The Strong Chromatic Index of Cacti

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Outline

- Introduction
  - What is Strong Chromatic Index?
  - First Studies of Strong Chromatic Index
  - The Strong Chromatic Index for Different Families of Graphs
- Main Results
  - The Strong Chromatic Index of Trees
  - The Strong Chromatic Index of Cacti (1)
  - The Strong Chromatic Index of Cacti (2)
  - The Strong Chromatic Index of Cacti (3)
- Future Works

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- Only **simple graphs**, that is, finite, undirected graphs without loops or multiple edges, are considered in this thesis.
- The **distance** between two edges e and e' in G is the minimum k for which there is a sequence $e_0, e_1, \ldots, e_k$ of edges such that $e_0 = e$, $e_k = e'$ and e_{i-1} shares a vertex with e_i for $1 \leq i \leq k$.

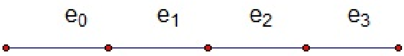

Figure : Example of distance = 3

Introduction - What is Strong Chromatic Index?

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- A **strong edge coloring** of a graph G is a function that assigns to each edge a color such that any two edges within distance two apart must receive different colors. In other words, for any edge xy, all edges containing x or y have different colors.

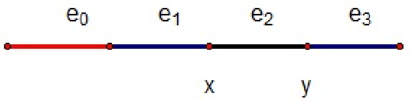

Figure : In this graph, because e_1 and e_3 has the same color, it is not a strong edge coloring.

Introduction - What is Strong Chromatic Index?

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- The **strong chromatic index** of a graph G is the minimum number $\chi'_s(G)$ of colors needed for a strong edge coloring of G.

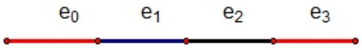

Figure : Example: $\chi'_s(P_4) = 3$

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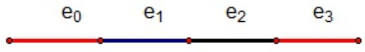

Figure : Example: $\chi'_s(P_4) = 3$

- We denote the maximum degree of a graph G by $\Delta(G)$ or Δ. For most types of graph colorings, the first question usually asked is to find an upper bound for the minimum number of colors necessary to color the graph in terms of the maximum degree Δ.

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Introduction - First Studies of Strong Chromatic Index

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- A trivial upper bound is that $\chi'_s(G) \leq 2\Delta^2 - 2\Delta + 1$ for any graph G of maximum degree Δ.
- Fouquet and Jolivet (1983) established a Brooks type upper bound $\chi'_s(G) \leq 2\Delta^2 - 2\Delta$, which is not true only for $G = C_5$. The following conjecture was posed by Erdős and Nešetřil (1988) and revised by Faudree, Gyárfás, Schelp and Tuza (1990):

Introduction - First Studies of Strong Chromatic Index

Conjectures 1

For any graph G of maximum degree Δ,

$$\chi'_s(G) \leq \begin{cases} \frac{5}{4}\Delta^2, & \textit{if } \Delta \textit{ is even}; \\ \frac{5}{4}\Delta^2 - \frac{1}{2}\Delta + \frac{1}{4}, & \textit{otherwise} \end{cases}$$

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- First, a 5-cycle C_5 on $\{x_1, x_2, x_3, x_4, x_5\}$ has $\Delta(C_5) = 2$ and $\chi'_s(C_5) = 5 = \frac{5}{4}\Delta(C_5)^2$.

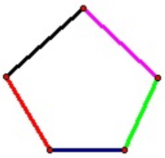

Figure : Example: $\chi'_s(C_5) = 5$

Introduction - First Studies of Strong Chromatic Index

Introduction - First Studies of Strong Chromatic Index

- More generally, if Δ is even, then the graph G obtained from C_5 by duplicating each vertex with an independent set of size $\frac{1}{2}\Delta$ is a graph of maximum degree Δ and $\chi'_s(G) = \frac{5}{4}\Delta^2$.

Introduction - First Studies of Strong Chromatic Index

- More generally, if Δ is even, then the graph G obtained from C_5 by duplicating each vertex with an independent set of size $\frac{1}{2}\Delta$ is a graph of maximum degree Δ and $\chi'_s(G) = \frac{5}{4}\Delta^2$.
- If Δ is odd, then the graph G obtained from C_5 by duplicating x_1 and x_3 (respectively, other vertices) with an independent set of size $\frac{1}{2}(\Delta + 1)$ (respectively, $\frac{1}{2}(\Delta - 1)$) is a graph of maximum degree Δ and $\chi'_s(G) = \frac{5}{4}\Delta^2 - \frac{1}{2}\Delta + \frac{1}{4}$.

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Introduction - The Strong Chromatic Index for Different Families of Graphs

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- For the n-cycle C_n: $\chi'_s(C_5) = 5$, $\chi'_s(C_n) = 3$ if n is a multiple of 3 and $\chi'_s(C_n) = 4$ otherwise.

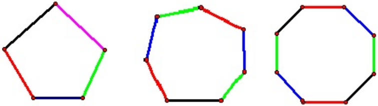

Figure : Examples of C_5, C_7, and C_8

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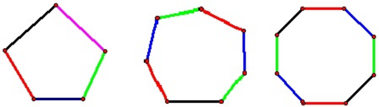

Figure : Examples of C_5, C_7, and C_8

- For planar graphs: Faudree *et al* (1990) proved that planar graphs with maximum degree Δ are strong $(4\Delta + 4)$-edge-colorable. They also exhibit a planar graph whose strong chromatic index is $4\Delta - 4$.

Introduction - The Strong Chromatic Index for Different Families of Graphs

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- For studies on Halin graphs, a subclass of planar graphs, A natural lower bound for the strong chromatic index is

$$\sigma(G) := \max_{xy \in E(G)} \{d_G(x) + d_G(y) - 1\}.$$

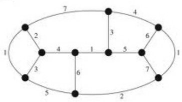

Figure : A Halin graph

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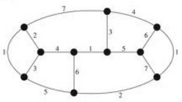

Figure : A Halin graph

- Faudree *et al* (1990) proved that $\chi'_s(T) = \sigma(T)$ for any tree T.

Introduction - The Strong Chromatic Index for Different Families of Graphs

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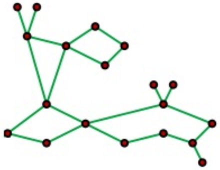
- In this thesis, we study strong chromatic edge coloring for cacti. A cactus is a connected graph whose blocks are edges or cycles.



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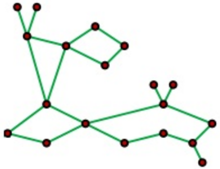

Figure : A cactus

- **Main Target: Find out the relationship between $\chi'_s(G)$ and $\sigma(G)$ for a cactus G.**

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Main Results - The Strong Chromatic Index of trees

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- Thus, we start with an alternative proof for the result of strong chromatic indices on trees, in order to use the same method for the proof techniques of our results on cacti.

Main Results - The Strong Chromatic Index of trees

Theorem 1

(Faudree et al (1990)) *If G is a tree, then $\chi'_s(G) = \sigma(G)$.*

Proof of Theorem 1:

We shall prove the theorem by using a mathematical induction on $|E(G)|$. For the case of G is a star centered at v, it is clear that $\chi'_s(G) = d_G(v) = \sigma(G)$.

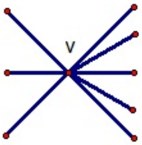

Figure : A star G

Main Results - The Strong Chromatic Index of trees

Proof of Theorem 1: (cont'd)

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- Now suppose that G is not a star. Then there is an edge xy such that G_1' and G_2' are nontrivial trees, where $G - xy$ is the disjoint union of G_1' and G_2' with G_1' containing x and G_2' containing y.

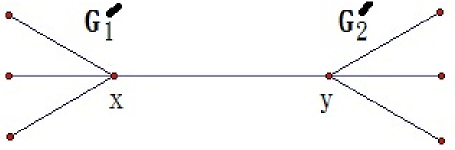

Figure : A tree G

Proof of Theorem 1: (cont'd)

Main Results - The Strong Chromatic Index of trees

Proof of Theorem 1: (cont'd)

- Suppose G_1 is obtained from G_1' by adding vertex y and edge xy, and G_2 is the graph obtained from G_2' by adding vertex x and edge xy. Now, G_1 and G_2 both have less numbers of edges than G.

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- By the induction hypothesis, graph G_i has a strong edge coloring f_i using at most $\sigma(G_i) \leq \sigma(G)$ colors for $1 \leq i \leq 2$.

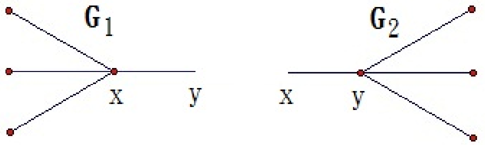

Figure : Two smaller tree G_1 and G_2

Main Results - The Strong Chromatic Index of trees

Proof of Theorem 1: (cont'd)

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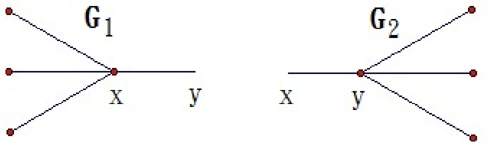
Proof of Theorem 1: (cont'd)

- If necessary, we may re-name the colors for f_i and assume that $f_1(xy) = f_2(xy) = \sigma(G)$, edges adjacent to x but other than xy are colored by $1, 2, \ldots, d_G(x) - 1$ for f_1, and edges adjacent to y but other than xy are colored by $\sigma(G) - 1, \sigma(G) - 2, \ldots, \sigma(G) - d_G(y) + 1$ for f_2.



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Main Results - The Strong Chromatic Index of trees

Proof of Theorem 1: (cont'd)

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- Notice that f_1 and f_2 together form an edge coloring f of G using $\sigma(G)$ colors. To see that f is a strong edge coloring, we only have to check that all edges containing x or y have different colors.

Proof of Theorem 1: (cont'd)

- Notice that f_1 and f_2 together form an edge coloring f of G using $\sigma(G)$ colors. To see that f is a strong edge coloring, we only have to check that all edges containing x or y have different colors.
- Since $d_G(x) + d_G(y) - 1 \leq \sigma(G)$, we have $d_G(x) - 1 < \sigma(G) - d_G(y) + 1$ and so the desired condition holds.

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- Since $d_G(x) + d_G(y) - 1 \leq \sigma(G)$, we have $d_G(x) - 1 < \sigma(G) - d_G(y) + 1$ and so the desired condition holds.
- As a consequence, $\chi'_s(G) \leq \sigma(G)$ and so $\chi'_s(G) = \sigma(G)$. $\square$

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Main Results - The Strong Chromatic Index of Cacti (1)

Now, we can use the same method to establish results on cacti.
We start from the easiest case.

Theorem 2

If G is a cactus in which the length of any cycle is divisible by six, then $\chi'_s(G) = \sigma(G)$.

Main Results - The Strong Chromatic Index of Cacti (1)

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- By the assumption, we know that n is a multiple of 6. Suppose $G - E(C)$ is the disjoint union of $G'_1, G'_2, \ldots, G'_n$, where G'_i contains x_i for $1 \leq i \leq n$.

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- Suppose G_i is the graph obtained from G'_i by adding vertices x_{i-1} and x_{i+1} and edges $x_{i-1}x_i$ and x_ix_{i+1} for $1 \leq i \leq n$.

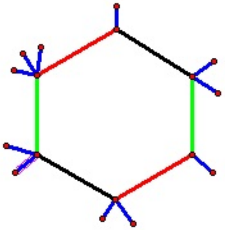

Figure : A cactus in which the length of any cycle is divisible by six

Main Results - The Strong Chromatic Index of Cacti (1)

Proof of Theorem 2: (cont'd)

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- Now, each G_i has less number of edges than G. By the induction hypothesis, graph G_i has a strong edge coloring f_i using at most $\sigma(G_i) \leq \sigma(G)$ colors for $1 \leq i \leq n$.

Main Results - The Strong Chromatic Index of Cacti (1)

Proof of Theorem 2: (cont'd)

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- Since n is a multiple of 6, the cycle C has a strong edge coloring f' using exactly 3 colors, say $\sigma(G), \sigma(G) - 1, \sigma(G) - 2$. If necessary, we may re-name the colors for f_i and assume that $f_i(x_{i-1}x_i) = f'(x_{i-1}x_i)$ and $f_i(x_ix_{i+1}) = f'(x_ix_{i+1})$.

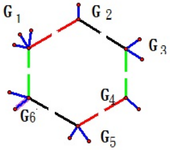

Figure : A cactus divided into 6 parts

Main Results - The Strong Chromatic Index of Cacti (1)

Proof of Theorem 2: (cont'd)

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Proof of Theorem 2: (cont'd)

- Also, for the case of i is odd, the edges adjacent to x_i but other than $x_{i-1}x_i$ and x_ix_{i+1} are colored by $1, 2, \ldots, d_G(x_i) - 2$ for f_i; and for the case of i is even, the edges adjacent to x_i but other than $x_{i-1}x_i$ and x_ix_{i+1} are colored by $\sigma(G) - 3, \sigma(G) - 4, \ldots, \sigma(G) - d_G(x_i)$ for f_i.

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- Notice that $f_1, f_2, \ldots, f_n$ together form an edge coloring f of G using $\sigma(G)$ colors. To see that f is a strong edge coloring, we only have to check that for $1 \leq i \leq n$ all edges containing x_i or x_{i+1} have different colors.

Main Results - The Strong Chromatic Index of Cacti (1)

Proof of Theorem 2: (cont'd)

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Proof of Theorem 2: (cont'd)

- Since $d_G(x_i) + d_G(x_{i+1}) - 1 \leq \sigma(G)$, we have $d_G(x_i) - 2 < \sigma(G) - d_G(x_{i+1})$ and $d_G(x_{i+1}) - 2 < \sigma(G) - d_G(x_i)$, and so the desired condition holds.

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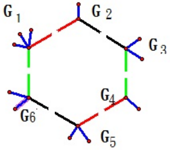
- As a consequence, $\chi'_s(G) \leq \sigma(G)$ and so $\chi'_s(G) = \sigma(G)$. □



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Main Results - The Strong Chromatic Index of Cacti (2)

After solving the case above, by a similar argument, we can prove a more general case.

Theorem 3

If G is a cacti in which the length of any cycle is even, then $\chi'_s(G) \leq \sigma(G) + 1$.

Main Results - The Strong Chromatic Index of Cacti (2)

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- By the assumption, we know that n is even. Suppose $G - E(C)$ is the disjoint union of $G'_1, G'_2, \ldots, G'_n$, where G'_i contains x_i for $1 \leq i \leq n$.

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Proof of Theorem 3: (cont'd)

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Proof of Theorem 3: (cont'd)

- Suppose G_i is the graph obtained from G'_i by adding vertices x_{i-1} and x_{i+1} and edges $x_{i-1}x_i$ and x_ix_{i+1} for $1 \le i \le n$.

Proof of Theorem 3: (cont'd)

- Suppose G_i is the graph obtained from G'_i by adding vertices x_{i-1} and x_{i+1} and edges $x_{i-1}x_i$ and x_ix_{i+1} for $1 \leq i \leq n$.
- Now, each G_i has less number of edges than G. By the induction hypothesis, graph G_i has a strong edge coloring f_i using at most $\sigma(G_i) + 1 \leq \sigma(G) + 1$ colors for $1 \leq i \leq n$.

Main Results - The Strong Chromatic Index of Cacti (2)

Proof of Theorem 3: (cont'd)

- Suppose G_i is the graph obtained from G_i' by adding vertices x_{i-1} and x_{i+1} and edges $x_{i-1}x_i$ and x_ix_{i+1} for $1 \leq i \leq n$.
- Now, each G_i has less number of edges than G. By the induction hypothesis, graph G_i has a strong edge coloring f_i using at most $\sigma(G_i) + 1 \leq \sigma(G) + 1$ colors for $1 \leq i \leq n$.
- Since n is even, the cycle C has a strong edge coloring f' using 4 colors, say $\sigma(G) + 1, \sigma(G), \sigma(G) - 1, \sigma(G) - 2$. If necessary, we may re-name the colors for f_i and assume that $f_i(x_{i-1}x_i) = f'(x_{i-1}x_i)$ and $f_i(x_ix_{i+1}) = f'(x_ix_{i+1})$.

Main Results - The Strong Chromatic Index of Cacti (2)

Proof of Theorem 3: (cont'd)

Main Results - The Strong Chromatic Index of Cacti (2)

Proof of Theorem 3: (cont'd)

- Also, for the case of i is odd, the edges adjacent to x_i but other than $x_{i-1}x_i$ and x_ix_{i+1} are colored by $1, 2, \ldots, d_G(x_i) - 2$ for f_i; and for the case of i is even, the edges adjacent to x_i but other than $x_{i-1}x_i$ and x_ix_{i+1} are colored by $\sigma(G) - 3, \sigma(G) - 4, \ldots, \sigma(G) - d_G(x_i)$ for f_i.

Proof of Theorem 3: (cont'd)

- Also, for the case of i is odd, the edges adjacent to x_i but other than $x_{i-1}x_i$ and x_ix_{i+1} are colored by $1, 2, \ldots, d_G(x_i) - 2$ for f_i; and for the case of i is even, the edges adjacent to x_i but other than $x_{i-1}x_i$ and x_ix_{i+1} are colored by $\sigma(G) - 3, \sigma(G) - 4, \ldots, \sigma(G) - d_G(x_i)$ for f_i.
- Notice that $f_1, f_2, \ldots, f_n$ together form an edge coloring f of G using $\sigma(G) + 1$ colors. To see that f is a strong edge coloring, we only have to check that for $1 \leq i \leq n$ all edges containing x_i or x_{i+1} have different colors.

Main Results - The Strong Chromatic Index of Cacti (2)

Proof of Theorem 3: (cont'd)

Main Results - The Strong Chromatic Index of Cacti (2)

Proof of Theorem 3: (cont'd)

- Since $d_G(x_i) + d_G(x_{i+1}) - 1 \leq \sigma(G)$, we have $d_G(x_i) - 2 < \sigma(G) - d_G(x_{i+1})$ and $d_G(x_{i+1}) - 2 < \sigma(G) - d_G(x_i)$, and so the desired condition holds.

Main Results - The Strong Chromatic Index of Cacti (2)

Proof of Theorem 3: (cont'd)

- Since $d_G(x_i) + d_G(x_{i+1}) - 1 \leq \sigma(G)$, we have $d_G(x_i) - 2 < \sigma(G) - d_G(x_{i+1})$ and $d_G(x_{i+1}) - 2 < \sigma(G) - d_G(x_i)$, and so the desired condition holds.
- As a consequence, $\chi'_s(G) \leq \sigma(G) + 1$. □

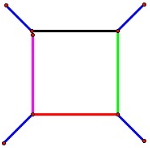

Figure : An example G such that $\chi'_s(G) = \sigma(G) + 1$

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Main Results - The Strong Chromatic Index of Cacti (3)

The above upper bound is not good for the case when a cactus has a cycle of odd length. For instance, suppose G is the graph obtained from a triangle by attaching at each vertex $\Delta - 2$ pendent edges. Then $\sigma(G) = 2\Delta - 1$ but $\chi'_s(G) = 3\Delta - 3$.

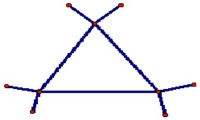

Figure : An example G such that $\chi'_s(G) > \sigma(G) + 1$

We now only have the following upper bound.

Theorem 4

If G is a cactus and G is not C_5, then $\chi'_s(G) \leq \lfloor \frac{3\sigma(G)-1}{2} \rfloor$.

Main Results - The Strong Chromatic Index of Cacti (3)

Proof of Theorem 4:

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Main Results - The Strong Chromatic Index of Cacti (3)

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- Now suppose G has a cycle $C = (x_1, x_2, \ldots, x_n, x_1)$, where x_i is adjacent to x_{i-1} and x_{i+1} for $1 \leq i \leq n$ by considering the indices for the vertices modulus n.

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- Without loss of generality, we may assume that $d_G(x_i) \leq d_G(x_n)$ for $1 \leq i \leq n$. In particular, $2d_G(x_1) \leq d_G(x_1) + d_G(x_n) \leq \sigma(G) + 1$ and so $d_G(x_1) \leq \lfloor \frac{\sigma(G)+1}{2} \rfloor$.

Main Results - The Strong Chromatic Index of Cacti (3)

Proof of Theorem 4: (cont'd)

Main Results - The Strong Chromatic Index of Cacti (3)

Proof of Theorem 4: (cont'd)

- Suppose $G - E(C)$ is the disjoint union of $G'_1, G'_2, \ldots, G'_n$, where G'_i contains x_i for $1 \le i \le n$. Suppose G_i is the graph obtained from G'_i by adding vertices x_{i-1} and x_{i+1} and edges $x_{i-1}x_i$ and x_ix_{i+1} for $1 \le i \le n$.

Main Results - The Strong Chromatic Index of Cacti (3)

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- Now, each G_i has less number of edges than G. By the induction hypothesis, graph G_i has a strong edge coloring f_i using at most $\lfloor \frac{3\sigma(G_i)-1}{2} \rfloor \leq \lfloor \frac{3\sigma(G)-1}{2} \rfloor$.

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Proof of Theorem 4: (cont'd)

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- Now, each G_i has less number of edges than G. By the induction hypothesis, graph G_i has a strong edge coloring f_i using at most $\lfloor \frac{3\sigma(G_i)-1}{2} \rfloor \le \lfloor \frac{3\sigma(G)-1}{2} \rfloor$.
- Let $m = \lfloor \frac{3\sigma(G)-1}{2} \rfloor$. Now we consider two cases.

Main Results - The Strong Chromatic Index of Cacti (3)

Proof of Theorem 4: (cont'd)

Main Results - The Strong Chromatic Index of Cacti (3)

Proof of Theorem 4: (cont'd)

- (i) If the cycle C is not C_5. Then C has a strong edge coloring f' using at most 4 colors, say $m, m-1, m-2, m-3$.

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Proof of Theorem 4: (cont'd)

- (i) If the cycle C is not C_5. Then C has a strong edge coloring f' using at most 4 colors, say $m, m-1, m-2, m-3$.
- If necessary, we may rename the colors for f_i and assume that $f_i(x_{i-1}x_i) = f'(x_{i-1}x_i)$ and $f_i(x_ix_{i+1}) = f'(x_ix_{i+1})$.

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- Also, for the case of i is odd but $i \neq n$, the edges adjacent to x_i but other than $x_{i-1}x_i$ and x_ix_{i+1} are colored by $1, 2, \ldots, d_G(x_i) - 2$ for f_i; for the case of $i = n$ is odd, the edges adjacent to x_n but other than $x_{n-1}x_n$ and x_nx_1 are colored by $d_G(x_1) - 1, d_G(x_1), \ldots, d_G(x_1) + d_G(x_n) - 4$ for f_n; and for the case of i is even, the edges adjacent to x_i but other than $x_{i-1}x_i$ and x_ix_{i+1} are colored by $m-4, m-5, \ldots, m-2-d_G(x_i)$ for f_i.

Main Results - The Strong Chromatic Index of Cacti (3)

Proof of Theorem 4: (cont'd)

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Proof of Theorem 4: (cont'd)

- Notice that $f_1, f_2, \ldots, f_n$ together form an edge coloring f of G using $\sigma(G) + 1$ colors. To see that f is a strong edge coloring, we only have to check that for $1 \leq i \leq n$ all edges containing x_i or x_{i+1} have different colors.

Main Results - The Strong Chromatic Index of Cacti (3)

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- For the case of n is even or $i \notin \{n-1, n\}$, since $d_G(x_i) + d_G(x_{i+1}) - 1 \leq \sigma(G) \leq m - 2$, we have $d_G(x_i) - 2 < m - 2 - d_G(x_{i+1})$ and $d_G(x_{i+1}) - 2 < m - 2 - d_G(x_i)$, and so the desired condition holds.

Main Results - The Strong Chromatic Index of Cacti (3)

Proof of Theorem 4: (cont'd)

Main Results - The Strong Chromatic Index of Cacti (3)

Proof of Theorem 4: (cont'd)

- For the case of n is odd and $i \in \{n-1, n\}$, since $d_G(x_1) \leq \lfloor \frac{\sigma(G)+1}{2} \rfloor$ and $d_G(x_{n-1}) + d_G(x_n) - 1 \leq \sigma(G)$, we have $d_G(x_1) + d_G(x_n) - 4 < m - 2 - d_G(x_{n-1})$, and so the desired condition holds.

Main Results - The Strong Chromatic Index of Cacti (3)

Proof of Theorem 4: (cont'd)

- For the case of n is odd and $i \in \{n-1, n\}$, since $d_G(x_1) \leq \lfloor \frac{\sigma(G)+1}{2} \rfloor$ and $d_G(x_{n-1}) + d_G(x_n) - 1 \leq \sigma(G)$, we have $d_G(x_1) + d_G(x_n) - 4 < m - 2 - d_G(x_{n-1})$, and so the desired condition holds.
- As a consequence, $\chi'_s(G) \leq m = \lfloor \frac{3\sigma(G)-1}{2} \rfloor$. $\square$

Main Results - The Strong Chromatic Index of Cacti (3)

Proof of Theorem 4: (cont'd)

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- (ii) If the cycle C is C_5. That means $n = 5$ in this case. Then C has a strong edge coloring f' using at most 5 colors, say $m, m-1, m-2, m-3, m-4$. We may assume that $f'(x_2x_3) = m$.

Proof of Theorem 4: (cont'd)

- (ii) If the cycle C is C_5. That means $n = 5$ in this case. Then C has a strong edge coloring f' using at most 5 colors, say $m, m-1, m-2, m-3, m-4$. We may assume that $f'(x_2x_3) = m$.
- If necessary, we may rename the colors for f_i and assume that $f_i(x_{i-1}x_i) = f'(x_{i-1}x_i)$ and $f_i(x_ix_{i+1}) = f'(x_ix_{i+1})$.

Main Results - The Strong Chromatic Index of Cacti (3)

Proof of Theorem 4: (cont'd)

- (ii) If the cycle C is C_5. That means $n = 5$ in this case. Then C has a strong edge coloring f' using at most 5 colors, say $m, m-1, m-2, m-3, m-4$. We may assume that $f'(x_2x_3) = m$.
- If necessary, we may rename the colors for f_i and assume that $f_i(x_{i-1}x_i) = f'(x_{i-1}x_i)$ and $f_i(x_ix_{i+1}) = f'(x_ix_{i+1})$.
- Also, for the case of i is odd but $i \neq 5$, the edges adjacent to x_i but other than $x_{i-1}x_i$ and x_ix_{i+1} are colored by $1, 2, \ldots, d_G(x_i) - 2$ for f_i;

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Proof of Theorem 4: (cont'd)

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Proof of Theorem 4: (cont'd)

- for the case of $i = 5$, the edges adjacent to x_5 but other than x_4x_5 and x_5x_1 are colored by m and $d_G(x_1) - 1, d_G(x_1), \ldots, d_G(x_1) + d_G(x_5) - 5$ for f_5 (since G is not C_5, we can guarantee that color m is used in this case); and for the case of i is even, the edges adjacent to x_i but other than $x_{i-1}x_i$ and x_ix_{i+1} are colored by $m-5, m-6, \ldots, m-2-d_G(x_i)$ for f_i.

Main Results - The Strong Chromatic Index of Cacti (3)

Proof of Theorem 4: (cont'd)

- for the case of $i = 5$, the edges adjacent to x_5 but other than x_4x_5 and x_5x_1 are colored by m and $d_G(x_1) - 1, d_G(x_1), \ldots, d_G(x_1) + d_G(x_5) - 5$ for f_5 (since G is not C_5, we can guarantee that color m is used in this case); and for the case of i is even, the edges adjacent to x_i but other than $x_{i-1}x_i$ and x_ix_{i+1} are colored by $m-5, m-6, \ldots, m-2-d_G(x_i)$ for f_i.
- Notice that $f_1, f_2, \ldots, f_5$ together form an edge coloring f of G using $\sigma(G) + 1$ colors.

Main Results - The Strong Chromatic Index of Cacti (3)

Proof of Theorem 4: (cont'd)

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Proof of Theorem 4: (cont'd)

- To see that f is a strong edge coloring, we only have to check that for $1 \leq i \leq 5$ all edges containing x_i or x_{i+1} have different colors.

Proof of Theorem 4: (cont'd)

- To see that f is a strong edge coloring, we only have to check that for $1 \le i \le 5$ all edges containing x_i or x_{i+1} have different colors.
- For the case of $i \notin \{4, 5\}$, since $d_G(x_i) + d_G(x_{i+1}) - 1 \le \sigma(G) \le m - 2$ (when $\sigma(G) \ge 5$), we have $d_G(x_i) - 2 < m - 2 - d_G(x_{i+1})$ and $d_G(x_{i+1}) - 2 < m - 2 - d_G(x_i)$, and so the desired condition holds.

Main Results - The Strong Chromatic Index of Cacti (3)

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- To see that f is a strong edge coloring, we only have to check that for $1 \leq i \leq 5$ all edges containing x_i or x_{i+1} have different colors.
- For the case of $i \notin \{4, 5\}$, since $d_G(x_i) + d_G(x_{i+1}) - 1 \leq \sigma(G) \leq m - 2$ (when $\sigma(G) \geq 5$), we have $d_G(x_i) - 2 < m - 2 - d_G(x_{i+1})$ and $d_G(x_{i+1}) - 2 < m - 2 - d_G(x_i)$, and so the desired condition holds.
- For the case of $i \in \{4, 5\}$, since $d_G(x_1) \leq \lfloor \frac{\sigma(G)+1}{2} \rfloor$ and $d_G(x_4) + d_G(5) - 1 \leq \sigma(G)$, we have $d_G(x_1) + d_G(5) - 5 < m - 2 - d_G(x_4)$, and so the desired condition holds. As a consequence, $\chi'_s(G) \leq m = \lfloor \frac{3\sigma(G)-1}{2} \rfloor$ in this case. □

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- **Future Works**

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- For a bipartite cactus G, how to check whether $\chi'_s(G) = \sigma(G)$ or $\chi'_s(G) = \sigma(G) + 1$?

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- For a bipartite cactus G, how to check whether $\chi'_s(G) = \sigma(G)$ or $\chi'_s(G) = \sigma(G) + 1$?
- After solving cacti, how about block-cactus graphs ?

Thank You for Your Attenion !