Permutation Patterns and ARM Identities

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Outline of this talk

- Permutation Patterns
- What's on Now
- Some of Our New Results
- Problems & Discussions

· Part of this talk is joint with

Fu (傅東山), Pan (潘業忠), Ting (丁建太), Yen (顏珮嵐).

Part 1

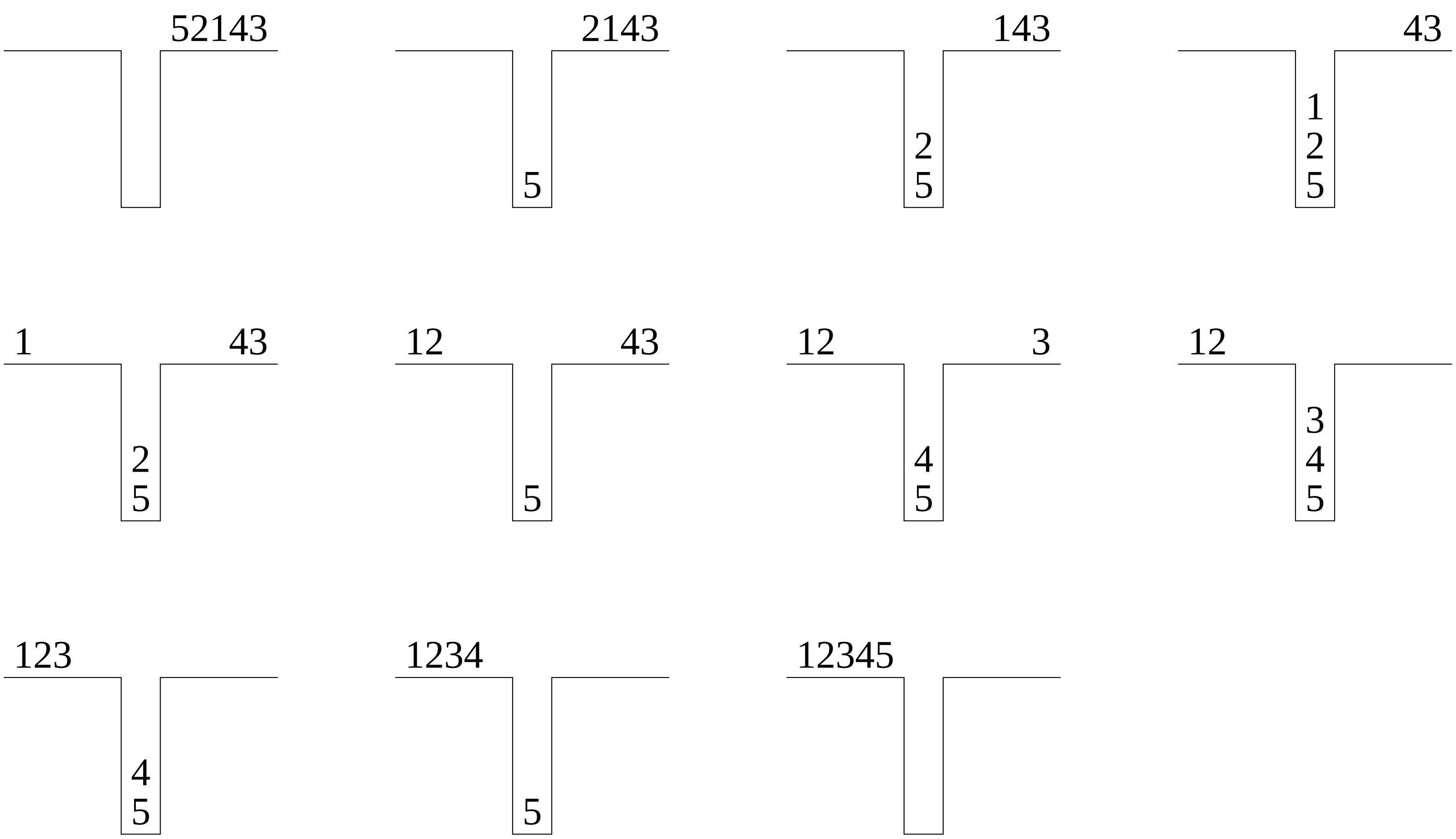
Permutation Patterns

(1) Stack Sortable

- Knuth considered the problem:

  What permutations are sortable by a stack??

- 52143 is stack sortable.



- 53421 is not stack sortable.

Theorem: [Knuth, 1968]

π is stack sortable $\iff$ π has no 231-pattern.

· 52143 has no 231 pattern.

· 53421 has 231 pattern(s).

Theorem: [Knuth, 1968] There are

$$\frac{1}{n+1}\binom{2n}{n} \qquad \text{(Catalan numbers again!)}$$

231-avoiding permutations of length n.

(2) Pattern avoiding permutations

- Given a pattern σ.
  - How many σ-avoiding permutations of length n are there?

- When $\sigma \in \mathfrak{S}_3$....

Theorem: [Schmidt, Simion, 1985] There are

$$\frac{1}{n+1}\binom{2n}{n}$$

123 (or $132, 213, 231, 312, 321$) -avoiding permutations of length n.

  - That means, when $\sigma \in \mathfrak{S}_3$, there is only 1 Wilf-class.

- When $\sigma \in \mathfrak{S}_4$
  - There are three 3 Wilf-classes: $1234, 1342, 1324$-avoiding.

Theorem: [Gessel, 1990] # 1234-avoiding permutations of length n is

$$2\sum_{k=0}^{n}\binom{2k}{k}\binom{n}{k}^2\frac{3k^2+2k+1-n-2kn}{(k+1)^2(k+2)(n-k+1)}.$$

Theorem: [Bona, 1997] # 1342-avoiding permutations of length n is

$$(-1)^{n-1}\left(\frac{7n^2-3n-2}{2}\right)+3\sum_{i=2}^{n}(-1)^{n-i}2^{i+1}\frac{(2i-4)!}{i!(i-2)!}\binom{n-i+2}{2}.$$

Open problem: # 1324-avoiding permutations of length n???

(3) Stanley-Wilf Conjecture

- The exact counting is usually not easy.
  - How about the the growth rate?
  - (**Stanley-Wilf conjecture**): is $\mathfrak{S}_n(\sigma) < k^n$ for some k?
  - This conjecture stimulates a lot of work.

Theorem: [Marcus, Tardos, 2004] The SW conjecture is correct. That is,

$$\mathfrak{S}_n(\sigma) < k^n$$

for some constant number k.

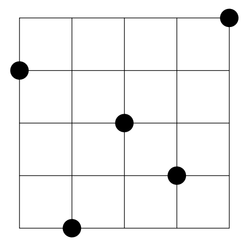
- Surprising elementary and extremely elegant solution.
- The exact value $\lim_{n\to\infty} \sqrt[n]{\mathfrak{S}_n(\sigma)}$ is again hard.

Part 2

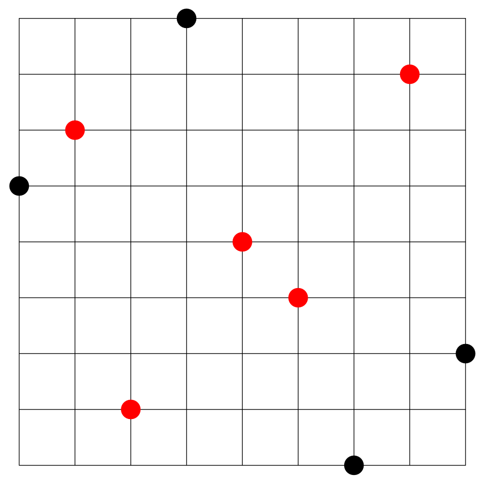
What’s on thereafter and now?

(1) Patterns

- New viewpoint of a “pattern":
  - 41325:



  - has 41325:



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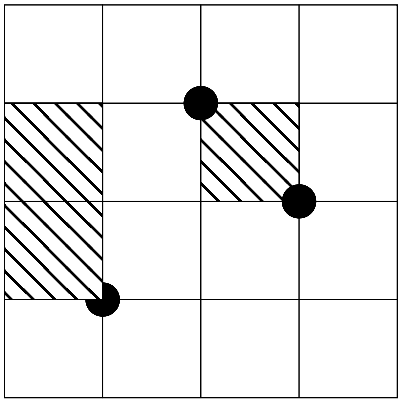
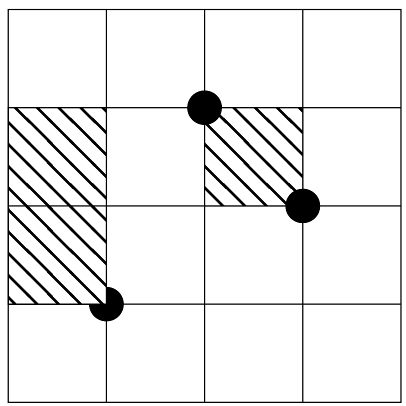
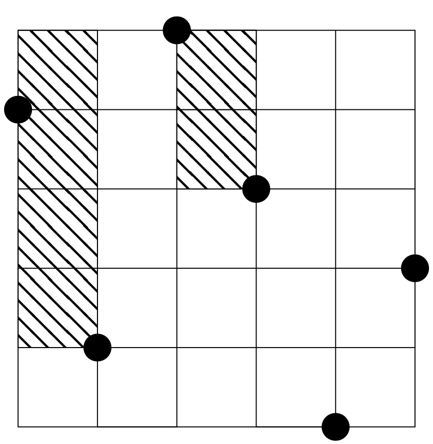
- Evolution of the concept of patterns:
  - · (classical) pattern: $2-1-3$
  - · generalized pattern: $2-13$,
  - · Barred pattern: $2-\overline{4}-1-3$
  - · Interval pattern: $[2-1-3)$,
  - · Mesh pattern (2011).
- A mesh pattern is a diagram, any square can be shaded:

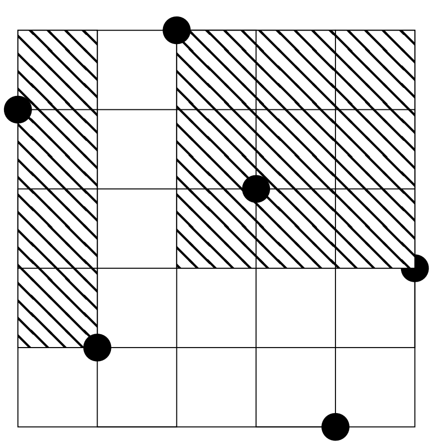


- The occurence of the mesh pattern:

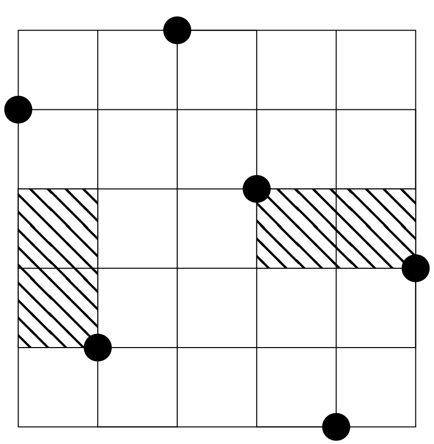




non−occurence



non−occurence

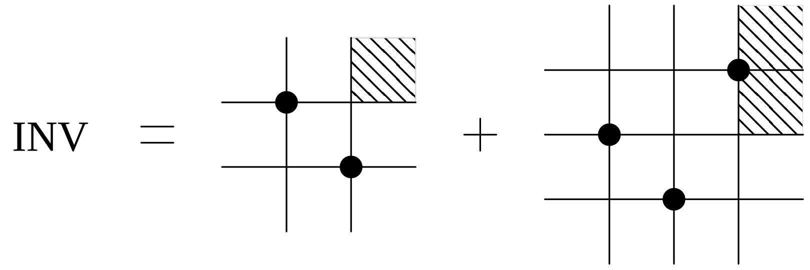


occurence

- Why mesh pattern?
  - · It includes (almost) all variations of ‘patterns’ above.
  - · It arises from the following striking theorem:

Theorem: [Brändén, Claesson, 2011, ElecJC]
Any permutation statistic may be represented uniquely as a, possibly infinite, linear combination of (classical) permutation patterns.

- · To provide explicit expansions, mesh patterns are introduced.



(2) Counting

- What we have at hand:
  - So many patterns.
  - So many classes. ($\mathfrak{S}_n$, $\mathfrak{A}_n$, Involutions, Derangements, etc.)

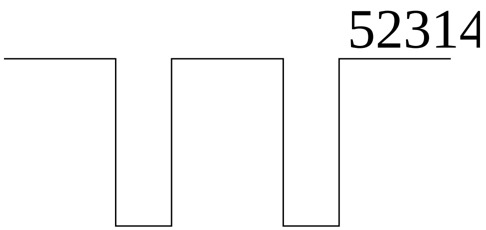
- **The Motto**:

$$\text{(patterns)} \times \text{(classes)}$$

  - Things to do:
    - avoiding some.
    - including some.
    - avoiding some and including some.
  - There are hundreds of papers.
  - Still very active, too many open problems.

(3) Stack sortable

- How about 2-stack sortable?
  - $\cdot$ 52314 is not 1-stack sortable, but is 2-stack sortable.



Theorem: [West, 1990]

π is 2-stack sortable $\iff$ π avoids $2341, 3\bar{5}241$

Theorem: [Zeilberger, 1992]

The # of 2-stack sortable permutations in $\mathfrak{S}_n$ is

$$\frac{2(3n)!}{(2n+1)!(n+1)!}$$

- How about 3-stack sortable?
  - Wildly open, until....

Theorem: [Ulfasson, 2011]

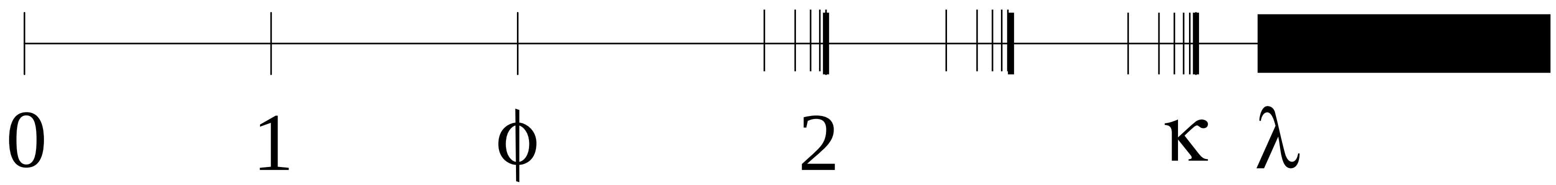
π is 3-stack sortable $\iff$ π avoids 10 (decorated) mesh patterns

Open Problem: How many 3-stack sortable permutations in $\mathfrak{S}_n$??

(4) Growth rate

- What number can be as an Stanley-Wilf limit (or Growth rate)?

Theorem: [Vatter, 2010, 2011]
Let the growth rate of class $\mathfrak{C}$ be $\limsup_{n\to\infty} \sqrt[n]{|\mathfrak{C}_n|}$. Then



· nothing from 1 to $\phi = \frac{\sqrt{5}-1}{2}$
· 2 is the first accumulation point.
· below $\kappa = 2.21$ one can characterize the growth rate.
· above $\lambda = 2.48$ every number can be the growth rate.

Open Problem: Fill the gap between κ and λ.

(5) Connection with other fields

Theorem: [Bousquet-Melou, Butler, 2007]
The Schubert variety X_π is factorial $\iff$ $\pi \in \mathfrak{S}(1324, 21\bar{3}54)$.

Theorem: [Ulfrasson, 2012]
The Schubert variety X_π is Gorenstein $\iff$ π avoids 2 'bivincular' patterns. (detail omitted).

Theorem: [Conklin, Woo, 2012]
• Bruhat graph B_π is planar iff $\pi \in \mathfrak{S}(321)$, and inv$(\pi) < 4$.
• Bruhat graph B_π is torodial iff $\pi \in \mathfrak{S}(3412)$, inv$(\pi) < 5$, and if inv$(\pi) = 4$ then $\pi \in \mathfrak{S}(321)$.

(6) Various "Avoiding"

Words: Words avoiding a shorter word.

· e.g. RNA sequence without A-G.

Trees: Trees avoiding a smaller tree.

· e.g. Binary tree avoiding a double V.

Posets: Posets avoiding a smaller poset.

· e.g. Posets avoiding $3+1$

· e.g. Posets avoiding a diamond — c.f. talk of Griggs yesterday.

Matrices: Matrices avoiding a smaller matrix.

· e.g. Alternating sign matrix avoiding $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

etc, etc...

(7) As a structure

- We have permutations avoiding some pattern(s).

Motto: Regard it as a new combinatorial object.

· For example:

Theorem: [Steingrimsson, Tenner, 2010]
$\sigma \in \tau$ defines a partial order on $\mathfrak{S}_n$. A large class of pairs (π_1, π_2) with Möbius function 0 is found.

Theorem: [Bona, 2012]
The total number 231 (or 321, or 213) patterns in all $\mathfrak{S}_n(132)$ is the same. (This is a surprising triple-symmetry.)

· Our new work falls into this category....

Part 3

Our work

321-avoiding

- Review:

Theorem [Simion, Schmidt, 1985]

$$|\mathfrak{S}_n(321)| = c_n,$$

i.e.

Theorem [Simion, Schmidt, 1985]

$$\sum_{\pi\in\mathfrak{S}_n(321)} 1 = c_n,$$

where $c_n = \frac{1}{n+1}\binom{2n}{n}$ is the Catalan number.

Sign-balance

- Amazingly, the signed counting is also a Catalan numbers.

Theorem [Simion, Schmidt, 1985]

$$\sum_{\pi\in\mathfrak{S}_n(321)} (-1)^{\mathrm{inv}(\pi)} = \begin{cases} c_{\frac{n-1}{2}}, \\ 0, \end{cases}$$

i.e.,

Theorem [Simion, Schmidt, 1985]

$$\begin{cases} \displaystyle\sum_{\pi\in\mathfrak{S}_{2n+1}(321)} (-1)^{\mathrm{inv}(\pi)} = \sum_{\pi\in\mathfrak{S}_n(321)} 1. \\ \displaystyle\sum_{\pi\in\mathfrak{S}_{2n}(321)} (-1)^{\mathrm{inv}(\pi)} = 0. \end{cases}$$

Adin-Roichman identities

- In 2004, Adin and Roichman gave a refinement.

$$\begin{aligned} \mathrm{ldes}(\pi) &:= \text{last descent of } \pi \\ &= \max\{1 \le i \le n-1 : \pi(i) > \pi(i+1)\} \\ &\quad (\mathrm{ldes}(\pi) := 0 \text{ if } \pi = \mathrm{id}) \end{aligned}$$

$\cdot$ $\mathrm{ldes}(216534) = 4.$

Adin-Roichman's identities

Theorem [Adin, Roichman, 2004]

$$\begin{cases} \sum_{\pi\in\mathfrak{S}_{2n+1}(321)} (-1)^{\mathrm{inv}(\pi)} q^{\mathrm{ldes}(\pi)} = \sum_{\pi\in\mathfrak{S}_n(321)} q^{2\mathrm{ldes}(\pi)} & (n \geq 0). \\ \sum_{\pi\in\mathfrak{S}_{2n}(321)} (-1)^{\mathrm{inv}(\pi)} q^{\mathrm{ldes}\pi} = (1-q) \sum_{\pi\in\mathfrak{S}_n(321)} q^{2\mathrm{ldes}(\pi)} & (n \geq 1). \end{cases}$$

n	1	2	3	4	5
$\sum(-1)^{\mathrm{inv}(\pi)} q^{\mathrm{ldes}(\pi)}$	1	$1-q$	1	$(1-q)(1+q^2)$	$1+q^2$
$\sum q^{\mathrm{ldes}(\pi)}$	1	$1+q$	—	—	—

· When $q = 1$, they reduces to Simon-Schmidt's identities.

Mansour's identities

At the same time, Mansour consider the $\mathfrak{S}_n(132)$.

$$\begin{aligned} \mathrm{find}(\pi) &:= \text{the index of the letter '1' in } \pi \\ &= \pi^{-1}(1) \end{aligned}$$

· $\mathrm{find}(216534) = 2$.

Theorem [Mansour, 2004] For $n \geq 1$,

$$\begin{cases} \displaystyle\sum_{\pi\in\mathfrak{S}_{2n+1}(132)} (-1)^{\mathrm{inv}(\pi)} q^{\mathrm{ldes}(\pi)} = \sum_{\pi\in\mathfrak{S}_{2n+1}(132)} q^{\mathrm{find}(\pi)-1}. \\ \displaystyle\sum_{\pi\in\mathfrak{S}_{2n}(132)} (-1)^{\mathrm{inv}(\pi)} q^{\mathrm{ldes}\pi} = (1-q) \sum_{\pi\in\mathfrak{S}_{n}(132)} q^{2(\mathrm{find}(\pi)-1)}. \end{cases}$$

$2n$ reduced to n phenomenon

- The Adin-Roichman-Mansour identities are essentially

  "$2n$ reduces to n phenomena"

  · "Signed enumerator on size $2n$" $=$ "enumerator on size n".

- "ARM-type identities"

  · bending the ARM — folding in half and double in thickness!

- We will present some new instances.

Part 3-1

Alternating permutations (321)

Alternating permutations

- $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ is alternating if
$$\pi_1 > \pi_2 < \pi_3 > \pi_4 < \ldots$$

- $\text{Alt}_n :=$ set of Alternating permutations of length n.
  - $\sum_{n\geq 0} |\text{Alt}_n| \frac{x^n}{n!} = \tan x + \sec x$.
  - $|\text{Alt}_n|_{n\geq 0} = 1, 1, 1, 2, 5, 16, 61, 272, 1385, 7936, \ldots$

- $\text{Alt}_n(321) :=$ Alt_n and avoiding 321.

Theorem [Deutsch, Reifegerste 2003, Mansour 2003]
$$|\text{Alt}_{2n}(321)| = |\text{Alt}_{2n-1}(321)| = \frac{1}{n+1}\binom{2n}{n}$$

- $|\text{Alt}_6(321)| = 5$. (214365, 215364, 314265, 315264, 415263).
- $|\text{Alt}_5(321)| = 5$. (21435, 21435, 31425, 31524, 41523).

Signed counting

- Motivation: signed counting of $\mathrm{Alt}_{2n}(321)$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\sum(-1)^{\mathrm{inv}}$	1	1	-1	0	0	-1	1	0	0	2	-2	0	0	-5	5

- **Theorem** [Pan, Ting, 2011, to appear]

- $$\sum_{\mathrm{Alt}_{4n+2}(321)} (-1)^{\mathrm{inv}(\pi)} = (-1)^{n+1} \sum_{\mathrm{Alt}_{2n}(321)} 1$$
- $$\sum_{\mathrm{Alt}_{4n+1}(321)} (-1)^{\mathrm{inv}(\pi)} = (-1)^{n} \sum_{\mathrm{Alt}_{2n}(321)} 1$$
- $$\sum_{\mathrm{Alt}_{4n}(321)} (-1)^{\mathrm{inv}(\pi)} = 0$$
- $$\sum_{\mathrm{Alt}_{4n-1}(321)} (-1)^{\mathrm{inv}(\pi)} = 0$$

ARM on $\mathrm{Alt}_n(321)$

- It is so similar to Simion-Schmidt's result.
  - Hence, it is natural to seek the ARM identities.

- **Theorem** [—, 2012]

  - $$\sum_{\mathrm{Alt}_{4n+2}(321)} (-1)^{\mathrm{inv}(\pi)} q^{\mathrm{lead}(\pi)} = (-1)^{n+1} \sum_{\mathrm{Alt}_{2n}(321)} q^{2\cdot\mathrm{lead}(\pi)}$$
  - $$\sum_{\mathrm{Alt}_{4n+1}(321)} (-1)^{\mathrm{inv}(\pi)} q^{\mathrm{lead}(\pi)} = (-1)^{n} \sum_{\mathrm{Alt}_{2n}(321)} q^{2\cdot\mathrm{lead}(\pi)}$$
  - $$\sum_{\mathrm{Alt}_{4n}(321)} (-1)^{\mathrm{inv}(\pi)} q^{\mathrm{lead}(\pi)} = (-1)^{n+1}(1-q) \sum_{\mathrm{Alt}_{2n}(321)} q^{2(\mathrm{lead}(\pi)-1)}$$
  - $$\sum_{\mathrm{Alt}_{4n-1}(321)} (-1)^{\mathrm{inv}(\pi)} q^{\mathrm{lead}(\pi)} = (-1)^{n}(1-q) \sum_{\mathrm{Alt}_{2n}(321)} q^{2(\mathrm{lead}(\pi)-1)}$$

Another

- ‘Ending’ ($\mathrm{end}(\pi) := \pi_n$) also works!

- **Theorem** [—, 2012]

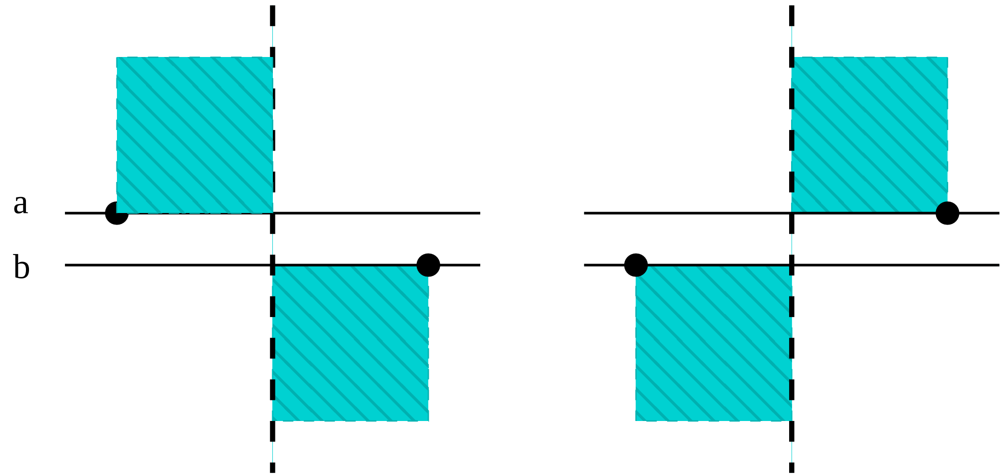
- $$\sum_{\mathrm{Alt}_{4n+2}(321)} (-1)^{\mathrm{inv}(\pi)} q^{\mathrm{end}(\pi)} = (-1)^{n+1} \sum_{\mathrm{Alt}_{2n}(321)} q^{2\cdot\mathrm{end}(\pi)}$$
- $$\sum_{\mathrm{Alt}_{4n+1}(321)} (-1)^{\mathrm{inv}(\pi)} q^{\mathrm{end}(\pi)} = (-1)^{n} \sum_{\mathrm{Alt}_{2n}(321)} q^{2\cdot\mathrm{end}(\pi)}$$
- $$\sum_{\mathrm{Alt}_{4n}(321)} (-1)^{\mathrm{inv}(\pi)} q^{\mathrm{end}(\pi)} = (-1)^{n}(1-q) \sum_{\mathrm{Alt}_{2n}(321)} q^{2(\mathrm{end}(\pi)-1)}$$
- $$\sum_{\mathrm{Alt}_{4n-1}(321)} (-1)^{\mathrm{inv}(\pi)} q^{\mathrm{end}(\pi)} = (-1)^{n+1}(1-q) \sum_{\mathrm{Alt}_{2n}(321)} q^{2(\mathrm{end}(\pi)-1)}$$

Part 3-2

Baxter permutations (321)

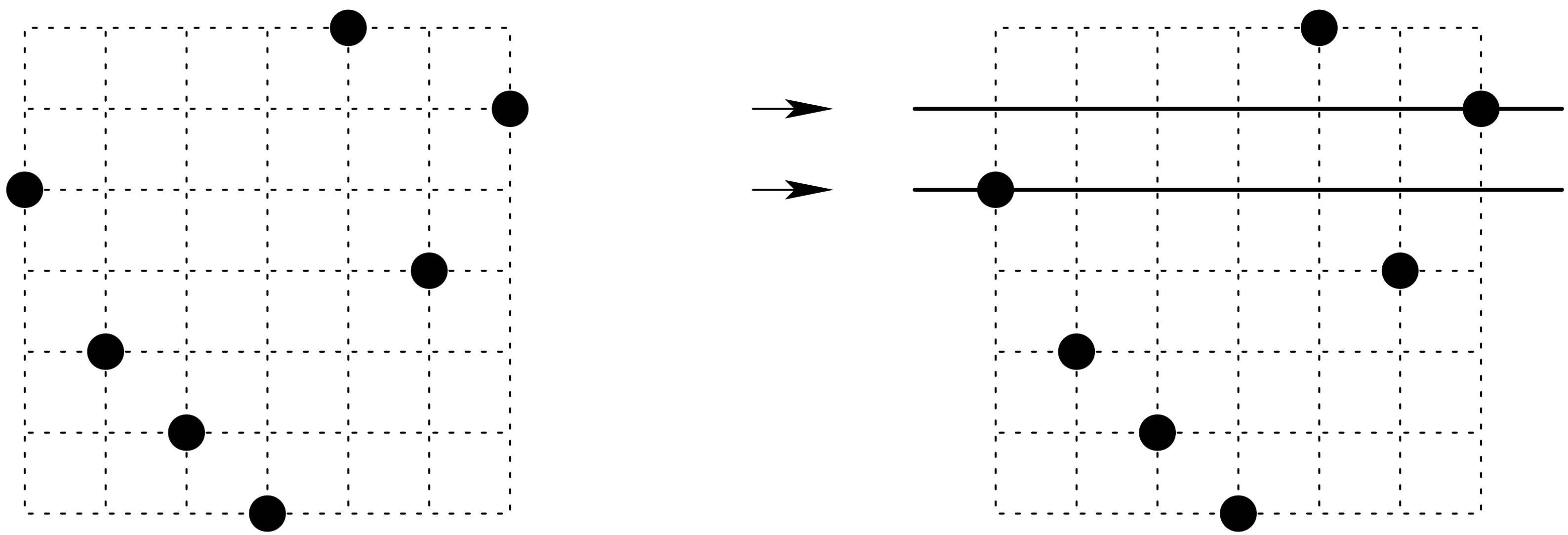
Baxter permutations

- Baxter permutatios:= $\mathfrak{S}_n(2\text{-}41\text{-}3, 3\text{-}14\text{-}2)$.
- $\pi \in \mathfrak{S}_n$ is Baxter if for all $1 \le a < b < c < d \le n$,
  - if $\pi_a + 1 = \pi_d$ and $\pi_b > \pi_d$, then $\pi_c > \pi_d$.
  - if $\pi_d + 1 = \pi_a$ and $\pi_c > \pi_a$, then $\pi_b > \pi_a$.
- For every 2 black dots whose height differ by 1,
  - ...the broken line can be found,
  - ...such that dots in between are in shaded area.



Baxter permutation

- 5321746 is **not** Baxter.



· $\mathrm{Bax}_n :=$ set of Baxter permutations of length n.

Theorem [Chung, Graham, Hoggatt, Kleitman, 1978]

$$|\mathrm{Bax}_n| = \frac{2}{n(n+1)^2}\sum_{k=1}^{n}\binom{n+1}{k-1}\binom{n+1}{k}\binom{n+1}{k+1}.$$

Baxter permutation avoiding 123

· $\text{Bax}_n(123) :=$ Bax_n and avoid 123.

Theorem [Mansour, Vajnovszki, 2007]

$$\sum_{n\geq 0} |\text{Bax}_n(123)| z^n = \frac{1-2z+z^2}{1-3z+2z^2-z^3}.$$

· $|\text{Bax}_n(123)|_{n\geq 0} = 1, 1, 2, 5, 12, 28, 65, \dots$.

· $|\text{Bax}_n(123)|_{n\geq 0} = p_{3n}$, a Padovan number, defined by

$(p_0, p_1, p_2) = (1, 0, 0)$ and $p_n = p_{n-2} + p_{n-3}$.

· $|\text{Bax}_n(321)| = |\text{Bax}_n(123)|$

ARM on $\mathrm{Bax}_n(321)$

- We have found an ARM-type identity on $\mathrm{Bax}_n(321)$.

Theorem [—, 2012, to appear] For $n \geq 0$, we have

$$\sum_{\pi \in \mathrm{Bax}_{2n+1}(321)} (-1)^{\mathrm{maj}(\pi)} p^{\mathrm{fix}(\pi)} q^{\mathrm{des}(\pi)} = p \cdot \sum_{\pi \in \mathrm{Bax}_n(321)} p^{2 \cdot \mathrm{fix}(\pi)} q^{2\mathrm{des}(\pi)}.$$

· The sign is controlled by maj (somewhat surprising)

$$\mathrm{maj}(\pi) := \sum_{\pi_i > \pi_{i+1}} i$$

· $\mathrm{maj}(536214) = 1 + 3 + 4 = 8$

On Bax$_{2n}(321)$

- For the even length, the corresponding ARM identity is a sum.

Theorem [—, 2012, to appear in ElecJC] For $n \geq 0$, we have

$$\sum_{\pi \in \mathrm{Bax}_{2n}(321)} (-1)^{\mathrm{maj}(\pi)} p^{\mathrm{fix}(\pi)} q^{\mathrm{des}(\pi)}$$

$$= (-1)^n \sum_{\pi \in \mathrm{Bax}_n(321)} p^{2\cdot \mathrm{fix}(\pi)} q^{2\mathrm{des}(\pi)} - q \sum_{i=0}^{n-1} \sum_{\pi \in \mathrm{Bax}_i(321)} p^{2\cdot \mathrm{fix}(\pi)} q^{2\mathrm{des}(\pi)}.$$

Part 3-3

Simsun permutations (312)

Simsun permutations

- $\pi \in \mathfrak{S}_n$ is **simsun** if for all $1 \leq k \leq n$,
  - $\cdot$ π restricted to $\{1, 2, \ldots k\}$ has no double descent.
  - $\cdot$ 6274351 is **not** simsun,
    - $\cdot$ 2431 has double descent 431.

- Let $|\mathrm{SS}_n|$ be the simsun permutations of length n.
  - $\cdot$ **Theorem** [Simion, Sundaram].

$$|\mathrm{SS}_n| = |\mathrm{Alt}_{n+1}|.$$

Simsun permutations

- π is **double simsun** if both π and π^{-1} are simsun.
  - 51324 is simsun but **not** double simsun,
    - since $(51324)^{-1} = 24351$ is not simsun.
- Let $|\mathrm{DS}_n|$:= double simsun permutations of length n.
  - $|\mathrm{DS}_n|$ is still unknown. However,....

Theorem [Chuang, Eu, Fu, Pan, 2012].

$$
\begin{aligned}
|\mathrm{DS}_n(123)| &= 2, \\
|\mathrm{DS}_n(132)| &= S_n, \\
|\mathrm{DS}_n(213)| &= S_n, \\
|\mathrm{DS}_n(231)| &= 2^{n-1}, \\
|\mathrm{DS}_n(312)| &= 2^{n-1}, \\
|\mathrm{DS}_n(321)| &= \tfrac{1}{n+1}\tbinom{2n}{n},
\end{aligned}
$$

- where S_n is the 'RNA secondary structrue number'.

ARMs on $\mathrm{DS}_n(312)$

Theorem [—, 2012]. For $n \geq 1$, we have

$$\sum_{\pi \in \mathrm{DS}_{2n+2}(312)} (-1)^{\mathrm{maj}(\pi)} \cdot q^{\mathrm{fix}(\pi)} = (-1+q^2) \sum_{\pi \in \mathrm{DS}_n(312)} q^{2\mathrm{fix}(\pi)}.$$

Theorem [—, 2012]. For $n \geq 2$, we have

$$\sum_{\pi \in \mathrm{DS}_{2n-1}(312)} (-1)^{\mathrm{maj}(\pi)} \cdot q^{\mathrm{lead}(\pi)} = \frac{2}{q(1+q^2)} \sum_{\pi \in \mathrm{DS}_n(312)} q^{2\mathrm{lead}(\pi)}.$$

Idea of Proof

$\textbf{Alt}_{4n+2}(321)$ **Sketch Proof**

- **Step 1**: Map everything to trees. $\text{Alt}_{2n} \longleftrightarrow \text{T}_n$
  - T_n := plane trees with n edges.
    - $\text{hsum}(T)$:= sum of heights of all nodes of T.
    - $\text{lmp}(T)$:=nodes on the leftmost path of T

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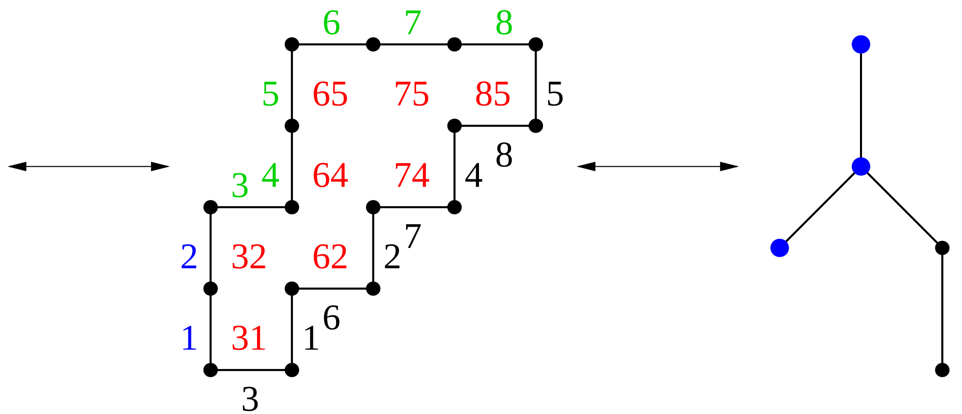
31, 32, 62, 64
65, 74, 75, 85



$$\text{inv}(\pi) \longleftrightarrow \text{hsum}(T)$$
$$\text{lead}(\pi) \longleftrightarrow \text{lmp}(T)$$

$\textbf{Alt}_{4n+2}(321)$ Sketch Proof

- **Step 2**: Look at idenities on trees.

We are to prove

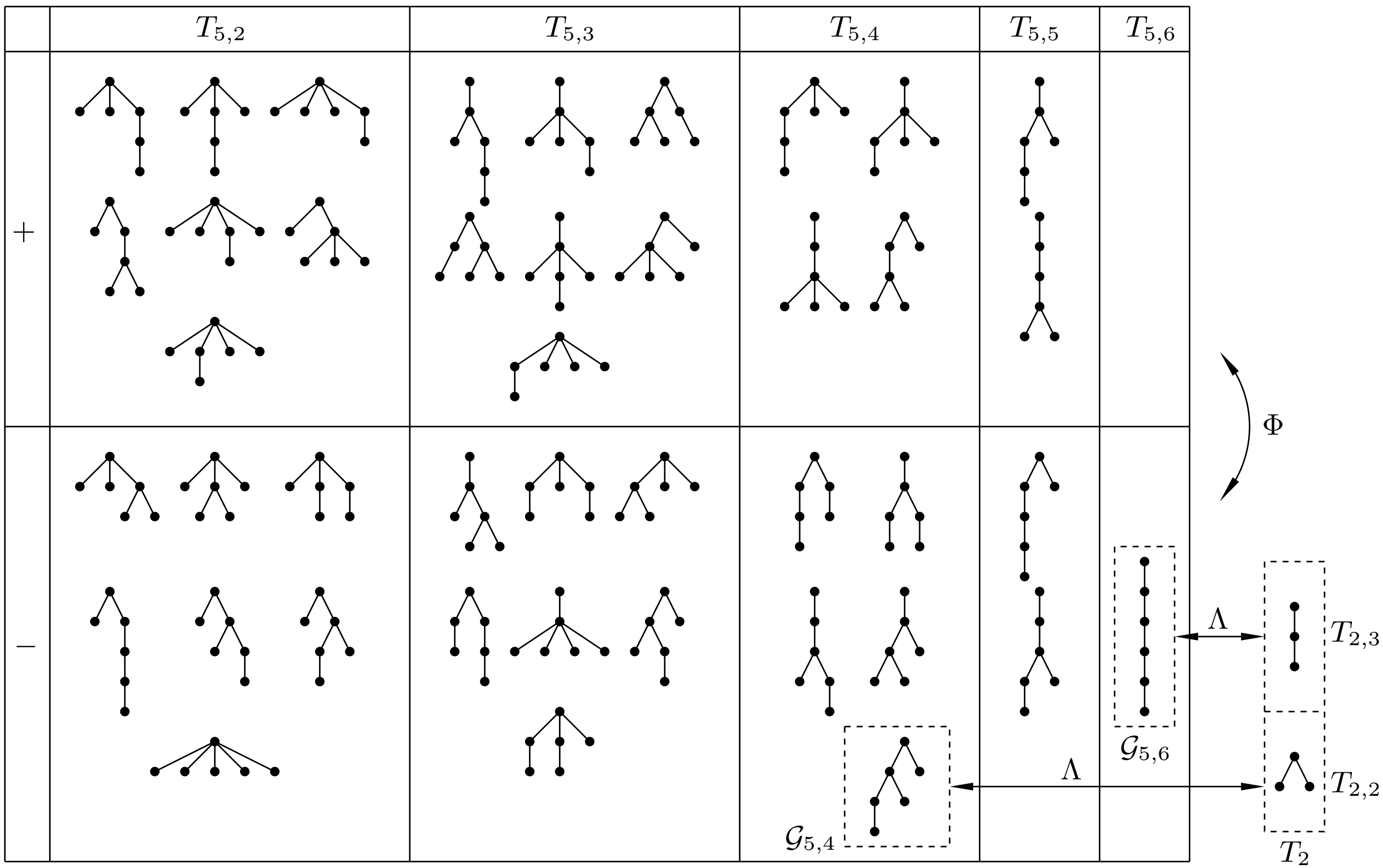
$$\sum_{\mathrm{Alt}_{10}(321)} (-1)^{\mathrm{inv}(\pi)} q^{\mathrm{lead}(\pi)} = (-1)^3 \sum_{\mathrm{Alt}_4(321)} q^{2\cdot \mathrm{lead}(\pi)}$$

which becomes

$$\sum_{\mathrm{T}_5} (-1)^{\mathrm{hsum}(T)} q^{\mathrm{lmp}(T)} = (-1) \sum_{\mathrm{T}_2} q^{2\cdot \mathrm{lmp}(T)}$$

$\mathbf{Alt}_{4n+2}(321)$ Sketch Proof

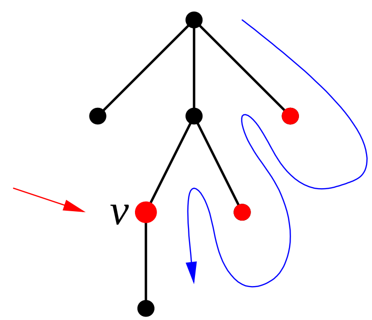
- **Step 3**: Devise an involution Φ (and a bijection Λ).



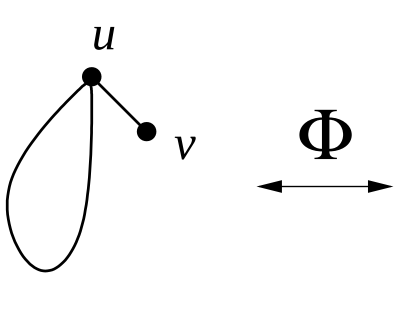
$$\sum_{\mathrm{T}_5} (-1)^{\mathrm{hsum}(T)} q^{\mathrm{lmp}(T)} = (-1) \sum_{\mathrm{T}_2} q^{2\cdot \mathrm{lmp}(T)}$$

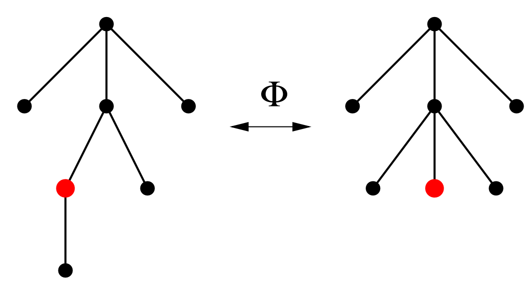
$\textbf{Alt}_{4n+2}(321)$ Sketch Proof

- Φ: an involution (to cancel w.r.t. hsum).
  - Find the last illegal vertex v via postorder
    - illegal vertex: (i) “leaf, but is not the first child” or
      (ii) “inner vertex, but is the first child".



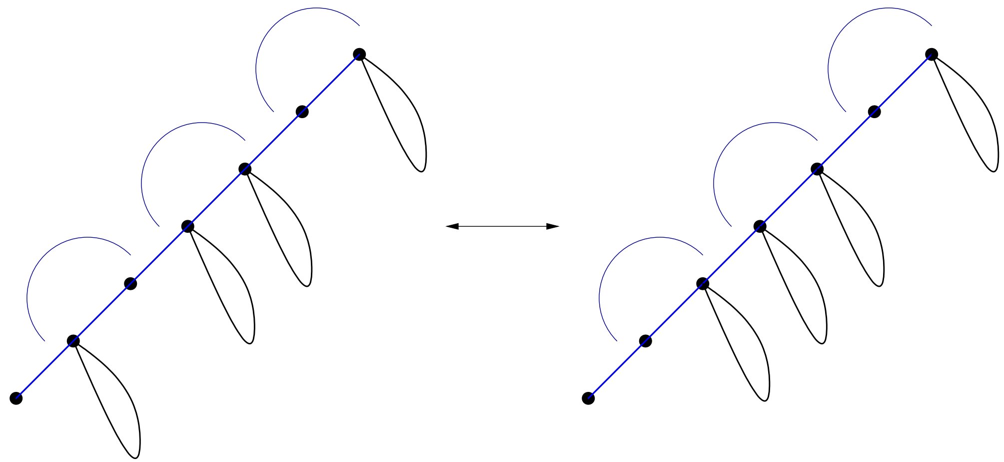
  - Do the following (hsum will change sign, lmp keep unchanged).



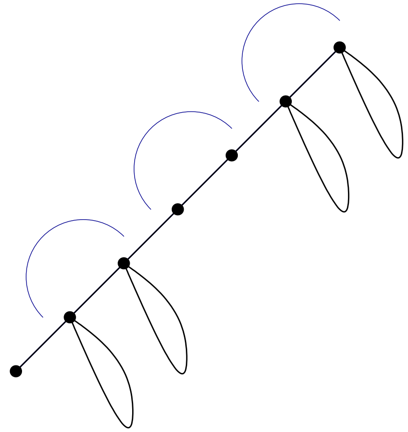


$\mathbf{Alt}_{4n+2}(321)$ Sketch Proof

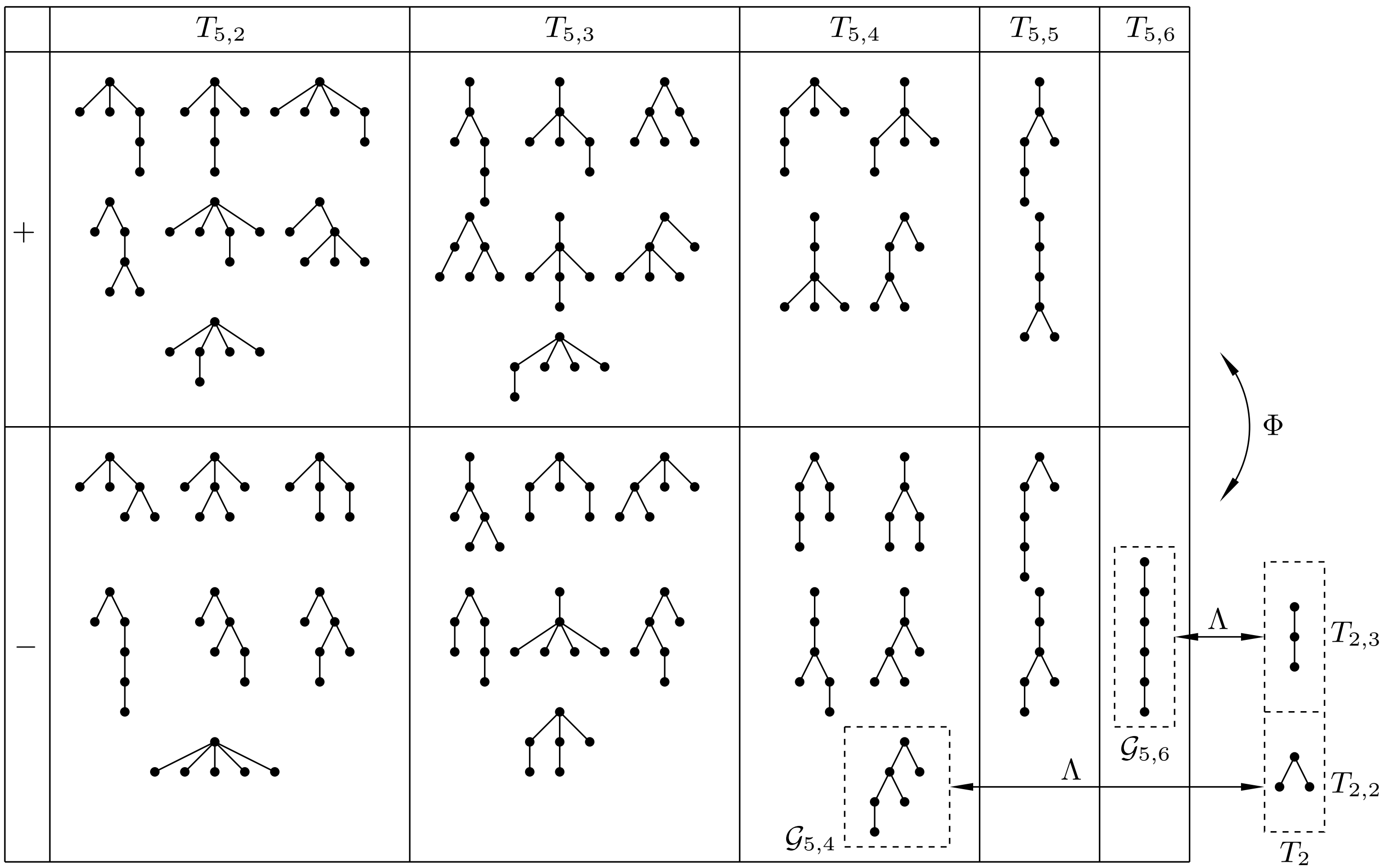
- If $\nexists$ illegal point, change the last (tree, vertex) $\longleftrightarrow$ (vertex, tree).



- If $\nexists$ illegal point, $\nexists$ (tree, vertex), $\nexists$ (vertex, tree), $\Rightarrow$ a fix point.



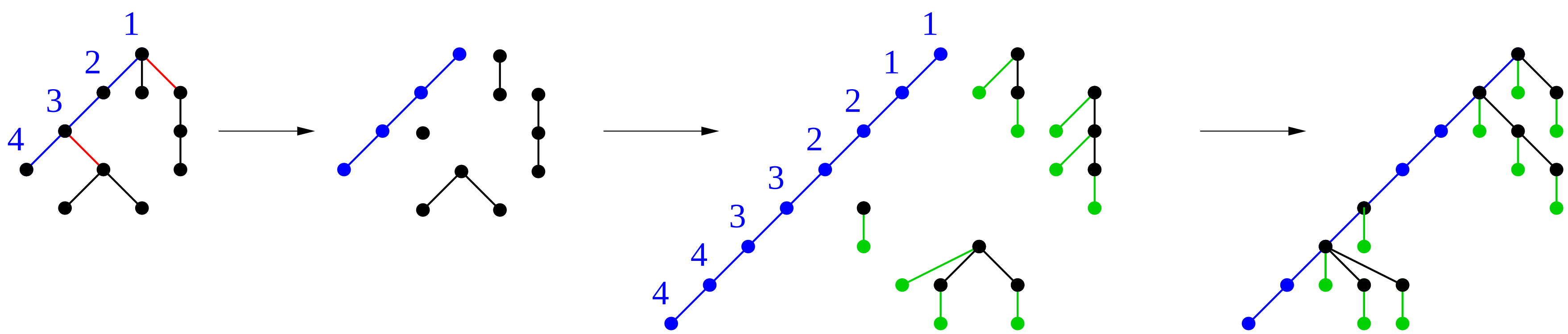
Alt$_{4n+2}(321)$ Sketch Proof



- The remaining task is to map fix points to T_2.
  - We need a bijection Λ: $T_2 \to T_5$

Alt$_{4n+2}(321)$ **Sketch Proof**

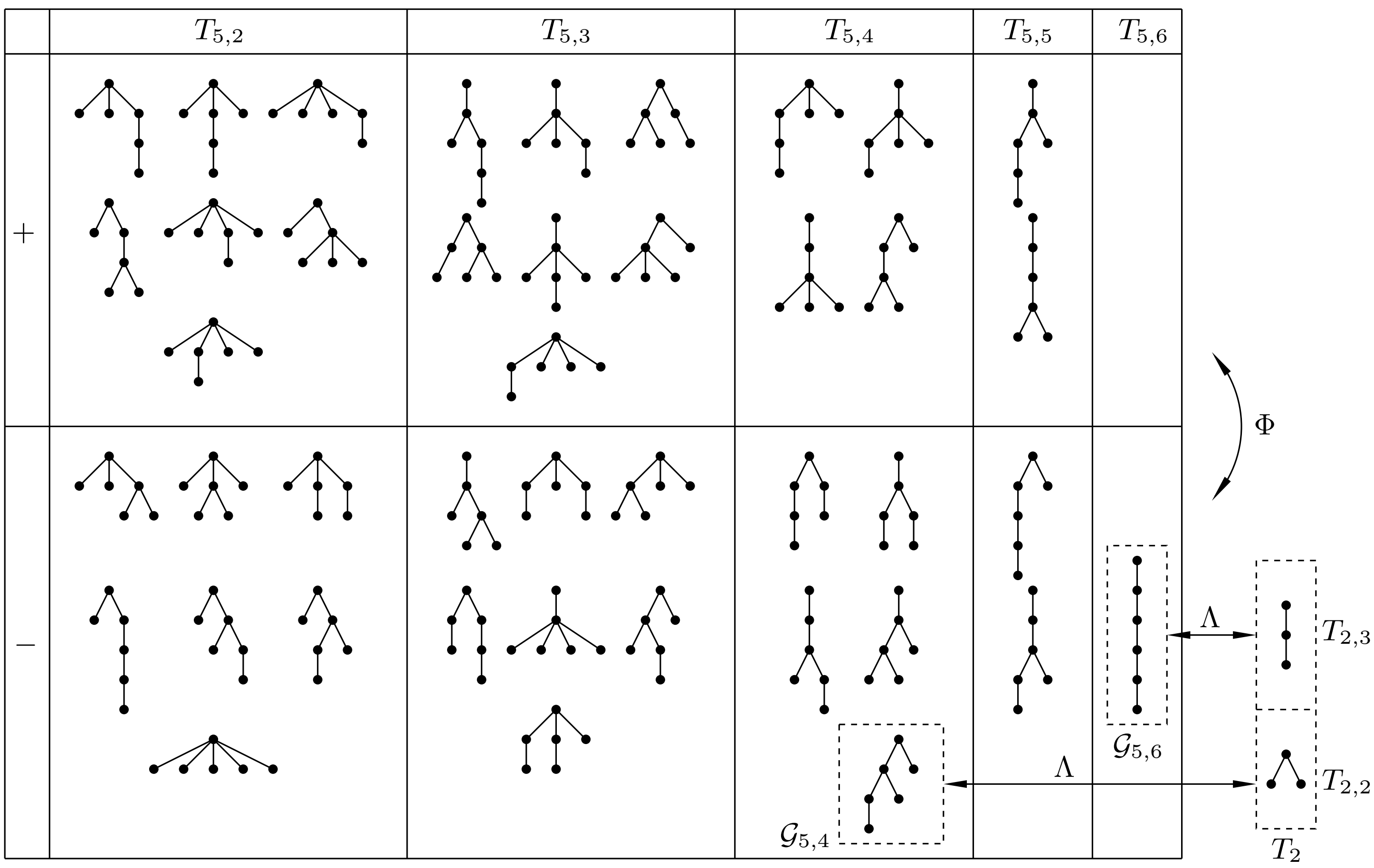
- Λ: a bijection (from T_n to T_{2n+1}).



- Idea:
  - Double the points on the left most path.
  - For each subtree, append a single edge on each vertex.

$\mathbf{Alt}_{4n+2}(321)$ Sketch Proof

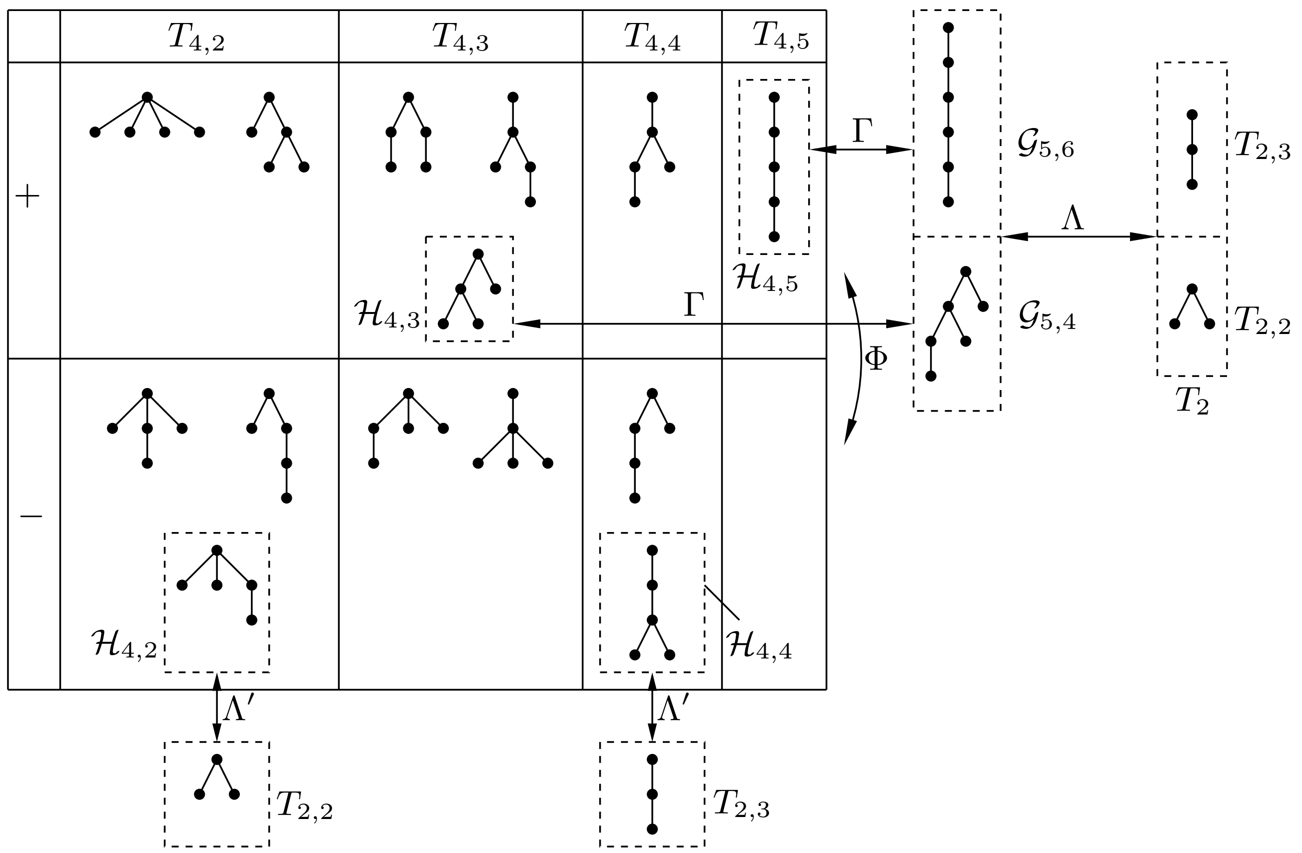
- Let us look the table again:



$$\sum_{\mathrm{T}_5} (-1)^{\mathrm{hsum}(T)} q^{\mathrm{lmp}(T)} = (-1) \sum_{\mathrm{T}_2} q^{2\cdot \mathrm{lmp}(T)}$$

$\text{Alt}_{4n}(321)$ Sketch Proof

- Proof of $\text{Alt}_{4n}(321)$:



$$\sum_{\text{T}_4}(-1)^{\text{hsum}(T)}q^{\text{lmp}(T)} = -\sum_{\text{T}_2}q^{2(\text{lmp}(T)-1)} + q\sum_{\text{T}_2}q^{2(\text{lmp}(T)-1)}$$

Sketch proof of $\text{Bax}_n(321)$

- Done by multivariable generating functions.
  - Let $b_n := b_n(t,p,q) = \sum_{\pi \in \text{Bax}_n(321)} t^{\text{maj}(\pi)} p^{\text{fix}(\pi)} q^{\text{des}(\pi)}$.
  - We are to prove $b_{2n+1}(-1,p,q) = p \cdot b_n(1,p^2,q^2)$.
- Very sketched proof:
  - $g_n := b_n(1,p,q)$

    $$g_n = (2+p)g_{n-1} - (1+2p-q)g_{n-2} + p \cdot g_{n-3}.$$

  - $h_n := b_n(-1,p,q)$

    $$h_{2n+1} = (2+p^2)h_{2n-1} - (1+2p^2-q^2)g_{2n-3} + p^2 \cdot h_{2n-5}.$$

  - $h_{1/3/5}(p,q) = p \cdot g_{0/1/2}(p^2,q^2)$ QED.
- **Problem:** Is there a combinatorial proof?

Sketch proof of $\mathrm{DS}_n(312)$

- Done by an involution on compostitions of $[n]$.

- Composition of $[n]$ with $\mathrm{DS}_n(312)$: [Eu et al, 2012]
  - $3+4+2 \longleftrightarrow 123|4567|89 \longleftrightarrow 312745698$

- We are to prove

$$\sum_{\pi \in \mathrm{DS}_{2n+2}(312)} (-1)^{\mathrm{maj}(\pi)} \cdot q^{\mathrm{fix}(\pi)} = (-1+q^2) \sum_{\pi \in \mathrm{DS}_n(312)} q^{2\mathrm{fix}(\pi)}.$$

$$\sum_{\pi \in \mathrm{DS}_{2n-1}(312)} (-1)^{\mathrm{maj}(\pi)} \cdot q^{\mathrm{lead}(\pi)} = \frac{2}{q(1+q^2)} \sum_{\pi \in \mathrm{DS}_n(312)} q^{2\mathrm{lead}(\pi)}.$$

- Too sketched proof:
  - Observer what maj, fix, lead means on compositions.
  - Design involutions (...not trivial...).

QED.

Part 5

Discussions

Naive believes were refuted over & over

- ...ARM appears only in $\mathfrak{S}_n(321)$?

  $\rightarrow$ NO, we have $\text{Alt}_n(321)$.

- ...ARM appears only in Catalan family?

  ...sign only controlled by inversion?

  $\rightarrow$ NO, we have $\text{Bax}_n(321)$.

- ...ARM appears only in 321-avoiding?

  $\rightarrow$ NO, we have $\text{DS}_n(312)$.

- ...If ARM appears, then $f(q)$ is a polynomial?

  $\rightarrow$ NO, $f(q) = \frac{2}{q(1+q^2)}$ in $\text{DS}_n(312)$.

- ...If ARM appears, then sign controlled by a Mahonian statistics?

  $\rightarrow$ NO, we have $(-1)^{\text{fix}}$ in $\text{DS}_n(312)$.

General Setting

- Conceptually, we are searching for

$$\mathrm{C}_n, (\mathsf{stat}_2, \mathsf{stat}_2)$$

such that essentially the following holds:

$$\sum_{\mathrm{C}_{2n}} (-1)^{\mathsf{stat}_1} q^{\mathsf{stat}_2} = f(q) \sum_{\mathrm{C}_n} q^{2 \cdot \mathsf{stat}_2},$$

where $f(q)$ is some rational polynomial.

- In this talk we present
  - $\cdot$ $\mathrm{Alt}_n(321)$, $(\mathrm{inv}, \mathrm{lead})$
  - $\cdot$ $\mathrm{Bax}_n(321)$, $(\mathrm{maj}, \mathrm{des})$
  - $\cdot$ $\mathrm{DS}_n(312)$, $(\mathrm{maj}, \mathrm{fix})$
  - $\cdot$ $\mathrm{DS}_n(312)$, $(\mathrm{maj}, \mathrm{lead})$

Are there others?

- Yes, we have about another 50^{+} nontrivial results.

- E.g.
  - $\mathrm{Alt}_n(321)$, $(\mathrm{chg}, \mathrm{ldes})$
  - $\mathrm{Alt}_n(321)$, $(\mathrm{cochg}, \mathrm{exc})$
  - $\mathrm{Bax}_n(321)$, $(\mathrm{chg}, \mathrm{des})$
  - $\mathrm{Bax}_n(321)$, $(\mathrm{imaj}, \mathrm{fix})$
  - $\mathrm{DS}_n(231)$, $(\mathrm{fix}, \mathrm{end})$
  - $\mathrm{DS}_n(231)$, $(\mathrm{maj}, \mathrm{inv})$
  - ...etc.

- However, ARM identities are relatively rare.
  - C_n, $(\mathrm{stat}_2, \mathrm{stat}_2)$ **seldom** produces ARM.

Discussions

- Encouraging:

  We have a bunch of nontrivial idenities.

- Disappointing:

  Proofs are done case by case.

- **Question:** Why is there ARM type identities?
  - What is lurking behind?
  - Is there a unifying/systemetic approach?
  - Can it be generalized to, say, Coxter groups?

- We are working on these questions, with some progress.

Thanks

And....

Happy Birthday to Gerard!

Welcome any discussions and collaboration!