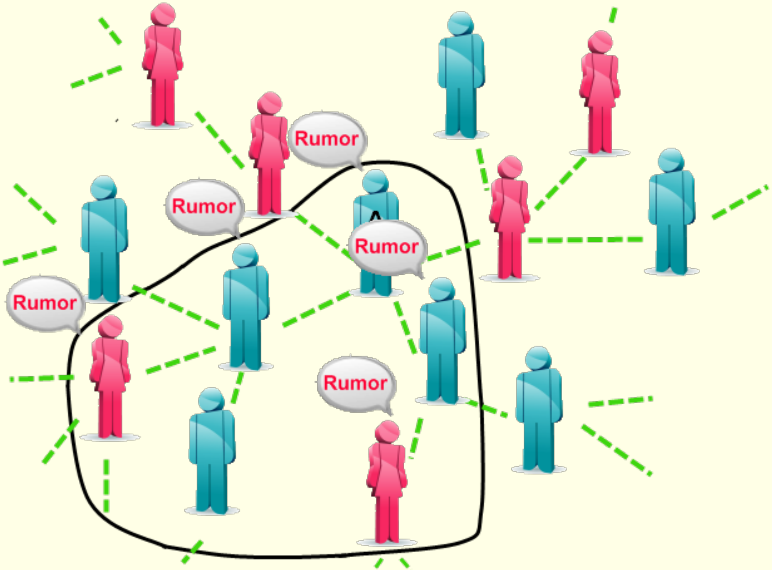
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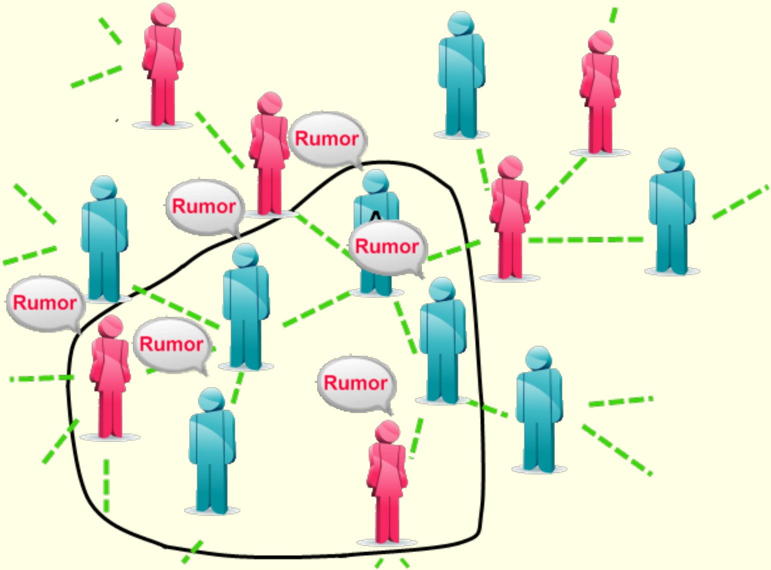
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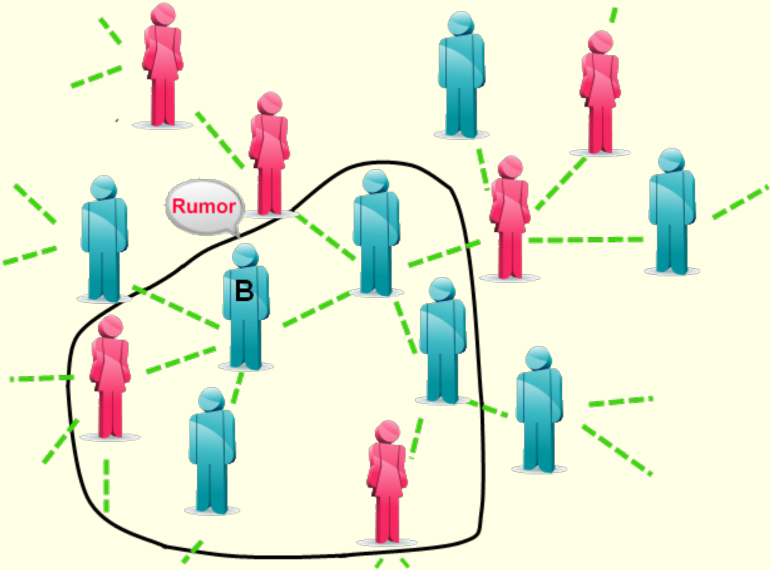
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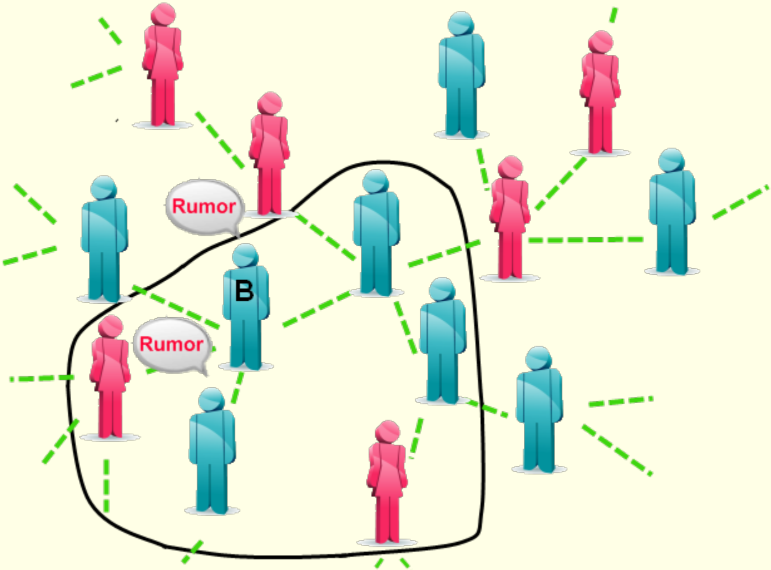
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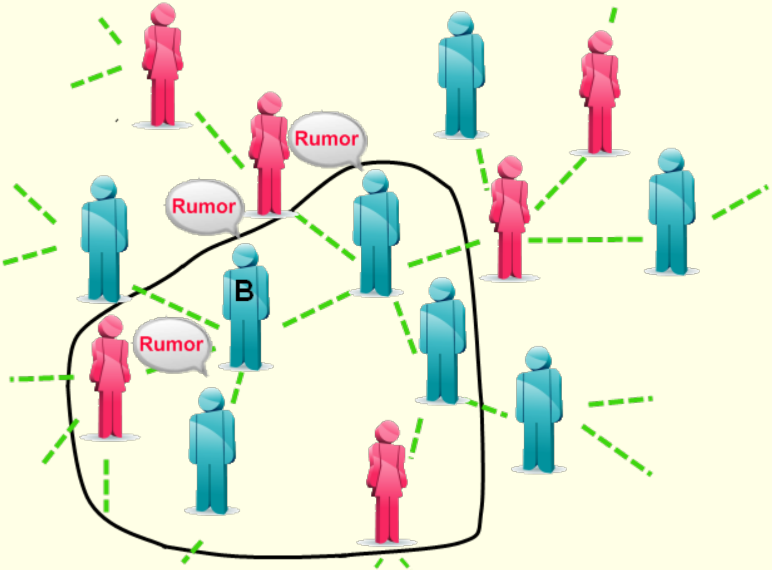
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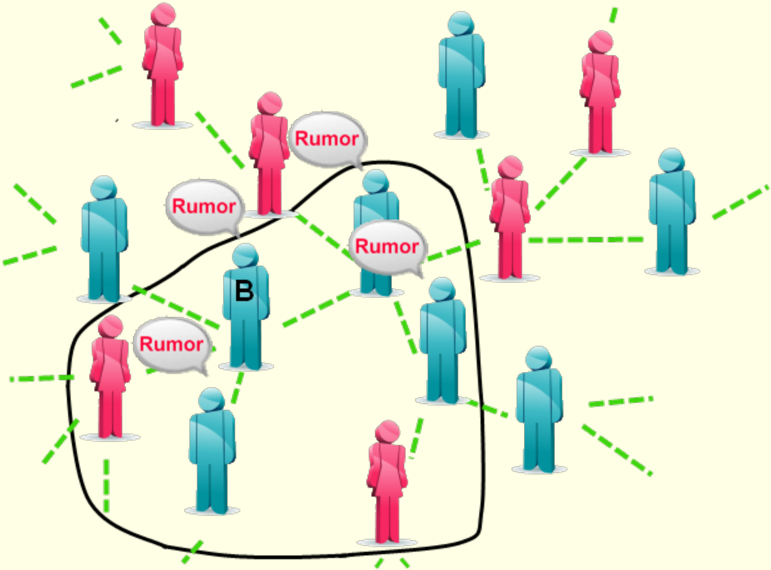
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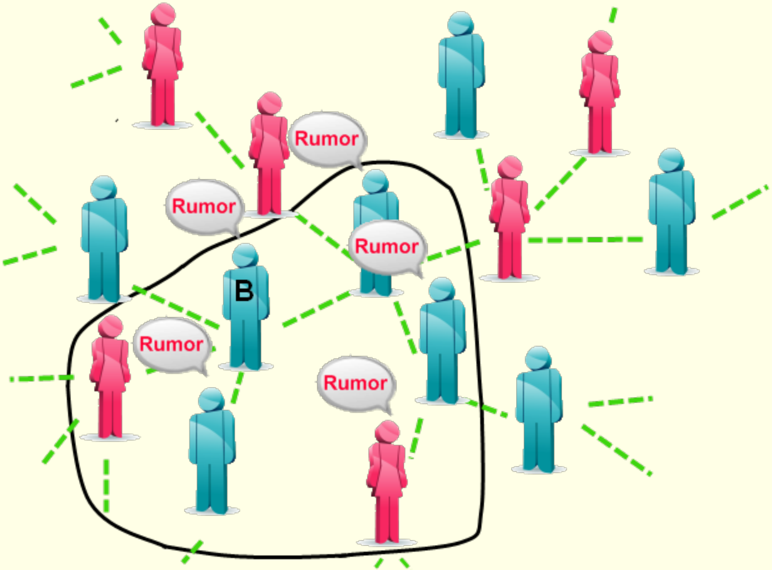
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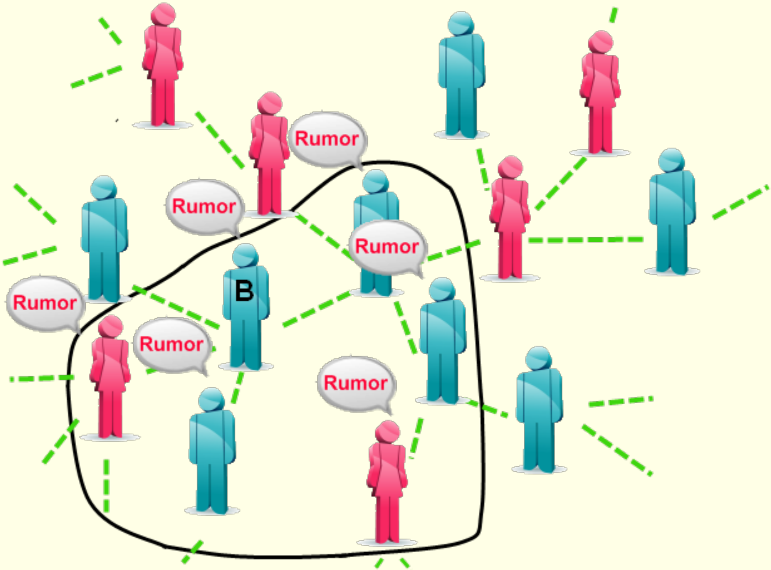
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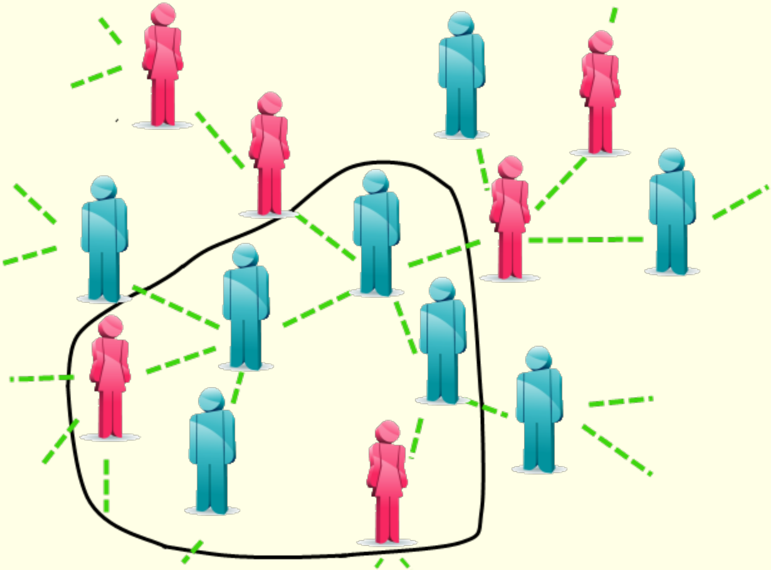
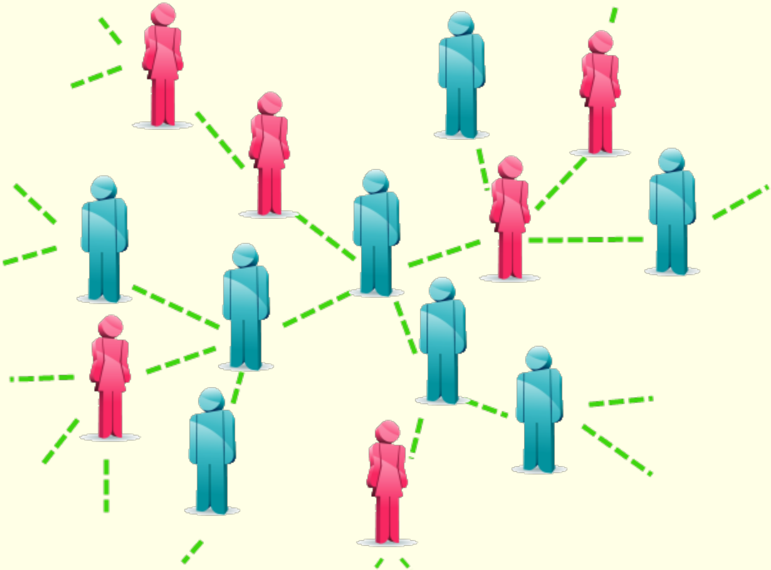
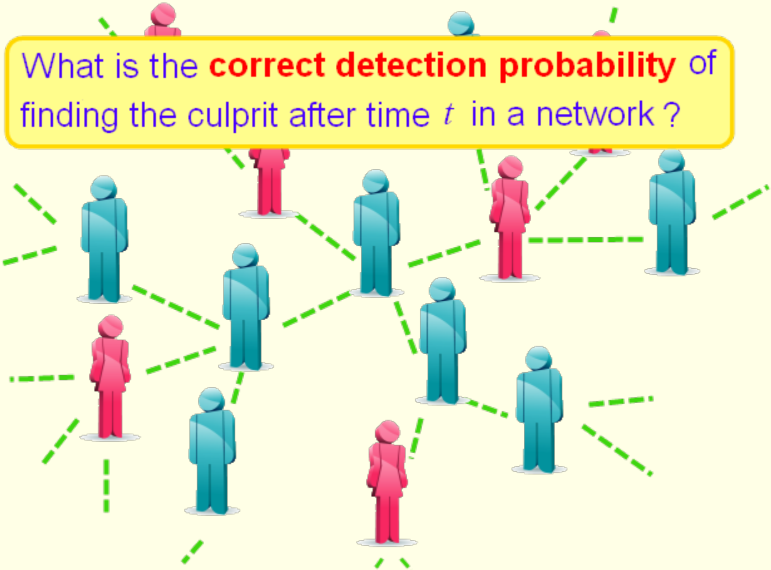




What is the **correct detection probability** of finding the culprit after time t in a network ?



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- A discrete-time model:

$$\begin{cases} S_{t+1} = S_t - \beta S_t I_t, \\ I_{t+1} = I_t + \beta S_t I_t, \\ I_0 = \text{A constant}. \end{cases}$$

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Using a fixed population, $S_t + I_t = N$.

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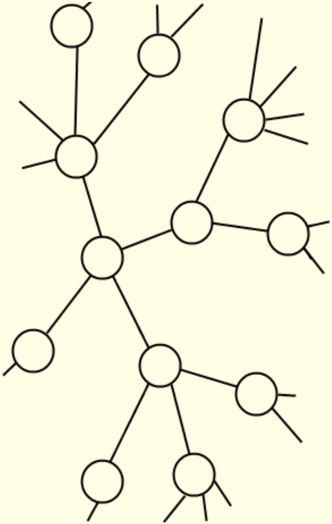
- $V(G)$: People.
- $E(G)$: Relationship between two people.

Rumor spreading model

Let G_t be a subgraph of order t of G.

This graph is compose of t infected vertices which are people who have known rumors at time t.



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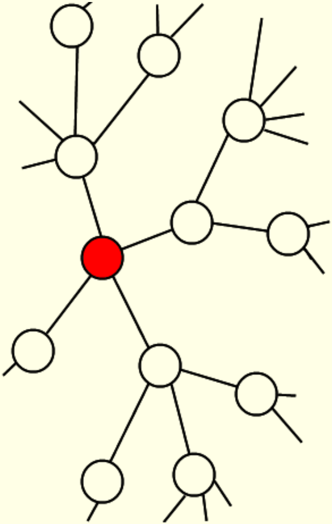

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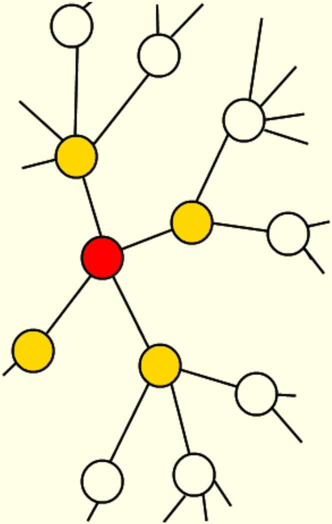
In each discrete time-step $t+1$, $t>0$, G_{t+1} develops from G_t by adding a vertex z with an edge with the following probability distribution:

$$\mathbf{P}_{t+1}(z) = \frac{1}{\sum_{v \in V(G_t)} d(v) - 2(t-1)}. \qquad (1)$$

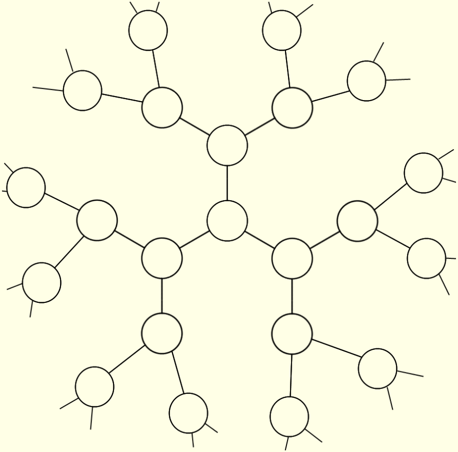


d-regular tree

- **Countably infinite** vertex set.
- **Every** vertex has d neighbors.

  $N(v)$: Neighborhood of v.

  $d(v)$: Degree of v.



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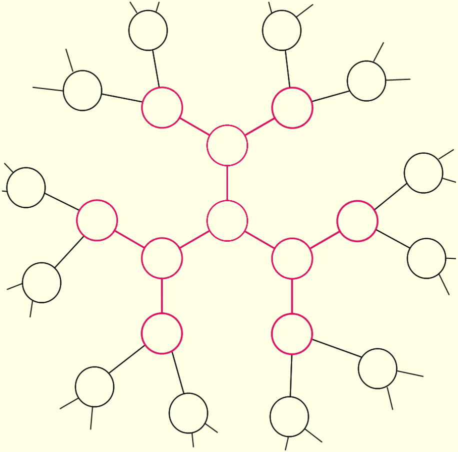

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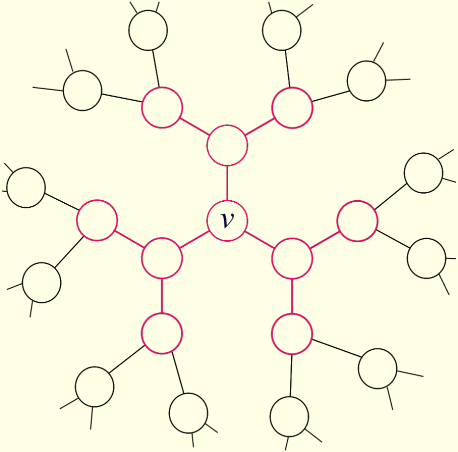

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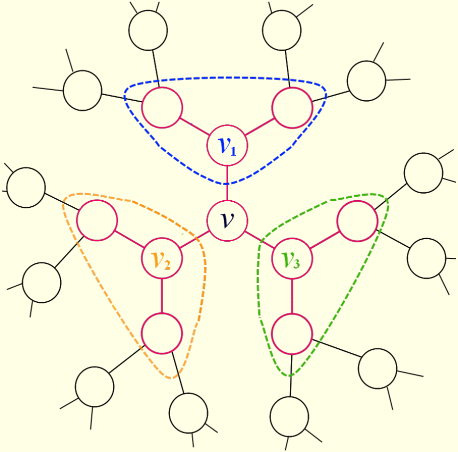
$(T^v_{v_1}, T^v_{v_2}, T^v_{v_3})$: Branches of T_v, and $t^v_{v_1} + t^v_{v_2} + t^v_{v_3} = t_v - 1$.



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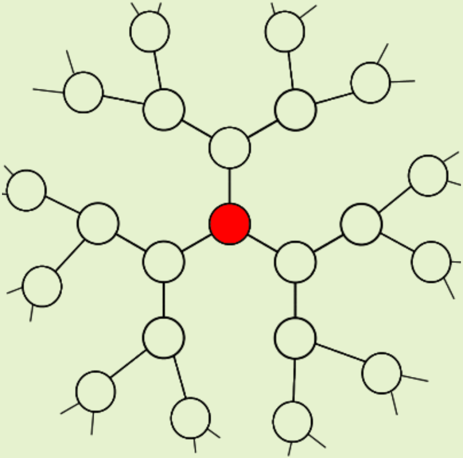
Proposition 2 (D. Shah and T. Zaman, 2011 [6])

Given an n vertices tree,

- Node v is the **only** rumor center if and only if $t_u^v < \frac{n}{2}$ for all $u \neq v$.
- If there is a node v such that $t_u^v = \frac{n}{2}$, then u and v both are rumor centers.

$d = 3, t = n = 5$



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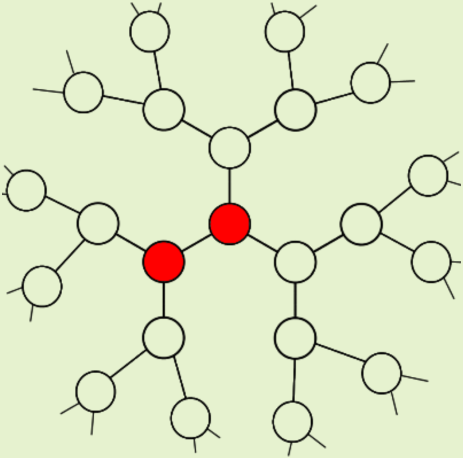

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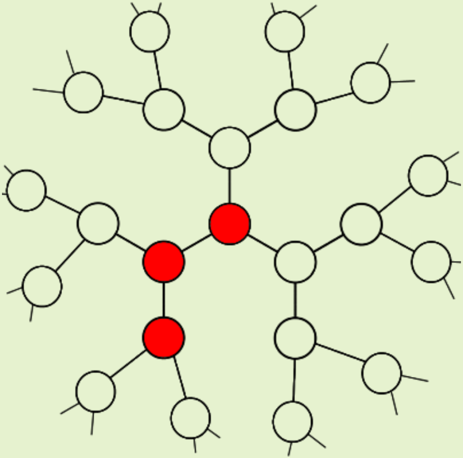

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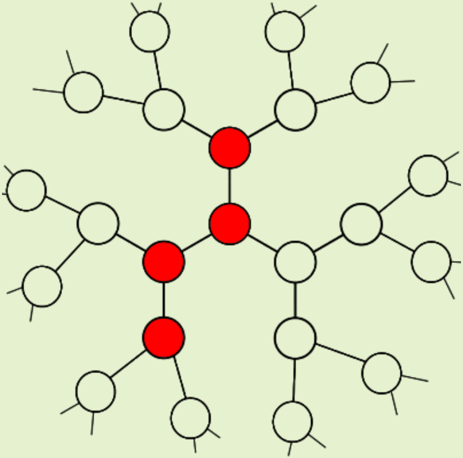

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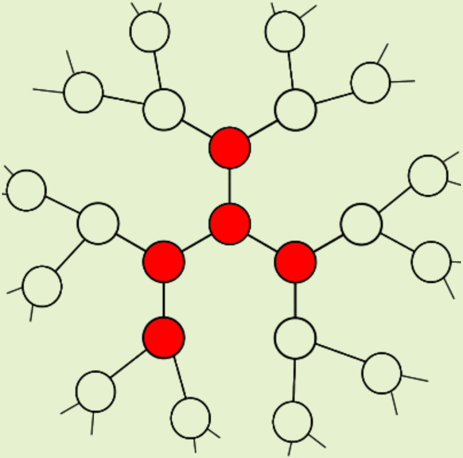

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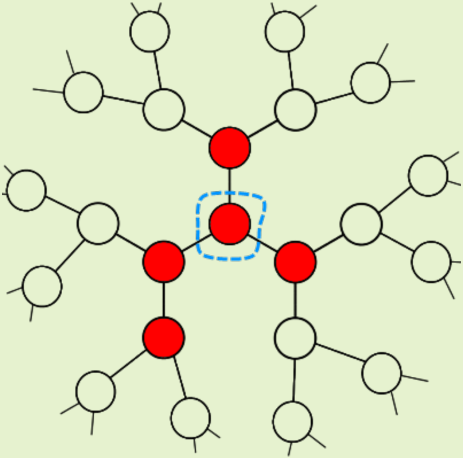

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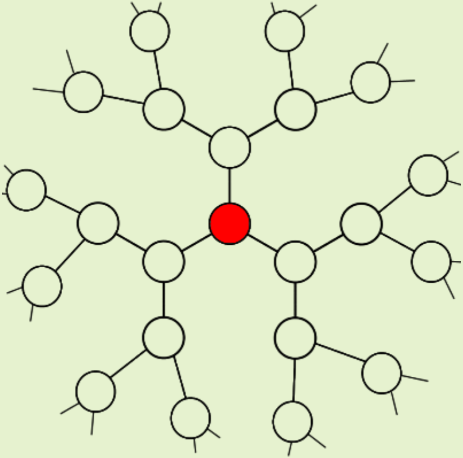

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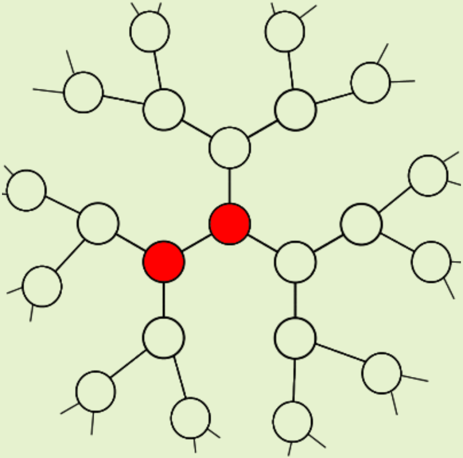

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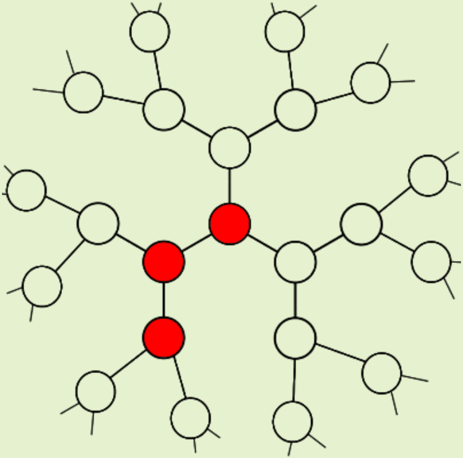

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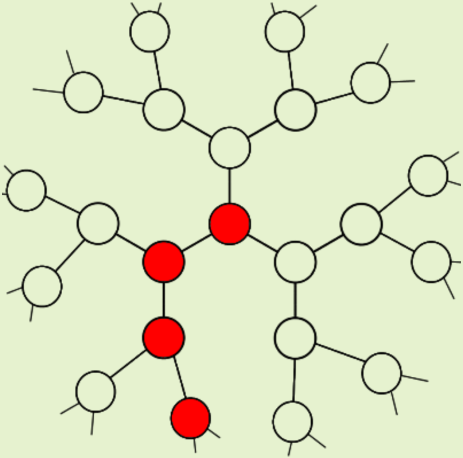

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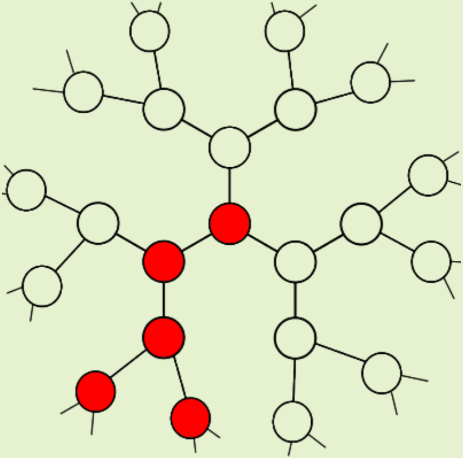

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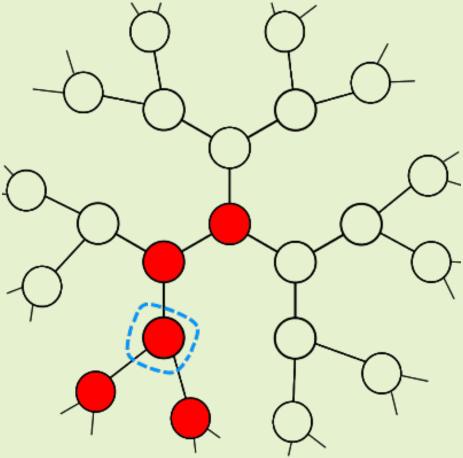

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Suppose the rumor has spread in a **2-regular tree**. Then we have that

$$\mathbf{P}(E_t) = O\left(\frac{1}{\sqrt{t}}\right).$$

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$$0 < \mathbf{P}\left(E_t\right) \leq \frac{1}{2}.$$

Definition 3

$$A_d = \{(a_1, a_2, \cdots, a_d) \mid 1 \leq a_i < \frac{n}{2}, \sum_{i=1}^{d} a_i = n - 1\}, \text{ and}$$

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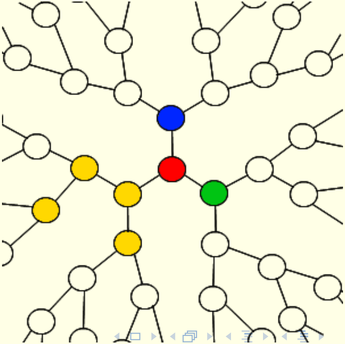

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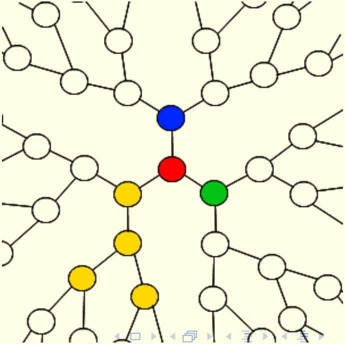

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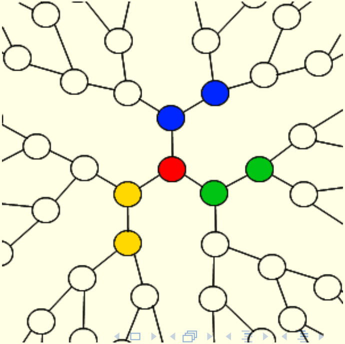

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Clearly, $A_d \subseteq B_d$. Moreover,

$$|B_d| = \binom{n-1-d+d-1}{d-1} = \binom{n-2}{d-1},$$

$$|A_d| = \binom{n-2}{d-1} - d \cdot \binom{\lfloor \frac{n}{2} \rfloor - 1}{d-1}.$$

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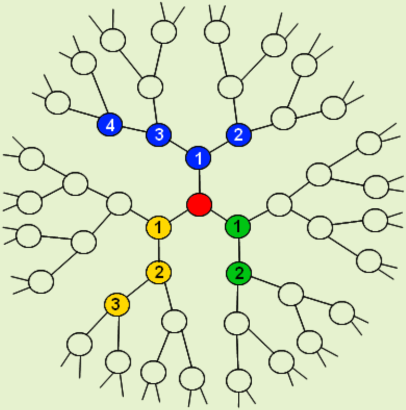
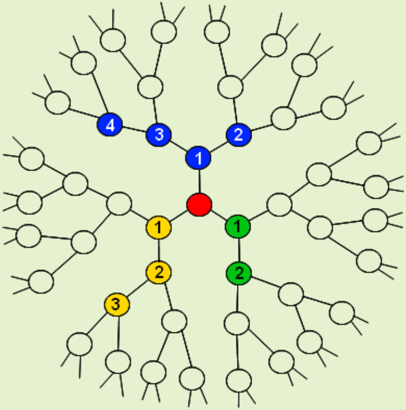
Given $(t^v_{v_1}, t^v_{v_2}, \cdots, t^v_{v_d})$, the total number of ways to spread a rumor is

$$\frac{(n-1)!}{t^v_{v_1}!t^v_{v_2}!\cdots t^v_{v_d}!} \cdot \prod_{k=1}^{d} \prod_{i=1}^{t^v_{v_k}} \left((d-2)(i-1)+1\right). \tag{2}$$



Given $(t_{v_1}^v, t_{v_2}^v, \cdots, t_{v_d}^v)$, the total number of ways to spread a rumor is

$$(n-1)! \prod_{k=1}^{d} \frac{\prod_{i=1}^{t_{v_k}^v} \left((d-2)(i-1)+1\right)}{t_{v_k}^v!}. \qquad (2)$$



Detection probability of d-regular trees

$$P_d(n) = \frac{\displaystyle\sum_{(t^v_{v_1}, t^v_{v_2}, \cdots, t^v_{v_d}) \in A_d} \left(\prod_{k=1}^{d} \frac{\prod_{k=1}^{t^v_{v_k}} \left((d-2)(i-1)+1\right)}{t^v_{v_k}!} \right)}{\displaystyle\sum_{(t^v_{v_1}, t^v_{v_2}, \cdots, t^v_{v_d}) \in B_d} \left(\prod_{k=1}^{d} \frac{\prod_{k=1}^{t^v_{v_k}} \left((d-2)(i-1)+1\right)}{t^v_{v_k}!} \right)}. \tag{3}$$

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Theorem 3

If G is a **3-regular tree**, then we have that

$$\lim_{t \to \infty} \mathrm{P}(E_t) = \frac{1}{4}.$$

Let

- $w_{dn} = \prod_{i=1}^{n} (d-2)(i-1) + 1.$

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- $f(x)$: The exponential generating function for the sequence $\{w_{dn}\}_{n=1}^{\infty}$

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And we have

- $(1-ax)^{-\frac{1}{a}} = \sum_{n=0}^{\infty} \binom{-\frac{1}{a}}{n} (-ax)^n = 1 + \sum_{n=1}^{\infty} \frac{\prod_{i=1}^{n} a(i-1)+1}{n!} x^n.$

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Hence we immediately know $f(x) = (1-ax)^{-\frac{1}{a}} - 1$ where $a = d-2$.

Detection probability of d-regular trees

Let $a = d - 2$,

$$P_d(n) = 1 - \frac{d \sum_{m \geq \frac{n}{2}}^{n-d} \frac{w_{dm}}{m!} \left([x^{n-1-m}] F_{d-1}(x) \right)}{[x^{n-1}] F_d(x)}. \quad (5)$$

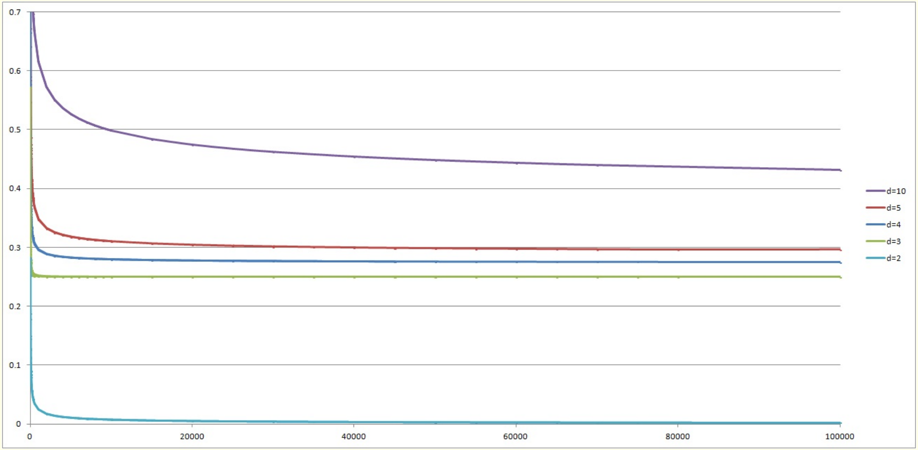
Where

$$w_{dn} = \prod_{i=1}^{n} \left(a\left(i - 1\right) + 1 \right),$$

$$[x^n] F_k(x) = \sum_{l=1}^{k} (-1)^{k-l} \binom{k}{l} \frac{\prod_{i=1}^{n} \left(l + a(i-1) \right)}{n!}.$$

Conclusion

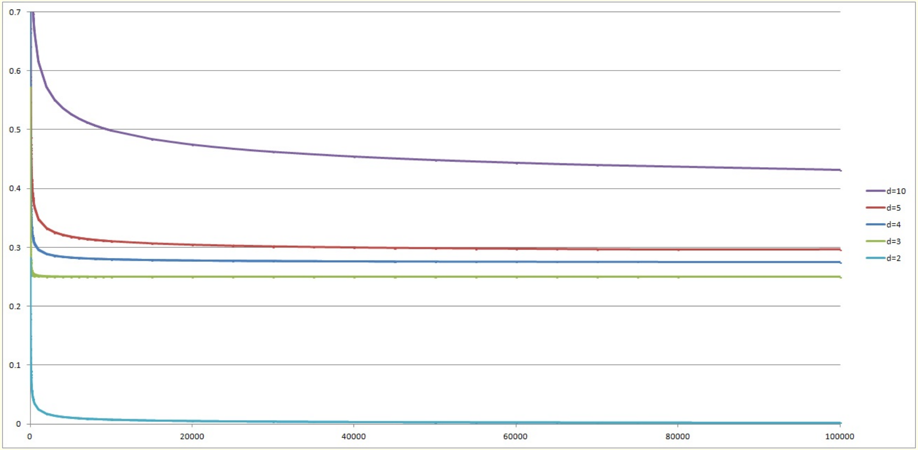
General behavior of detection probability for several d's and $n \leq 100,000$.



Conclusion

General behavior of detection probability for several d's and $n \leq 100,000$.

- $\frac{1}{2} \geq P_d(n) \geq P_{d'}(n) \geq \frac{1}{4}$ if $d \geq d' \geq 3$



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Thank you for listening.

祝 張鎮華教授

六十大壽

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