Relation between Graphs

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WGTC2012, Kaohsiung

Motivation

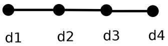
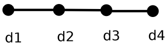


Figure: Domain interaction graph

Relation: containing relation

$$\{d_2, d_3 \in p_1; d_4 \in p_2; d_1, d_3 \in p_3; d_1 \in p_4\}$$

Motivation



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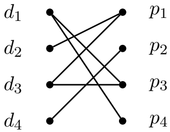
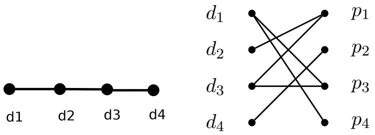


Figure: Relation

We can define a protein interaction graph by any two proteins have interaction iff there are two domains which belong to them respectively.



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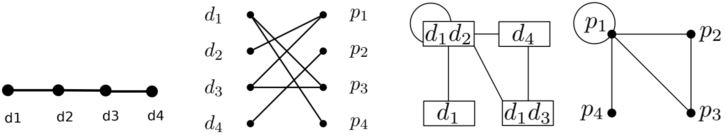
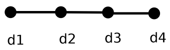


Figure: From Domain Interaction Network to Protein Interaction Network

Find relation between graphs



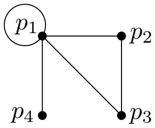


Figure: Domain interaction network Protein Interaction Network

relation R =

$$\{(d_2, p_1), (d_3, p_1), (d_4, p_2), (d_1, p_3), (d_3, p_3), (d_1, p_4)\}$$

Basic Definitions

Definition

The (binary) relation R between two sets A and B is a subset of $A \times B$.

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Given a graph $G = (V_G, E_G)$, a finite set B and a binary relation $R \subset V_G \times B$, we define a new graph $H = G * R$ whose vertex set is B and for arbitrary $u, v \in B$, $(u, v) \in E_H$ iff one can find $x, y \in V_G$, such that $(x, u), (y, v) \in R$ and $(x, y) \in E_G$.

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Definition

Given two graphs G and H, if there exists a $R \subset V_G \times V_H$, such that $G * R = H$, we say R is a relation from G to H.

Definition

Graph homomorphism $G \to H$ is a mapping $f\colon V_G \to V_H$ such that $(x, y) \in E_G$ implies $(f(x), f(y)) \in E_H$.

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Definition

Surjective multihomomorphism is a multihomomorphism such that pre-image of every vertex in H is non-empty and for every edge (u, v) in H we can find an edge (x, y) in G satisfying $u \in f(x)$, $v \in f(y)$.

Relation VS Surjective multihomomorphism

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Definition

R is **full** if for every u, there is $(u, v) \in R$.

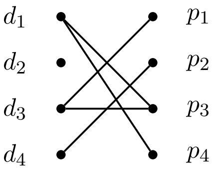

Figure: Non-full relation

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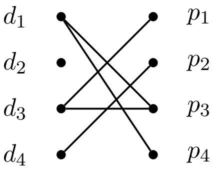

Figure: Non-full relation

If R is full and $G * R = H$, then R is a surjective multihomomorphism.

Relation VS Homomorphism

- If there is a surjective homomorphism $f : G \to H$, then $G * R = H$ for R corresponding to f.
- If $G * R = H$ and R is full, then there is a homomorphism $f : G \to H$.

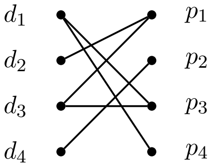

Figure: Homomorphism

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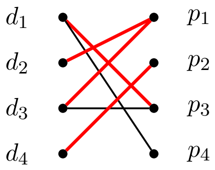

Figure: Homomorphism

Compose and Decompose

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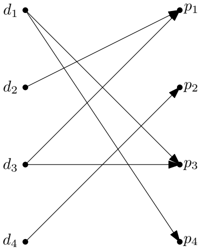
Graph relation R can be decomposed to a relation R_D duplicating vertices and a relation R_C contracting vertices.



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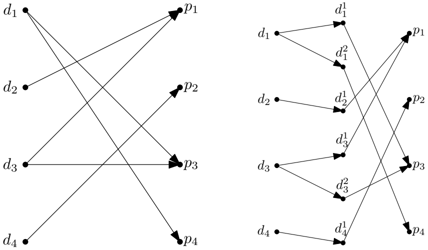
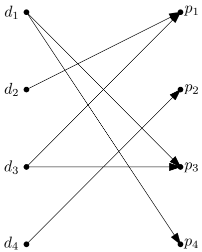
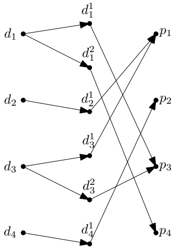


Figure: Decomposition


d_1
d_2
d_3
d_4
p_1
p_2
p_3
p_4


d_1
d_2
d_3
d_4
d_1^1
d_1^2
d_2^1
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p_1
p_2
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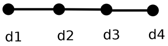
p_4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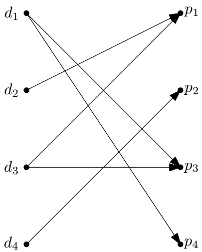
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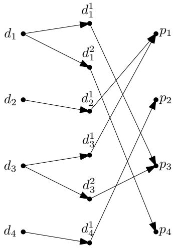
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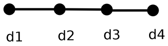
p_3
p_4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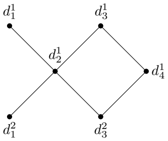
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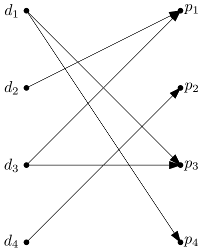
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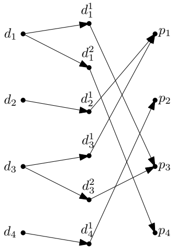
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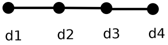
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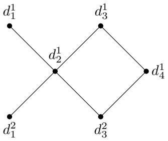
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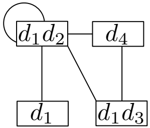
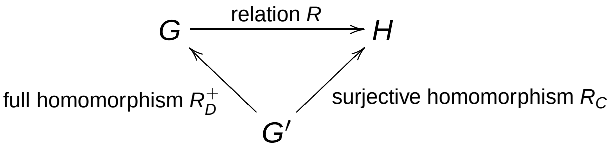

d_1d_2
d_4
d_1
d_1d_3



Full homomorphism: preserve both edge and non-edge

Surjective homomorphism: contraction

Let $\leq_{\mathrm{P}}^{\mathrm{Tur}}$ indicate polynomial time Turing reduction.
Fix a graph H,

- *Homomorphism problem* $\textsc{Hom}(H)$
- *Full relation problem* $\textsc{Ful-Rel}(H)$
- *Surjective homomorphis problem* $\textsc{Sur-Hom}(H)$

Theorem

For finite H our relation problem sits in the following relationship:

$$\textsc{Hom}(H) \leq_P^{Tur} \textsc{Ful-Rel}(H) \leq_P^{Tur} \textsc{Sur-Hom}(H).$$

$G * R = G$

Theorem

All solutions of $G * R = G$ are automorphisms if and only if G has property that the neighborhoods of vertices do not contain each other.

In graph homomorphism, core of graph G is the smallest subgraph of G which G has homomorphism to.

Core of graph G is unique up to isomorphism, so we can it the core, denote it by G_{core}.

Example

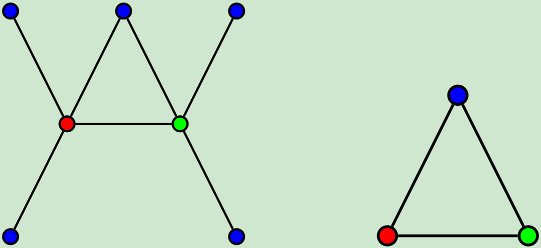


Figure: G and its core

Why core?

- G and G_{core} are homomorphically equivalent, i.e.,$G \to G_{\text{core}}$ and $G_{\text{core}} \to G$.
- Test whether G has homomorphism to H is equivalent to testing $G_{\text{core}} \to H_{\text{core}}$.
- Cores are the smallest graphs (in the number of vertices) in the class of homomorphism equivalence.
- Core is the smallest representative element in category theory.

Outline

Definition

Two graphs G and H are relationally equivalent, $G \backsim H$, if there are relations R and S such that $G * R = H$ and $H * S = G$.
If R and S are full, we write $G \backsim_f H$.

An R-core of a graph G, $G_{\text{R-core}}$, is the smallest graph in the same equivalence class of $\backsim_f$ as G.

Example

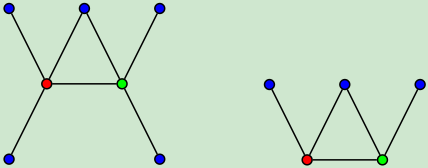


Figure: G and its R-core

R-core is unique up to isomorphism.

Proposition

$G_{\text{R-core}}$ is isomorphic to an induced subgraph of G.

If G is an R-core, then every relation R such that $G * R = G$ satisfies the Hall condition and thus contains a monomorphism.

Proposition

$G_{R\text{-core}}$ can be characterized by an algorithm, which removes all vertices $v \in G$ such that
(1) the neighborhood of v is union of neighborhood of some other vertices $v_1, v_2, \leq v_n$ and
(2) there is vertex u such that $N_G(v) \subseteq N_G(u)$.

Example

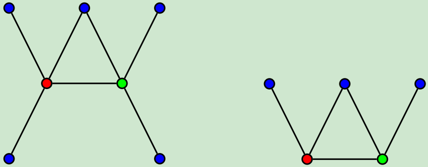


Figure: G and its R-core

Outline

Definition

A graph $G_{\text{R-reduced}}$ is R-reduced graph of G iff it is the smallest subgraph of G which has relation to G.

Proposition

a graph is R-reduced if and only if it is a graph core.

Example

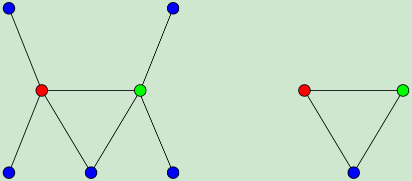


Figure: G and its R-reduced graph

Outline

1. Motivation
2. Homomorphisms and Relations
3. Computational Complexity
4. Core, Cocore, R-core
   - R-core
   - R-reduced graph
   - Cocore

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Figure: From football to mathematics

Definition

A cocore of graph G is the smallest subgraph of G which has relation to G.

Cocore is unique up to isomorphism.

Example

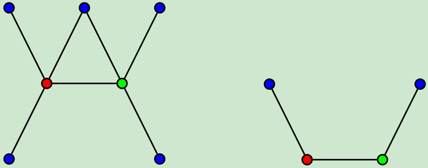


Figure: G and its cocore

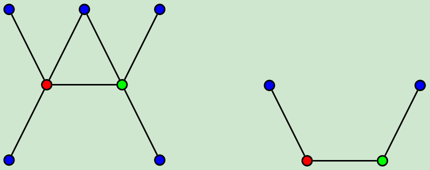
Proposition

G is a cocore if and only if any vertex neighborhood is not the union of other vertex neighborhoods.

Proposition

Cocore of G is the smallest member of the equivalence class of relational equivalence $\backsim$.

Example



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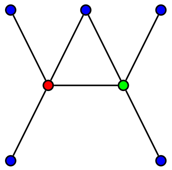

Figure: G

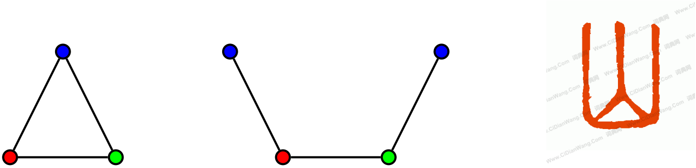

Figure: core cocore R-core

Thank you!