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New zero-free regions for Dedekind zeta-functions at small and large ordinates [☆]

Sourabhashis Das [a], Swati Gaba [b], Ethan Simpson Lee [c], Aditi Savalia [d], Peng-Jie Wong [e,*]

[a] *Department of Pure Mathematics, University of Waterloo, Ontario, Canada*
[b] *Department of Mathematics, University of Connecticut, United States of America*
[c] *University of the West of England, School of Computing and Creative Technologies, Coldharbour Lane, Bristol, BS16 1QY, United Kingdom*
[d] *Department of Mathematics, Indian Institute of Technology Bombay, Mumbai, India*
[e] *Department of Applied Mathematics, National Sun Yat-Sen University, Kaohsiung City, Taiwan*

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ABSTRACT

Given a number field $L \neq \mathbb{Q}$, we obtain new and explicit zero-free regions for Dedekind zeta-functions of L, which refine the previous works of Ahn–Kwon, Kadiri, and Lee. In particular, for low-lying zeros, we extend Kadiri's result to all number fields while improving the main constant for all L of degree $n_L \geq 6$.

1. Introduction

Suppose throughout that L is a number field with degree $n_L \geq 2$ and discriminant Δ_L; the absolute discriminant of L will be denoted by $d_L = |\Delta_L|$. Let $\zeta_L(s)$ be the Dedekind zeta-function associated to L.

The error term in the Chebotarëv Density Theorem (CDT), a vast generalisation of the prime number theorem, the prime number theorem for arithmetic progressions, and the prime ideal theorem, is closely connected to a certain sum over the non-trivial zeros $\varrho = \beta + i\gamma$, with $0 < \beta < 1$, of $\zeta_L(s)$ via an explicit

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* Corresponding author.

E-mail addresses: s57das@uwaterloo.ca (S. Das), swati.gaba@uconn.edu (S. Gaba), ethan.lee@uwe.ac.uk (E.S. Lee), aditis@math.iitb.ac.in, aditisav1995@gmail.com (A. Savalia), pjwong@math.nsysu.edu.tw (P.-J. Wong).

URLs: https://souravasis.wixsite.com/sourabhdas (S. Das), https://math.uconn.edu/person/swati-gaba/ (S. Gaba), https://sites.google.com/view/ethansleemath/home (E.S. Lee), https://sites.google.com/view/aditisavalia/home (A. Savalia), https://www-math.nsysu.edu.tw/~pjwong/ (P.-J. Wong).

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formula. Recall that a *zero-free region* for $\zeta_L(s)$ is a region in the critical strip $\{s \in \mathbb{C} : 0 < \operatorname{Re} s < 1\}$ that contains no non-trivial zeros. In fact, zero-free regions for $\zeta_L(s)$ are central ingredients that enable one to prove effective bounds for the error in the CDT; see [2,11] for further details. So, any refinements one can make to the zero-free regions for $\zeta_L(s)$ will automatically improve the error in the CDT, among other consequences. For example, [1,8,10] gave other applications of zero-free regions for $\zeta_L(s)$. In light of these, the purpose of this article is to update the latest explicit zero-free regions for $\zeta_L(s)$.

There are infinitely many non-trivial zeros $\varrho = \beta + i\gamma$ of $\zeta_L(s)$, and the Generalised Riemann Hypothesis (GRH) postulates that every ϱ satisfies $\beta = \frac{1}{2}$. Platt and Trudgian [17] have verified that if $L = \mathbb{Q}$, then the GRH in this setting, namely the Riemann Hypothesis (RH), is valid for the region $|\gamma| \le 3 \cdot 10^{12}$. When L is a quadratic extension with $d_L \le 400\,000$, the verification of the GRH, up to the height $10^8/d_L$, follows from the work of Platt [16] on numerical computations of the GRH for Dirichlet L-functions (cf. Remark 3 below). Furthermore, Tollis [20] has verified that the GRH is true for $\zeta_L(\sigma + it)$ in the region $|t| \le 92$ for number fields L with $n_L = 3$ and $d_L \le 239$, and in the region $|t| \le 40$ for number fields L with $n_L = 4$ and $d_L \le 320$. In general, computations verifying the GRH are limited and only available in some special cases.

To obtain a deeper understanding of the non-trivial zeros of $\zeta_L(s)$ that fall outside of these regions where the GRH has been verified, we introduce several new zero-free regions for $\zeta_L(s)$ when $L \neq \mathbb{Q}$ (equivalently $n_L \ge 2$). In particular, we establish separate zero-free regions for the non-trivial zeros $\varrho = \beta + i\gamma$ of $\zeta_L(s)$ when $|\gamma| > 1$, $0 < |\gamma| \le 1$, and $\gamma = 0$.

Remark 1. We restrict our attention to number fields L such that $n_L \ge 2$, since $n_L = 1$ implies $L = \mathbb{Q}$, and $\zeta_L(s) = \zeta(s)$ is the Riemann zeta-function, whose zeros have been well-studied. For example, the lowest-lying zero of $\zeta(s)$ is $\frac{1}{2} + 14.13472\ldots i$, and de la Vallée Poussin [3] famously proved that $\zeta(\sigma + it) \neq 0$ in the region $t \ge T$ and

$$\sigma \ge 1 - \frac{1}{R \log t}, \tag{1.1}$$

where T and R are positive constants. This is commonly referred to as the classical zero-free region for $\zeta(s)$. Over the years, many authors (including Westphal [21], Stečkin [19], Rosser–Schoenfeld [18], Kondrat'ev [9], Kadiri [5], and Mossinghoff–Trudgian [14]) have refined these values of R. Mossinghoff, Trudgian, and Yang [15] established the latest admissible value for R in (1.1); these values being $R = 5.558691$ and $T = 2$.

1.1. Zeros with large ordinate

For $n_L \ge 2$, we can compute absolute constants (C_1, C_2, C_3, C_4, T) such that $\zeta_L(\sigma + it) \neq 0$ in the region

$$\sigma \ge 1 - \frac{1}{C_1 \log d_L + C_2 n_L \log|t| + C_3 n_L + C_4} \quad \text{and} \quad |t| \ge T. \tag{1.2}$$

Historically, Lagarias and Odlyzko generalised de la Vallée Poussin's approach into the number field setting in [11], which requires a non-negative, even, trigonometric polynomial. However, their proof did not use Stečkin's key idea from [19] or attempt to find a more favourable trigonometric polynomial. Kadiri generalised Stečkin's approach into the number field setting in [6] to establish (1.2) with the values

$$(C_1, C_2, C_3, C_4, T) = (12.55, 9.69, 3.03, 58.63, 1).$$

Further, by choosing a more favourable trigonometric polynomial, inserting new parameters into Kadiri's method, and refining important bounds for certain gamma factors, Lee (the third-named author) proved in [12] that (1.2) is true with the refined constants

$$(C_1, C_2, C_3, C_4, T) = (12.2411, 9.5347, 0.05017, 2.2692, 1).$$

Our first result is presented in Theorem 1.1, which refines these computations for number fields L with degree $n_L \geq 3$. Most of the refinement in Theorem 1.1 comes from two sources:

(a) A new bound, Lemma 2.1, for certain gamma factors that refines [13, Lem. 2].
(b) New choices of non-negative, even, trigonometric polynomials in the method.

Theorem 1.1. *If L is a number field of degree $n_L \geq 3$, then $\zeta_L(\sigma + it)$ is non-zero in the region (1.2) with*

$$(C_1, C_2, C_3, C_4, T) = (12.21124, 9.54177, -11.59548, 4.57803, 1).$$

If $n_L \geq n_0 \geq 3$, then further computations (for any $n_0 \leq 21$) are presented in Tables 9, 10, and 11.

Remark 2. It is not unreasonable for C_3 to be negative. To see this, observe that if $n_L = 3$, then $\log d_L \geq \log 23$ and $12.21124 \log 23 - 11.59548 \cdot 3 > 3.50183$. Indeed, our refinements are designed to "waste" as little as possible, and this negative constant demonstrates that there is very little waste in our method (especially by comparison to previous results).

Remark 3. If L is a number field of degree $n_L = 2$, then there exists a Dirichlet character χ_L modulo d_L such that $\zeta_L(s) = \zeta(s)L(s, \chi_L)$, where $L(s, \chi_L)$ is the Dirichlet L-function attached to χ_L. By extending the above-mentioned works of Stečkin and McCurley, Kadiri [4,7] further introduced an extra "smoothing" to establish that if χ is a Dirichlet character modulo q, then $L(s, \chi)$ has at most one zero in the region

$$\operatorname{Re} s \geq 1 - \frac{1}{\mathcal{R} \log(q \max\{1, |\operatorname{Im}(s)|\})},$$

with $\mathcal{R} = 6.44$ (for all $q \geq 3$) and $\mathcal{R} = 5.60$ (for any $3 \leq q \leq 400\,000$). Applying this with $q = d_L$, we see that if $n_L = 2$, then $\zeta_L(\sigma + it)$ has at most one zero in the region

$$\sigma \geq 1 - \frac{1}{\mathcal{R} \log(d_L \max\{1, |t|\})}.$$

One can extend our Theorem 1.1 to also cover number fields with $n_L = 2$; however, the preceding analysis provides a stronger outcome in this case.

1.2. Zeros with small ordinate

If $n_L \geq 2$, then one can compute absolute constants $A > 0$ and $A' > 0$ such that $\zeta_L(\sigma + it) \neq 0$ in the region

$$\sigma \geq 1 - \frac{1}{A \log d_L + A' n_L \log(|t| + 2)} \quad \text{and} \quad |t| \leq 1, \tag{1.3}$$

with the exception of at most one real zero β_1. This zero, if it exists, is called an *exceptional* zero. If d_L is sufficiently large, then Kadiri proved in [6, Thm. 1.1] that $A = 12.74$ and $A' = 0$ are admissible for the region (1.3); Lee (the third-named author) refined these computations to $A = 12.44$ and $A' = 0$ in [12, Thm. 2]. For all number fields $L \neq \mathbb{Q}$, Ahn and Kwon proved that $\zeta_L(\sigma + it) \neq 0$ in the region (1.3) with $A = A' = 29.57$ in [1, Prop. 6.1].

Our next result (Theorem 1.2) refines the computations in [6,12] and removes the "sufficiently large" condition that was previously enforced when $A' = 0$. It shall be worthwhile remarking that our result is even superior when $n_L \geq 6$. These improvements originated from the above-mentioned sources (a) and (b), as well as refinements to the optimisation process, which is necessary to balance certain parameters that naturally appear in the calculations.

Theorem 1.2. *If L is a number field of degree $n_L \geq 2$, then $\zeta_L(\sigma + it)$ has at most one zero, namely the exceptional zero (if it exists), in the region (1.3), with $A' = 0$, and A is given by*

n_L	2	3	4	5	6	≥ 7
A	16.01983	19.55293	16.72207	13.71235	11.78180	11.51910

Recall that the Deuring–Heilbronn phenomenon roughly asserts that if an exceptional zero β_1 exists, then the zero-free region for $\zeta_L(s)$ can be enlarged. We refer the interested reader to [8] for an explicit statement of the phenomenon as well as the references therein. Inspired by this, the following result describes the strongest available zero-free region that would be true, if an exceptional zero β_1 exists.

Theorem 1.3. *Let L be a number field with $n_L \geq 2$ and $\mathscr{L} = \log d_L$. If a real exceptional zero β_1 presents in the region $[1 - \frac{\nu}{\mathscr{L}}, 1)$, then $\zeta_L(\sigma + it)$ is non-zero in the region (1.3), except for $\sigma + it = \beta_1$, with $A' = 0$, and with admissible values of A for $\nu = 0.05$ given in*

n_L	2	3	4	5	6	7	8	9	≥ 10
A	2.57422	3.31591	2.85120	2.31221	1.96009	1.85345	1.82945	1.80640	1.80519

and admissible values of A for $\nu = 0.5$ given in

n_L	2	3	4	5	6	7	≥ 8
A	6.03399	8.25229	6.42206	4.66816	3.65941	3.58128	3.57670

Remark 4. (i) In each of our results so far, we have presented successive computations until the outcomes stopped meaningfully improving. An analogous comment applies to the computations presented in the next three tables.
(ii) Our argument to establish Theorem 1.3 in Section 4.5 works for the general situation that an exceptional zero β_1 presents in the region $[1 - \frac{\nu}{\mathscr{L}}, 1)$ with $\nu \in (0, \frac{1}{2}]$. The choice $\nu = \frac{1}{20}$ is particularly interesting, because it was assumed in [8, Sec. 3.4] to study the least prime problem in the CDT.

1.3. Zeros on the real line

Assuming d_L is sufficiently large, Kadiri [6, p. 146] established that $\zeta_L(\sigma)$ admits at most one zero in the region

$$\sigma \geq 1 - \frac{1}{A'' \log d_L}, \tag{1.4}$$

with $A'' = 1.6110$. Our next theorem extends her work, notably we remove the "sufficiently large" condition.

Theorem 1.4. *If L is a number field with $n_L \geq 2$, then $\zeta_L(\sigma)$ admits at most one real zero in the region (1.4) with A'' given as*

n_L	2	3	4	5	6	≥ 7
A''	1.61094	1.93173	1.88178	1.69958	1.61857	1.61094

Our final result (Theorem 1.5) demonstrates the impact of the existence of a real exceptional zero β_1 on the real zeros.

Theorem 1.5. *Let L be a number field with $n_L \geq 2$ and $\mathscr{L} = \log d_L$. If a real exceptional zero β_1 presents in the region $[1 - \frac{\nu}{\mathscr{L}}, 1)$, then $\zeta_L(\sigma)$ is non-vanishing in the region (1.4), except for $\sigma = \beta_1$, with admissible A'' for $\nu = 0.05$ given in*

n_L	2	3	4	≥ 5
A''	0.47863	0.48380	0.48074	0.47863

and with A'' for $\nu = 0.5$ given in

n_L	2	3	4	5	6	≥ 7
A''	1.32086	1.86631	1.77210	1.45550	1.33079	1.32086

1.4. Structure

The remainder of this paper is structured as follows. In Section 2, we introduce several useful notations and preparatory observations which will be required throughout this paper. Building upon these observations, we prove Theorems 1.1, 1.2, 1.3, 1.4, and 1.5 in Sections 3, 4, and 5.

2. Set-up and preparatory observations

Our initial set-up is the same as that used by Kadiri and Lee to prove [6, Thm. 1.1] and [12, Thm. 1], respectively, following a similar shape to Stečkin's argument in [19]. To begin, recall that the Dedekind zeta function $\zeta_L(s)$ associated to the number field L is defined by

$$\zeta_L(s) = \sum_{\mathfrak{a} \neq 0} \mathrm{N}(\mathfrak{a})^{-s} = \prod_{\mathfrak{p}} (1 - \mathrm{N}(\mathfrak{p})^{-s})^{-1},$$

for $\mathrm{Re}\, s > 1$. Here, $\mathrm{N}(\mathfrak{a})$ is the norm of $\mathfrak{a}$, the sum is over non-zero integral ideals of L, and the product is over prime ideals of the ring of integers $\mathcal{O}_L$ of L. It is known that $\zeta_L(s)$ has an analytic continuation to a meromorphic function on $\mathbb{C}$ with only a simple pole at $s = 1$, and its zeros $\rho = \beta + i\gamma$ encode deep arithmetic information of L. The logarithmic derivative of ζ_L is

$$-\frac{\zeta_L'}{\zeta_L}(s) = \sum_{\mathfrak{a} \neq 0} \frac{\Lambda(\mathfrak{a})}{\mathrm{N}(\mathfrak{a})^s},$$

where $\Lambda(\mathfrak{a})$ is the number field analogue of the von Mangoldt function:

$$\Lambda(\mathfrak{a}) = \begin{cases} \log \mathrm{N}(\mathfrak{p}) & \text{if } \mathfrak{a} = \mathfrak{p}^k \text{ for some prime ideal } \mathfrak{p} \text{ of } \mathcal{O}_L \text{ and } k \geq 1; \\ 0 & \text{otherwise.} \end{cases}$$

Let r_1 and r_2 be the number of real and complex places (respectively) of L, and note that $n_L = r_1 + 2r_2$. The completed zeta function $\xi_L(s)$ is

$$\xi_L(s) = s(s-1) d_L^{s/2} \gamma_L(s) \zeta_L(s),$$

where

$$\gamma_L(s) = \left(\pi^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right)\right)^{r_2}\left(\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\right)^{r_1+r_2}. \tag{2.1}$$

Recall that $\xi_L(s)$ extends to an entire function of order 1 and satisfies the functional equation

$$\xi_L(s) = \xi_L(1-s). \tag{2.2}$$

Next, we let $t \in \mathbb{R}$. The following definitions hold for the rest of this paper:

- $\kappa = \frac{1}{\sqrt{5}}$,
- $1 < \sigma < 1+\varepsilon$ for some $0 < \varepsilon \le 0.15$,
- $s_k = \sigma + ikt$ such that $k \in \mathbb{N}$, and
- $s_k' = \sigma_1(\sigma) + ikt$ such that $\sigma_1(\sigma) = \frac{1+\sqrt{1+4\sigma^2}}{2}$.

The parameter ε will be chosen optimally later. Further, the choices of κ and $\sigma_1(\sigma)$ are motivated by the original work of Stečkin [19]. To prove our results, we isolate a non-trivial zero $\varrho_0 = \beta_0 + i\gamma_0$ of $\zeta_L(s)$ such that $\beta_0 > 1-\varepsilon \ge 0.85$, and choose a polynomial $p_n(\varphi)$ from the class P_n of non-negative, even, trigonometric polynomials of degree $n \ge 2$, which is defined by

$$P_n := \left\{p_n(\varphi) = \sum_{k=0}^{n} a_k \cos(k\varphi) : p_n(\varphi) \ge 0 \text{ for all } \varphi,\ a_k \ge 0 \text{ and } a_0 < a_1\right\}.$$

Now, consider the function

$$S(\sigma, t) = \sum_{k=0}^{n} a_k f_L(\sigma, kt),$$

in which

$$f_L(\sigma, kt) = -\operatorname{Re}\left(\frac{\zeta_L'}{\zeta_L}(s_k) - \kappa\frac{\zeta_L'}{\zeta_L}(s_k')\right) = \sum_{0\ne\mathfrak{p}\subset\mathcal{O}_L} \Lambda(\mathfrak{p})(N(\mathfrak{p})^{-\sigma} - \kappa N(\mathfrak{p})^{-\sigma_1(\sigma)})\cos(kt\log(N(\mathfrak{p})),$$

because $\sigma > 1$ and $\sigma_1(\sigma) > \sigma_1(1) > 1$. The choices for $\sigma_1(\sigma)$ and κ that we have made, combined with the non-negativity of $p_n(\varphi)$ implies that $S(\sigma, t) \ge 0$. On the other hand, the explicit formula in [11, Eqn. (8.3)] tells us that

$$-\operatorname{Re}\frac{\zeta_L'}{\zeta_L}(s) = -\sum_{\rho\in Z_L}\operatorname{Re}\frac{1}{s-\rho} + \frac{\log d_L}{2} + \operatorname{Re}\frac{1}{s} + \operatorname{Re}\frac{1}{s-1} + \operatorname{Re}\frac{\gamma_L'}{\gamma_L}(s), \tag{2.3}$$

where γ_L was defined in (2.1) and Z_L denotes the set of non-trivial zeros of ζ_L. Following [6,19], to ease the notation, we let

$$F(s, z) = \operatorname{Re}\left(\frac{1}{s-z} + \frac{1}{s-1+\bar{z}}\right).$$

The functional equation (2.2) tells us that non-trivial zeros are symmetric with respect to the critical line, so we also have

$$-\sum_{\rho=\beta+i\gamma\in Z_L}\operatorname{Re}\left(\frac{1}{s_k-\rho} - \frac{\kappa}{s_k'-\rho}\right) = -{\sum_{\beta\ge\frac{1}{2}}}'(F(s_k,\rho) - \kappa F(s_k',\rho)),$$

where $\sum'_{\beta\geq\frac{1}{2}} = \frac{1}{2}\sum_{\beta=\frac{1}{2}} + \sum_{\frac{1}{2}<\beta\leq 1}$. It follows from (2.3) that

$$0 \leq S(\sigma, t) = S_1 + S_2 + S_3 + S_4, \tag{2.4}$$

where $F(s,z) = \mathrm{Re}(\frac{1}{s-z} + \frac{1}{s-1+\bar{z}})$,

$$S_1 = -\sum_{k=0}^{n} a_k \sum_{\varrho\in Z_L} \mathrm{Re}\Big(\frac{1}{s_k-\varrho} - \frac{\kappa}{s_k'-\varrho}\Big) = -\sum_{k=0}^{n} a_k \sum_{\beta\geq\frac{1}{2}}{}' (F(s_k,\varrho) - \kappa F(s_k',\varrho)),$$

$$S_2 = \frac{1-\kappa}{2}\Big(\sum_{k=0}^{n} a_k\Big)\log d_L,$$

$$S_3 = \sum_{k=0}^{n} a_k \left(F(s_k,1) - \kappa F(s_k',1)\right), \text{ and}$$

$$S_4 = \sum_{k=0}^{n} a_k \,\mathrm{Re}\Big(\frac{\gamma_L'(s_k)}{\gamma_L(s_k)} - \kappa\frac{\gamma_L'(s_k')}{\gamma_L(s_k')}\Big).$$

We shall denote $\mathscr{L} = \log d_L$ throughout, and consider the following five cases separately:

Case 1: $\gamma_0 > 1$
Case 2: $\frac{d_2}{\mathscr{L}} < \gamma_0 \leq 1$
Case 3: $\frac{d_1}{\mathscr{L}} < \gamma_0 \leq \frac{d_2}{\mathscr{L}}$
Case 4: $0 < \gamma_0 \leq \frac{d_1}{\mathscr{L}}$
Case 5: $\gamma_0 = 0$

Henceforth, we shall further assume $t = \gamma_0$ (we can use these interchangeably) and in the fifth case, we consider two real zeros $\rho_1 = \beta_1 + i\gamma_1$ and $\rho_2 = \beta_2 + i\gamma_2$ such that $\beta_1 \leq \beta_2$. We consider each case in the upcoming sections. For convenience, we also introduce some intermediary results in Sections 2.1-2.4, which we will require later.

Remark 5. (i) Our approach to prove Theorems 1.1-1.3 involves three methods, each of which is best suited to Cases 1-2, Case 3, and Case 4, respectively. The boundaries between these regions are transition points where one method becomes more effective than the others.
(ii) Our eventual bounds for Cases 1-3 will depend on the choice of the polynomial, whereas our bounds for Cases 4-5 will not depend on this choice.
(iii) It is worthwhile to remark that the primary innovation from Stečkin's argument in [19] is the lower bound

$$F(s_k,\rho) - \kappa F(s_k',\varrho) \geq 0,$$

where s_k, s_k', κ, and ϱ are each defined as above. This enables one to safely discard all zeros $\varrho \neq \varrho_0$ from the primed sum in S_1, which is an important step in the derivation of our bound (3.1) for S_1.

2.1. Minimum discriminants

Suppose that L is a number field with degree n_L and absolute discriminant d_L. Table 1 presents lower bounds for the size of d_L at given choices of n_L. These computations are imported from the appendix in [8].

Table 1
Computations for $d_{\min}(n_L)$ such that $d_L \geq d_{\min}(n_L)$ for every number field of a given degree $n_L \geq 2$.

n_L	$d_{\min}(n_L)$	n_L	$d_{\min}(n_L)$	n_L	$d_{\min}(n_L)$	n_L	$d_{\min}(n_L)$
2	3	7	184 607	12	$2.74 \cdot 10^{10}$	17	$3.70 \cdot 10^{16}$
3	23	8	1 257 728	13	$7.56 \cdot 10^{11}$	18	$2.73 \cdot 10^{17}$
4	117	9	$2.29 \cdot 10^{7}$	14	$5.43 \cdot 10^{12}$	19	$9.03 \cdot 10^{18}$
5	1 609	10	$1.56 \cdot 10^{8}$	15	$1.61 \cdot 10^{14}$	20	$6.74 \cdot 10^{19}$
6	9 747	11	$3.91 \cdot 10^{9}$	16	$1.17 \cdot 10^{15}$	≥ 21	10^{n_L}

2.2. Bounds for gamma factors

The following technical lemma refines [13, Lem. 2]. We will use this to refine the bounds on certain gamma factors that are important down the line.

Lemma 2.1. *Let $k \geq 1$ and $\delta \in \{0, 1\}$. We have*

$$\frac{1}{2}\operatorname{Re}\Big(\frac{\Gamma'}{\Gamma}\Big(\frac{s_k+\delta}{2}\Big) - \kappa\frac{\Gamma'}{\Gamma}\Big(\frac{s_k'+\delta}{2}\Big)\Big)$$
$$= \frac{1-\kappa}{2}\log\frac{kt}{2} + \Xi(\sigma, k, t, \delta) + \frac{\theta_1}{6}\Big(\frac{1}{(\sigma+\delta)^2+(kt)^2}\Big) + \frac{\theta_2\kappa}{6}\Big(\frac{1}{(\sigma_1(\sigma)+\delta)^2+(kt)^2}\Big),$$

for some $|\theta_i| \leq 1$, where

$$\Xi(\sigma, k, t, \delta) := \frac{1}{4}\log\Big(1+\Big(\frac{\sigma+\delta}{kt}\Big)^2\Big) - \frac{\kappa}{4}\log\Big(1+\Big(\frac{\sigma_1(\sigma)+\delta}{kt}\Big)^2\Big)$$
$$- \frac{\sigma+\delta}{2((\sigma+\delta)^2+(kt)^2)} + \kappa\frac{\sigma_1(\sigma)+\delta}{2((\sigma_1(\sigma)+\delta)^2+(kt)^2)}.$$

Proof. We know that

$$\frac{\Gamma'}{\Gamma}(s) = \log s - \frac{1}{2s} - 2\int_0^\infty \frac{v}{(s^2+v^2)(e^{2\pi v}-1)}\,dv; \tag{2.5}$$

see [22, p. 251]. It follows from (2.5) that

$$\operatorname{Re}\frac{\Gamma'}{\Gamma}\Big(\frac{s_k+\delta}{2}\Big) = \log\Big|\frac{s_k+\delta}{2}\Big| + \operatorname{Re}\Big(\frac{1}{s_k+\delta}\Big) - 2\operatorname{Re}\int_0^\infty \frac{v}{((\frac{s_k+\delta}{2})^2+v^2)(e^{2\pi v}-1)}dv. \tag{2.6}$$

Moreover, we have

$$\operatorname{Re}\int_0^\infty \frac{v}{(z^2+v^2)(e^{2\pi v}-1)}dv \leq \Big|\int_0^\infty \frac{v}{(z^2+v^2)(e^{2\pi v}-1)}\,dv\Big|,$$

which is

$$\leq \int_0^\infty \Big|\frac{v}{(z^2+v^2)(e^{2\pi v}-1)}\Big|\,dv \leq \frac{1}{|z^2|}\int_0^\infty \frac{v}{e^{2\pi v}-1}\,dv = \frac{1}{24|z|^2}.$$

Therefore, we can write

$$\mathrm{Re}\int_0^\infty \frac{v}{(z^2+v^2)(e^{2\pi v}-1)}dv = \frac{\theta}{24|z|^2},$$

wherein $|\theta| \leq 1$, and so we can insert $z = \frac{s_k+\delta}{2}$ into this observation to obtain

$$-2\,\mathrm{Re}\int_0^\infty \frac{v}{((\frac{s_k+\delta}{2})^2+v^2)(e^{2\pi v}-1)}dv = \frac{\theta}{3|s_k+\delta|^2} = \frac{\theta}{3((\sigma+\delta)^2+k^2t^2)}.$$

In addition, we know

$$\log\left|\frac{s_k+\delta}{2}\right| = \log\frac{kt}{2} + \frac{1}{2}\log\left(1+\left(\frac{\sigma+\delta}{kt}\right)^2\right) \text{ and } \mathrm{Re}\left(\frac{1}{s_k+\delta}\right) = \frac{\sigma+\delta}{(\sigma+\delta)^2+k^2t^2}.$$

We insert these observations into (2.6) to yield the expected bounds. □

Using Lemma 2.1, we can also prove the following result.

Lemma 2.2. *Let $1 < \sigma < 1.15$, $|t| \leq 1$, $1 \leq k \leq 46$, $\delta \in \{0,1\}$, $s_k = \sigma + ikt$, and $s_k' = \sigma_1(\sigma) + ikt$. Then*

$$\frac{1}{2}\,\mathrm{Re}\left(\frac{\Gamma'}{\Gamma}\left(\frac{s_k+\delta}{2}\right) - \kappa\frac{\Gamma'}{\Gamma}\left(\frac{s_k'+\delta}{2}\right)\right) \leq \mathfrak{d}(\delta,k), \tag{2.7}$$

where values for $\mathfrak{d}(\delta,k)$ are presented in Table 8.

Proof. Using Lemma 2.1, we can bound the left-hand side of (2.7) from above by

$$\frac{1}{4}\log\left(\frac{(\sigma+\delta)^2+(kt)^2}{4}\right) - \frac{\kappa}{4}\log\left(\frac{(\sigma_1(\sigma)+\delta)^2+(kt)^2}{4}\right) - \frac{1}{2}\left(\frac{\sigma+\delta}{(\sigma+\delta)^2+(kt)^2}\right)$$
$$+\frac{\kappa}{2}\left(\frac{\sigma_1(\sigma)+\delta}{(\sigma_1(\sigma)+\delta)^2+(kt)^2}\right) + \frac{1}{6}\left(\frac{1}{(\sigma+\delta)^2+(kt)^2}\right) + \frac{\kappa}{6}\left(\frac{1}{(\sigma_1(\sigma)+\delta)^2+(kt)^2}\right).$$

If $\delta = 0$ and $k = 1$ or $\delta = 1$ and $1 \leq k \leq 46$, then this upper bound is majorised when $\sigma = 1.15$ and $t = 1$. If $\delta = 0$ and $2 \leq k \leq 46$, then this upper bound is majorised when $\sigma = 1$ and $t = 1$. Inserting these choices, we compute the explicit values presented in Table 8. □

Remark 6. Lemma 2.2 is applied in Section 4, where we address zeros in the interval $0 < |\gamma_0| \leq 1$. To simplify our analysis in that section, we do not choose ε optimally. Therefore, even though slight numerical improvements would be achieved by asserting $\sigma \leq 1+\varepsilon$ and choosing $\varepsilon < 0.15$ in Lemma 2.2, we take $\varepsilon = 0.15$ in this lemma to match the boundaries in those cases.

2.3. Bounds for Cases 2-4

In addition to the preceding results, we import the following results from [6, Eqns. (2.21), (2.23)-(2.27), (2.32), and (2.33)], which will be important in our treatment of Cases 2-4.

Lemma 2.3. *Let $\varrho_0 = \beta_0 + i\gamma_0$ be a non-trivial zero of $\zeta_L(s)$ such that $\beta_0 \geq 0.85$ and $\gamma_0 \geq 0$. For $k = 1$ and $\gamma_0 \leq 1$, one has*

$$-\sideset{}{'}\sum_{\substack{\varrho\in Z_L\\ \beta\geq 1/2}} \left(F(\sigma+i\gamma_0,\varrho) - \kappa F(\sigma_1(\sigma)+i\gamma_0,\varrho)\right) \leq -\frac{1}{\sigma-\beta_0}. \tag{2.8}$$

Let $\alpha_1 = 1.3951$. *For* $k \neq 1$ *and* $\gamma_0 \leq 1$, *one has two possible bounds:*

$$-\sum_{\substack{\varrho \in Z_L \\ \beta \geq 1/2}}{}' \left(F(\sigma + ik\gamma_0, \varrho) - \kappa F(\sigma_1(\sigma) + ik\gamma_0, \varrho)\right) \leq -\frac{\sigma - \beta_0}{(\sigma - \beta_0)^2 + (k-1)^2\gamma_0^2} + \alpha_1, \tag{2.9}$$

and

$$\begin{aligned} -\sum_{\substack{\varrho \in Z_L \\ \beta \geq 1/2}}{}' & \left(F(\sigma + ik\gamma_0, \varrho) - \kappa F(\sigma_1(\sigma) + ik\gamma_0, \varrho)\right) \\ & \leq -\frac{\sigma - \beta_0}{(\sigma - \beta_0)^2 + (k-1)^2\gamma_0^2} - \frac{\sigma - \beta_0}{(\sigma - \beta_0)^2 + (k+1)^2\gamma_0^2} + 2\alpha_1. \end{aligned} \tag{2.10}$$

Finally, for $0 < \gamma_0 \leq 1$, *one has*

$$-\sum_{\substack{\varrho \in Z_L \\ \beta \geq 1/2}}{}' \left(F(\sigma, \varrho) - \kappa F(\sigma_1, \varrho)\right) \leq -2\frac{\sigma - \beta_0}{(\sigma - \beta_0)^2 + \gamma_0^2} + 2\alpha_1. \tag{2.11}$$

Remark 7. Equation (2.8) will be invoked in Cases 2 and 3. In addition, Equations (2.9), (2.10), and (2.11) will be used, respectively, in Cases 2, 3, and 4.

Lemma 2.4. *Let* $\alpha_2 = 0.0215$ *and* $\alpha_3 = 1.5166$. *We have*

$$F(\sigma, 1) - \kappa F(\sigma_1, 1) \leq \frac{1}{\sigma - 1} + \alpha_2. \tag{2.12}$$

Further, if $k \geq 1$ *and* $0 < \gamma_0 < 1$, *then*

$$F(\sigma + ik\gamma_0, 1) - \kappa F(\sigma_1(\sigma) + ik\gamma_0, 1) \leq \frac{\sigma - 1}{(\sigma - 1)^2 + (k\gamma_0)^2} + \alpha_3. \tag{2.13}$$

Lemma 2.5. *If* $k = 0$ *or* $\gamma_0 = 0$, *then*

$$\operatorname{Re}\left(\frac{\gamma_L'}{\gamma_L}(\sigma) - \kappa\frac{\gamma_L'}{\gamma_L}(\sigma_1)\right) \leq \left(d(0) - \frac{(1-\kappa)\log\pi}{2}\right)n_L, \tag{2.14}$$

where $d(0) = -0.0512$. *If* $0 < \gamma_0 \leq 1$ *and* $k \geq 1$, *then*

$$\operatorname{Re}\left(\frac{\gamma_L'}{\gamma_L}(\sigma + ik\gamma_0) - \kappa\frac{\gamma_L'}{\gamma_L}(\sigma_1 + ik\gamma_0)\right) \leq \left(d(k) - \frac{(1-\kappa)\log\pi}{2}\right)n_L, \tag{2.15}$$

where $d(k) = \max\{\mathfrak{d}(0,k), \mathfrak{d}(1,k)\}$ *with* $\mathfrak{d}(\cdot,k)$ *the same as in Lemma 2.2. (Explicit values of* $d(k)$, *for* $k \leq 46$, *can be calculated using Table 8.)*

2.4. Candidates for the choice of polynomial

In [6, Sec. 3.1], Kadiri chose a degree $n = 4$ polynomial, denoted by $p_{4,1}$, to prove (1.2) with $(C_1, C_2, C_3, C_4, T) = (12.55, 9.69, 3.03, 58.63, 1)$. Lee used the degree $n = 16$ polynomial from [14] in [12] to establish (1.2) with

$$(C_1, C_2, C_3, C_4, T) = (12.2411, 9.5347, 0.05017, 2.2692, 1).$$

This refinement to Kadiri's method was one of the biggest sources of refinement in [12], in part because the polynomial was chosen using simulated annealing to improve the latest zero-free region for the Riemann zeta-function at the time. Prior to this, Kondrat'ev had used a degree $n = 8$ polynomial in [9] to beat Stečkin's result in [19]. Recently, Mossinghoff, Trudgian, and Yang have unveiled higher degree polynomials (of degree $n \in \{40, 46\}$) in [15], which were also chosen using simulated annealing to refine the latest zero-free regions for the Riemann zeta-function. The coefficients of these polynomials are presented in Tables 12-13. It seems natural, therefore, that these polynomials can also be used to refine the result in [12].

3. Case 1: large ordinates

In this section, bring forward all of the notations and set-up from Section 2. Our first result, namely Theorem 1.1 will follow by considering Case 1, that is $\gamma_0 > 1$. In particular, we bound S_1, S_3, and S_4 in Section 3.1, apply these bounds to obtain an upper bound for β_0 in Section 3.2, and complete our proof of Theorem 1.1 in Section 3.3. Note that S_2 will be computed directly, so there is no need to bound it.

3.1. Bounds for S_i

To begin, we import [12, Lem. 4], which tells us

$$S_1 \leq -\frac{a_1}{\sigma - \beta_0}, \tag{3.1}$$

where β_0 is the real part of the non-trivial zero ϱ_0 that we isolated earlier. The bound (3.1) is originally derived using [6, Lem. 2.3], by following the arguments in [12, Sec. 2.1]. Next, we bound S_3 and S_4 in the following lemmas.

Lemma 3.1. *Suppose that* $h(\sigma) = \frac{1}{\sigma} - \frac{\kappa}{\sigma_1} - \frac{\kappa}{\sigma_1(\sigma)-1}$ *and*

$$\begin{aligned}\Sigma_k(\sigma, t) &:= F(\sigma + ikt, 1) - \kappa F(\sigma_1(\sigma) + ikt, 1)\\ &= \frac{\sigma}{\sigma^2 + k^2t^2} + \frac{\sigma - 1}{(\sigma-1)^2 + k^2t^2} - \kappa\frac{\sigma_1}{\sigma_1{}^2 + k^2t^2} - \kappa\frac{\sigma_1(\sigma) - 1}{(\sigma_1(\sigma)-1)^2 + k^2t^2}.\end{aligned}$$

If $\alpha_\varepsilon = h(1+\varepsilon)$ *and* $\varepsilon \leq 0.15$, *then*

$$S_3 \leq a_0\Big(\frac{1}{\sigma - 1} + \alpha_\varepsilon\Big) + \sum_{k=1}^{n} a_k \Sigma_k(1+\varepsilon, 1).$$

Proof. Consider the cases $k = 0$ and $k > 0$ separately, and follow the arguments in [12, Sec. 2.2]. Note that α_ε increases as ε grows; its maximum $0.02146\ldots$ is realised at $\varepsilon = 0.15$. □

Lemma 3.2. *Suppose that*

$$d_\sigma = \frac{1}{2}\max_{\delta\in\{0,1\}}\Big\{\operatorname{Re}\Big(\frac{\Gamma'}{\Gamma}\Big(\frac{\sigma+\delta}{2}\Big) - \kappa\frac{\Gamma'}{\Gamma}\Big(\frac{\sigma_1(\sigma)+\delta}{2}\Big)\Big)\Big\}.$$

Moreover, we write $t > T_0 \geq 1$,

$$\mathcal{S}_1(k,\varepsilon) = \max_{\delta\in\{0,1\}}\big\{\mathcal{C}_1(k,\delta,\varepsilon)\big\}, \quad \textit{and} \quad \mathcal{S}_2(k,\varepsilon) = \max_{\delta\in\{0,1\}}\big\{\mathcal{C}_2(k,\delta,\varepsilon)\big\}$$

such that

$$\mathcal{C}_1(k,\delta,\varepsilon) = \frac{1-\kappa}{2}\log\frac{k}{2} + \Xi_2(1+\varepsilon,k,T_0,\delta) + \frac{1}{6}\Big(\frac{1}{(1-\delta)^2+k^2T_0^2}\Big) + \frac{\kappa}{6}\Big(\frac{1}{(\sigma_1(1)-\delta)^2+k^2T_0^2}\Big),$$

$$\mathcal{C}_2(k,\delta,\varepsilon) = \frac{1-\kappa}{2}\log\frac{k}{2} + \mathcal{A}(k,T_0,\delta,\varepsilon) + \frac{1}{6}\Big(\frac{1}{(1-\delta)^2+k^2T_0^2}\Big) + \frac{\kappa}{6}\Big(\frac{1}{(\sigma_1(1)-\delta)^2+k^2T_0^2}\Big),$$

where

$$\Xi_2(\sigma,k,t,\delta) = \frac{1}{4}\log\Big(1+\Big(\frac{\sigma+\delta}{kt}\Big)^2\Big) - \frac{\kappa}{4}\log\Big(1+\Big(\frac{\sigma_1(\sigma)+\delta}{kt}\Big)^2\Big),$$

$$\Xi(\sigma,k,t,\delta) = \Xi_2(\sigma,k,t,\delta) - \frac{\sigma+\delta}{2((\sigma+\delta)^2+k^2t^2)} + \frac{\kappa}{2}\frac{\sigma_1(\sigma)+\delta}{(\sigma_1(\sigma)+\delta)^2+k^2t^2}, \quad \textit{and}$$

$$\mathcal{A}(k,t,\delta,\varepsilon) = \begin{cases} 0 & \textit{if } \delta = 0 \textit{ or } \delta = 1 \textit{ and } k \notin \{1,2\}; \\ \Xi(1+\varepsilon,k,t,1) & \textit{if } \delta = 1 \textit{ and } k = 1; \\ \Xi(1.15,k,t,1) & \textit{if } \delta = 1 \textit{ and } k = 2. \end{cases}$$

If $\mathcal{S}(k,\varepsilon) = \min\{\mathcal{S}_1(k,\varepsilon), \mathcal{S}_2(k,\varepsilon)\}$*, then*

$$\operatorname{Re}\Big(\frac{\gamma_L'(s_k)}{\gamma_L(s_k)} - \kappa\frac{\gamma_L'(s_k')}{\gamma_L(s_k')}\Big) \leq \begin{cases} n_L\left(d_{1+\varepsilon} - \frac{1-\kappa}{2}\log\pi\right) & \textit{if } k=0; \\ n_L\left(\frac{1-\kappa}{2}(\log t + \log(\frac{k}{\pi})) + \mathcal{S}(k,\varepsilon)\right) & \textit{if } k \neq 0. \end{cases}$$

Consequently, we have

$$S_4 \leq a_0 n_L\Big(d_{1+\varepsilon} - \frac{1-\kappa}{2}\log\pi\Big) + \sum_{k=1}^{n} a_k n_L\Big(\frac{1-\kappa}{2}\Big(\log t + \log\Big(\frac{k}{\pi}\Big)\Big) + \mathcal{S}(k,\varepsilon)\Big).$$

Proof. On the interval $\sigma \in [1, 1+\varepsilon]$, we observe that $d_\sigma \leq d_{1+\varepsilon}$. Consider the cases $k = 0$ and $k > 0$ separately and follow the detailed arguments laid out in [12, Sec. 2.3], using Lemma 2.1 instead of [13, Lem. 2]. □

Remark 8. (i) Most of the notations in Lemma 3.2 are chosen to mirror the notations in [12] as closely as possible. This way, the interested reader can follow the arguments in [12, Sec. 2.3] with minimal differences. In fact, the only differences between our statements are that their notation $d_\varepsilon(0)$ is updated to $d_{1+\varepsilon}$, which we feel is a more natural notation, and the definitions for $\mathcal{C}_i(k,\delta,\varepsilon)$, which reflect that Lemma 2.1 has been applied in place of [13, Lem. 2].
(ii) In future work, one could extend Lemma 3.2 to hold for $t > T_0$, where $T_0 < 1$. Our decision to assert $T_0 \geq 1$ is made, because for this case we are refining the zero-free regions presented by Kadiri [6] and Lee [12] in the range $|t| > 1$. It is unclear (but likely) that one could improve our eventual result for Case 2 by considering $T_0 < 1$, but we do not investigate this in this paper.

3.2. Completing the argument

Suppose that $r > 0$, and σ is chosen such that

$$\sigma - 1 = r(1-\beta_0).$$

Insert the upper bounds for S_i from (3.1), Lemma 3.1, and Lemma 3.2 into (2.4) and rearrange the resulting inequality to see

$$\beta_0 \le 1 - \frac{\frac{a_1}{1+r} - \frac{a_0}{r}}{c_1 \log d_L + c_2 n_L \log t + c_3 n_L + c_4}, \tag{3.2}$$

where

$$c_1 = \frac{1-\kappa}{2} \sum_{k=0}^{n} a_k,$$

$$c_2 = \frac{1-\kappa}{2} \sum_{k=1}^{n} a_k,$$

$$c_3 = a_0\left(d_{1+\varepsilon} - \frac{1-\kappa}{2}\log\pi\right) + \sum_{k=1}^{n} a_k\left(\frac{1-\kappa}{2}\log\left(\frac{k}{\pi}\right) + \mathcal{S}(k,\varepsilon)\right), \text{ and}$$

$$c_4 = \alpha_\varepsilon a_0 + \sum_{k=1}^{n} a_k \Sigma_k(1+\varepsilon, 1).$$

The maximum value of $\frac{a_1}{1+r} - \frac{a_0}{r}$ occurs at $r = \frac{\sqrt{a_0}}{\sqrt{a_1}-\sqrt{a_0}}$. Therefore, dividing the numerator and denominator of (3.2) by

$$M = \frac{a_1}{1 + \frac{\sqrt{a_0}}{\sqrt{a_1}-\sqrt{a_0}}} - \frac{a_0}{\frac{\sqrt{a_0}}{\sqrt{a_1}-\sqrt{a_0}}},$$

we see that

$$\beta_0 \le 1 - \frac{1}{\frac{c_1}{M}\log d_L + \frac{c_2}{M} n_L \log t + \frac{c_3}{M} n_L + \frac{c_4}{M}} \quad \text{for all} \quad |\gamma_0| > 1;$$

this is (1.2) with

$$(C_1, C_2, C_3, C_4, T) = \left(\frac{c_1}{M}, \frac{c_2}{M}, \frac{c_3}{M}, \frac{c_4}{M}, 1\right). \tag{3.3}$$

Clearly, the choice of ε influences the definitions of c_3 and c_4, in that these values are minimised when ε is minimised. Therefore, we should choose ε to be as small as possible, but also enforce the condition

$$\varepsilon > \frac{1}{\frac{c_1}{M}\log d_L + \frac{c_3}{M} n_L + \frac{c_4}{M}}, \tag{3.4}$$

which will prevent the condition $\beta_0 > 1 - \varepsilon$ from being violated. So, to find the largest admissible constants in a zero-free region of the form (1.2) for number fields L with $n_L \ge n_0 \ge 3$, all that remains is to choose ε and the polynomial appropriately. To this end, we note that

$$\frac{1}{\frac{c_1}{M}\log d_L + \frac{c_3}{M} n_L + \frac{c_4}{M}} \le \frac{1}{\frac{c_1}{M}\log d_{\min}(n_0) + \frac{c_3}{M} n_0 + \frac{c_4}{M}},$$

where $d_{\min}(n_0)$ is the smallest permissible discriminant for a number field with $n_L \ge n_0$. Recall that admissible values for $d_{\min}(n_0)$ are presented in Table 1.

Remark 9. (i) Alternatively, one could choose ε arbitrarily, but then the result would only be valid when t satisfies $|t| \geq T$ such that

$$\varepsilon > \frac{1}{\frac{c_1}{M}\log d_{\min}(n_0) + \frac{c_2}{M}n_0\log T + \frac{c_3}{M}n_0 + \frac{c_4}{M}}.$$

For a fixed choice of ε, it is an easy problem to find an explicit value for T such that this would hold for all $|t| \geq T$, and hence we opt to leave this as an exercise for the reader.
(ii) Another avenue for future research is to implement a choice of ε depending on t. We did not pursue this, because it would have no effect on the constants c_1 or c_2, which are the major contributors to our final results.

3.3. Computations

We describe our algorithm to compute new admissible values for C_i in (1.2) with $T = 1$ and $n_L \geq n_0 \geq 3$. That is, fixing a choice of polynomial (from the polynomials introduced in Section 2.4) and n_0, we follow this process:

$\varepsilon \leftarrow 0.0001$
while $\left(\frac{c_1}{M}\log d_{\min}(n_0) + \frac{c_3(\varepsilon)}{M}n_0 + \frac{c_4(\varepsilon)}{M}\right)^{-1} \geq \varepsilon$ **do**
  $\varepsilon \leftarrow \varepsilon + 0.0001$
end while

The outcome of this process will be the least ε (up to four decimal places) such that (3.4) is satisfied. Once this choice of ε has been determined, insert it into (3.3) to yield admissible computations for C_i such that (1.2) is true with $T = 1$.

We computed values following the preceding logic and using the polynomials p_n of degree $n \in \{8, 16, 40, 46\}$ from Kondrat'ev [9], Mossinghoff–Trudgian [14], and Mossinghoff–Trudgian–Yang [15] (whose coefficients are presented in Tables 12-13). In the end, there were two outcomes that might be considered the "best", depending on whether the reader places more importance on the constant C_1 or C_2 being minimised. If it is more important that C_1 is minimal, then we determined that p_{46} is the best polynomial to choose. On the other hand, if it is more important that C_2 is minimal, then we determined that p_{16} is the best polynomial to choose. Our results under these two choices (and p_{40} for comparison) are presented in Tables 9, 10, and 11.

Remark 10. Better (and faster) algorithms can be implemented to locate the true optimal choice for ε. For example, using a binary search. However, we found that optimising ε to more than four decimal places did not yield noticeable improvements, so we implemented the simplified algorithm presented here.

4. Cases 2-4: complex zeros with small ordinates

Bring forward all of the notations (including s_k, s_k') and set-up from Section 2. In addition, we let $n_L \geq n_0$, $\mathscr{L} = \log d_L$, $\mathscr{L} \geq \mathscr{L}_0 > 0$ where $\mathscr{L}_0 = \log(d_{\min}(n_0))$,

$$\sigma - 1 = \frac{r}{\mathscr{L}}, \quad \text{and} \quad 1 - \beta_0 = \frac{c}{\mathscr{L}}. \tag{4.1}$$

Since $1 < \sigma < 1 + \varepsilon$ for some $0 < \varepsilon \leq 0.15$ and $\mathscr{L} \geq \mathscr{L}_0$, we naturally have

$$0 < r < 0.15\mathscr{L}_0.$$

In this section, we focus on Cases 2-4, corresponding to the range $0 < \gamma_0 \leq 1$. Specifically, for real numbers d_1 and d_2 satisfying $0 < d_1 \leq d_2 < \log d_L$, we recall:

- **Case 2:** $\frac{d_2}{\mathscr{L}} < \gamma_0 \leq 1$
- **Case 3:** $\frac{d_1}{\mathscr{L}} < \gamma_0 \leq \frac{d_2}{\mathscr{L}}$
- **Case 4:** $0 < \gamma_0 \leq \frac{d_1}{\mathscr{L}}$

4.1. Case 2

To begin, let r and c be taken as in (4.1). The symmetry of the zeros with respect to the critical line enables one to write

$$-\sum_{\rho=\beta+i\gamma\in Z_L} \operatorname{Re}\Big(\frac{1}{s_k-\rho} - \frac{\kappa}{s_k'-\rho}\Big) = -\sum_{\beta\geq\frac{1}{2}}{}' (F(s_k,\rho) - \kappa F(s_k',\rho)),$$

where $\sum'_{\beta\geq\frac{1}{2}} = \frac{1}{2}\sum_{\beta=\frac{1}{2}} + \sum_{\frac{1}{2}<\beta\leq 1}$. Using (2.8) for $k=1$, (2.9) for $k=0,2,3,\dots,n$, and this relation, we have

$$\begin{aligned} S_1 &= -\sum_{k=0}^{n} a_k \sum_{\varrho\in Z_L} \operatorname{Re}\Big(\frac{1}{s_k-\varrho} - \frac{\kappa}{s_k'-\varrho}\Big) \\ &\leq -\frac{a_1}{\sigma-\beta_0} - \frac{a_0(\sigma-\beta_0)}{(\sigma-\beta_0)^2+\gamma_0^2} - \sum_{k=2}^{n} \frac{a_k(\sigma-\beta_0)}{(\sigma-\beta_0)^2+(k-1)^2\gamma_0^2} + \alpha_1 a_0 + \alpha_1\sum_{k=2}^{n} a_k. \end{aligned}$$

Secondly, we apply Lemma 2.4 (i.e., (2.12) for $k=0$ and (2.13) otherwise) to obtain

$$\begin{aligned} S_3 &= \sum_{k=0}^{n} a_k\left(F(s_k,1) - \kappa F(s_k',1)\right) \\ &\leq \frac{a_0}{\sigma-1} + a_0\alpha_2 + \sum_{k=1}^{n}\frac{a_k(\sigma-1)}{(\sigma-1)^2+k^2\gamma_0^2} + \alpha_3\sum_{k=1}^{n} a_k. \end{aligned}$$

Thirdly, it follows from Lemma 2.5 (i.e., (2.14) for $k=0$ and (2.15) otherwise) that

$$S_4 = \sum_{k=0}^{n} a_k \operatorname{Re}\Big(\frac{\gamma_L'(s_k)}{\gamma_L(s_k)} - \kappa\frac{\gamma_L'(s_k')}{\gamma_L(s_k')}\Big) \leq \sum_{k=0}^{n} a_k\Big(d(k) - \frac{1-\kappa}{2}\log\pi\Big)n_L.$$

Our computations for $d(k)$ confirm that the coefficient of n_L is negative, so we have

$$S_4 \leq \sum_{k=0}^{n} a_k\Big(d(k) - \frac{1-\kappa}{2}\log\pi\Big)n_0. \tag{4.2}$$

Therefore, (2.4) and the above inequalities yield

$$\begin{aligned} 0 \leq{}& \frac{a_0}{\sigma-1} - \frac{a_1}{\sigma-\beta_0} + \frac{a_1(\sigma-1)}{(\sigma-1)^2+\gamma_0^2} - \frac{a_0(\sigma-\beta_0)}{(\sigma-\beta_0)^2+\gamma_0^2} + \frac{1-\kappa}{2}\mathscr{L}\sum_{k=0}^{n} a_k \\ &+ \sum_{k=2}^{n} a_k\Big(\frac{\sigma-1}{(\sigma-1)^2+k^2\gamma_0^2} - \frac{\sigma-\beta_0}{(\sigma-\beta_0)^2+(k-1)^2\gamma_0^2}\Big) \end{aligned} \tag{4.3}$$

$$+ \alpha_1 a_0 + \alpha_1 \sum_{k=2}^{n} a_k + \alpha_2 a_0 + \alpha_3 \sum_{k=1}^{n} a_k + \sum_{k=0}^{n} a_k \Big(d(k) - \frac{1-\kappa}{2} \log \pi \Big) n_0.$$

Using these observations, we prove the following lemma.

Lemma 4.1. *Assume that*

$$0 < \frac{a_0}{a_1 - a_0} c < r < 1 \quad \textit{and} \quad d_2 > \frac{\sqrt{r(r+c)}}{2}. \tag{4.4}$$

Suppose that $\gamma_0 \in (\frac{d_2}{\mathscr{L}}, 1)$*. Let* $\mathcal{U}(x)$ *be the unit step function defined by*

$$\mathcal{U}(x) = \begin{cases} 1 & \textit{if } x \geq 0, \\ 0 & \textit{otherwise}, \end{cases} \tag{4.5}$$

and set

$$A_{n_0} = \alpha_1 a_0 + \alpha_1 \sum_{k=2}^{n} a_k + \alpha_2 a_0 + \alpha_3 \sum_{k=1}^{n} a_k + \sum_{k=0}^{n} a_k \Big(-\frac{1-\kappa}{2} \log \pi + d(k) \Big) n_0. \tag{4.6}$$

We have

$$0 \leq \mathscr{L} \Big(\frac{a_0}{r} - \frac{a_1}{r+c} + \frac{a_1 r}{r^2 + d_2^2} - \frac{a_0 (r+c)}{(r+c)^2 + d_2^2} + \frac{1-\kappa}{2} \sum_{k=0}^{n} a_k + \frac{\mathcal{U}(A_{n_0}) A_{n_0}}{\mathscr{L}} \Big).$$

Proof. Note that, for any $k \geq 2$ and any $\gamma_0 \in (\frac{d_2}{\mathscr{L}}, 1)$, by taking

$$a = \sigma - 1, \quad b = \sigma - \beta_0, \quad x = k\gamma_0,$$

the second condition of (4.4) implies $(b-a)(x^2 - ab) > 0$. Thus, [6, Lem. 3.1(ii)] implies

$$\frac{\sigma - 1}{(\sigma-1)^2 + k^2 \gamma_0^2} - \frac{\sigma - \beta_0}{(\sigma - \beta_0)^2 + (k-1)^2 \gamma_0^2} \leq 0$$

for $k \in \{2, 3, \ldots, n\}$; this observation extends [6, Eqn. (3.4)]. Hence, the terms in the second row of (4.3) may be discarded.

Furthermore, the first condition of (4.4) satisfies the hypothesis of [6, Lem. 3.1(iii)] with

$$a = \sigma - 1 = \frac{r}{\mathscr{L}}, \quad b = \sigma - \beta_0 = \frac{r+c}{\mathscr{L}}, \quad q = \frac{a_1}{a_0}, \quad \text{and} \quad x = \gamma_0.$$

Thus, [6, Lem. 3.1(iii)] implies

$$\frac{a_1(\sigma-1)}{(\sigma-1)^2 + \gamma_0^2} - \frac{a_0(\sigma - \beta_0)}{(\sigma-\beta_0)^2 + \gamma_0^2} \leq \Big(\frac{a_1 r}{r^2 + d_2^2} - \frac{a_0(r+c)}{(r+c)^2 + d_2^2} \Big) \mathscr{L}, \tag{4.7}$$

in the interval $\gamma_0 \in (\frac{d_2}{\mathscr{L}}, 1)$. Finally, the RHS of (4.3) is positive for $\gamma_0 \in (\frac{d_2}{\mathscr{L}}, 1)$. The result follows from this observation, (4.3), and (4.7). □

It follows from Lemma 4.1 and $\mathscr{L} \geq \mathscr{L}_0 > 0$ that

$$0 \leq \frac{a_0}{r} - \frac{a_1}{r+c} + \frac{a_1 r}{r^2 + d_2^2} - \frac{a_0(r+c)}{(r+c)^2 + d_2^2} + M_1, \tag{4.8}$$

where

$$M_1 = \frac{1-\kappa}{2} \sum_{k=0}^{n} a_k + \frac{\mathcal{U}(A_{n_0}) A_{n_0}}{\mathscr{L}_0}. \tag{4.9}$$

Therefore, (4.8) holds if and only if

$$\frac{a_1}{r+c} + \frac{a_0(r+c)}{(r+c)^2 + d_2^2} \leq \frac{a_0}{r} + \frac{a_1 r}{r^2 + d_2^2} + M_1,$$

which is equivalent to

$$a_1 + a_0\left(1 + \left(\frac{d_2}{r+c}\right)^2\right)^{-1} \leq (r+c)\left(\frac{a_0}{r} + \frac{a_1 r}{r^2 + d_2^2} + M_1\right).$$

It then follows that

$$c \geq \frac{a_1 + a_0\left(1 + \left(\frac{d_2}{r+c}\right)^2\right)^{-1} - \left(\frac{a_0}{r} + \frac{a_1 r}{r^2+d_2^2} + M_1\right) r}{\frac{a_0}{r} + \frac{a_1 r}{r^2+d_2^2} + M_1}. \tag{4.10}$$

Note that by the first part of (4.4), we have

$$a_0\left(1 + \left(\frac{d_2}{r+c}\right)^2\right)^{-1} < a_0\left(1 + \left(\frac{d_2}{r + \frac{(a_1-a_0)r}{a_0}}\right)^2\right)^{-1} = a_0\left(1 + \left(\frac{a_0 d_2}{a_1 r}\right)^2\right)^{-1}.$$

Thus, (4.10) holds if r, d_2 and c satisfy

$$c \geq r \cdot \frac{a_1 + a_0\left(1 + \left(\frac{a_0 d_2}{a_1 r}\right)^2\right)^{-1} - r\left(\frac{a_0}{r} + \frac{a_1 r}{r^2+d_2^2} + M_1\right)}{a_0 + \frac{a_1}{1+(d_2/r)^2} + M_1 r} =: \tau_1(r, d_2). \tag{4.11}$$

For such an instance, as $(1-\beta_0)\mathscr{L} = c$, we have $\beta_0 = 1 - \frac{c}{\mathscr{L}}$ and thus

$$\beta_0 < 1 - \frac{\tau_1(r, d_2)}{\mathscr{L}}. \tag{4.12}$$

Remark 11. We remark that directly computing the precise values of r, d_2, and c to satisfy (4.10) is complicated and somewhat inefficient for achieving a proper balance with Cases 3 and 4. So, we instead derive a sufficient condition in (4.11) that guarantees that (4.10) holds for the values of r, d_2, and c obtained in (4.11). This is precisely the reason we introduced the auxiliary quantities $\tau_1(r, d_2)$ in (4.12) for this case, $\tau_2(r, d_1, d_2)$ in (4.23) for Case 3, and $\tau_3(r, d_1)$ in (4.30) for Case 4, and then perform our computations in terms of them.

Table 2 presents choices of r, d_2, and one of the polynomials p_n of degree $n \in \{8, 16, 40, 46\}$ that were discussed in Section 2.4 such that $\tau_1(r, d_2)^{-1}$ is minimised. These values were obtained using straightforward two-variable optimisation (accurate to five decimal places), similar to the approach described in Section 3.3,

Table 2
Optimised parameter choices and computations for τ_1.

n_L	polynomial	r	d_2	$\tau_1(r,d_2)^{-1}$
2	p_{40}	0.201	0.70445	16.018313
3	p_{16}	0.165	0.56927	19.552923
4	p_8	0.195	0.68000	16.722068
5	p_8	0.240	0.83500	13.712344
6	p_8	0.275	0.91803	11.781799
≥ 7	p_{46}	0.280	0.98024	11.519094

and ensuring that the parameters r, d_2 and $c = \tau_1(r, d_2)$ satisfy condition (4.4). Note that the verification works of Platt [16] and Tollis [20], described in the introduction, mean that we can safely assert

$$d_L \geq \begin{cases} 400\,001 & \text{if } n_L = 2, \\ 240 & \text{if } n_L = 3, \\ 321 & \text{if } n_L = 4, \\ d_{\min}(n_L) & \text{if } n_L > 4, \end{cases} \tag{4.13}$$

in these computations. Recall that values for $d_{\min}(n_L)$ are presented in Table 1. In the end, we expect the computations from this case to have the worst numerical outcome from Cases 2-4, so we fix the choices of d_2 associated to each n_L from this case moving forwards.

4.2. Case 3

Let r and c be taken as in (4.1) and suppose that $0 < \frac{a_0}{a_1 - a_0} c < r < 1$. Similar to Case 2, using (2.8) for $k = 1$, and (2.10) otherwise, we get

$$\begin{aligned} S_1 \leq & -\frac{a_1}{\sigma - \beta_0} + 2\alpha_1 a_0 + 2\alpha_1 \sum_{k=2}^{n} a_k - \frac{2a_0(\sigma-\beta_0)}{(\sigma-\beta_0)^2 + \gamma_0^2} \\ & - \sum_{k=2}^{n} a_k \Big(\frac{\sigma - \beta_0}{(\sigma-\beta_0)^2 + (k-1)^2\gamma_0^2} + \frac{\sigma-\beta_0}{(\sigma-\beta_0)^2 + (k+1)^2\gamma_0^2} \Big). \end{aligned}$$

Next, we apply Lemma 2.4 (i.e., (2.12) for $k = 0$ and (2.13) otherwise) to deduce

$$S_3 \leq \frac{a_0}{\sigma - 1} + a_0\alpha_2 + \sum_{k=1}^{n} \frac{a_k(\sigma-1)}{(\sigma-1)^2 + k^2\gamma_0^2} + \alpha_3 \sum_{k=1}^{n} a_k.$$

In addition, as argued in the previous section, applying Lemma 2.5 with (2.14) for $k = 0$ and (2.15) otherwise, together with our computations for $d(k)$ confirming that the coefficient of n_L is negative, we again have the upper bound (4.2) for S_4.

Combining all the above inequalities with (2.4), we deduce

$$\begin{aligned} 0 \leq & \frac{a_0}{\sigma-1} - \frac{a_1}{\sigma-\beta_0} + \frac{a_1(\sigma-1)}{(\sigma-1)^2+\gamma_0^2} - \frac{2a_0(\sigma-\beta_0)}{(\sigma-\beta_0)^2+\gamma_0^2} + \frac{1-\kappa}{2} \Big(\sum_{k=0}^{n} a_k \Big) \mathscr{L} \\ & + \sum_{k=2}^{n} a_k \Big(\frac{\sigma-1}{(\sigma-1)^2 + k^2\gamma_0^2} - \frac{\sigma-\beta_0}{(\sigma-\beta_0)^2+(k-1)^2\gamma_0^2} - \frac{\sigma-\beta_0}{(\sigma-\beta_0)^2+(k+1)^2\gamma_0^2} \Big) \\ & + 2\alpha_1 a_0 + 2\alpha_1 \sum_{k=2}^{n} a_k + \alpha_2 a_0 + \alpha_3 \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} a_k \Big(d(k) - \frac{1-\kappa}{2} \log \pi \Big) n_0. \end{aligned} \tag{4.14}$$

Here, $k \geq 2$ and $\gamma_0 \in (\frac{d_1}{\mathscr{L}}, \frac{d_2}{\mathscr{L}})$, so we have

$$\begin{aligned}&\frac{\sigma-1}{(\sigma-1)^2+k^2\gamma_0^2}-\frac{\sigma-\beta_0}{(\sigma-\beta_0)^2+(k-1)^2\gamma_0^2}-\frac{\sigma-\beta_0}{(\sigma-\beta_0)^2+(k+1)^2\gamma_0^2}\\&\qquad\leq\Big(\frac{r}{r^2+k^2d_1^2}-\frac{r+c}{(r+c)^2+(k-1)^2d_2^2}-\frac{r+c}{(r+c)^2+(k+1)^2d_2^2}\Big)\mathscr{L}.\end{aligned} \tag{4.15}$$

Remark 12. As may be observed, (4.14) closely mirrors the inequality given in [6, Pg. 143, Eq. (3.8)], except for the fifth term on the right of [6, Eq. (3.8)], which appears to be an error. This issue also occurs in [12]. In addition, (4.15) corresponds to the inequality stated below [6, Eq. (3.8)], but we correct a typographical error there by placing $\mathscr{L}$ in the numerator (instead of having $\frac{1}{\mathscr{L}}$). Last but not least, we shall remark that for d_L sufficiently large, one would get $\frac{1}{c} = 12.972$ instead of 12.7301 in [6, Case 3] and $\frac{1}{c_B} = 12.6811$ instead of 12.43355 in [12]. Nevertheless, our Theorem 1.2 assures that these previously-claimed values are still valid for $n_L \geq 6$.

Similar to the derivation of (4.7) which required the condition $0 < \frac{a_0}{a_1-a_0}c < r < 1$ to satisfy the hypothesis of [6, Lem. 3.1(iii)], we deduce that

$$\frac{a_1(\sigma-1)}{(\sigma-1)^2+\gamma_0^2}-\frac{a_0(\sigma-\beta_0)}{(\sigma-\beta_0)^2+\gamma_0^2}$$

decreases with $\gamma_0 \in (\frac{d_1}{\mathscr{L}}, \frac{d_2}{\mathscr{L}})$. Thus,

$$\frac{a_1(\sigma-1)}{(\sigma-1)^2+\gamma_0^2}-\frac{a_0(\sigma-\beta_0)}{(\sigma-\beta_0)^2+\gamma_0^2}\leq\Big(\frac{a_1r}{r^2+d_1^2}-\frac{a_0(r+c)}{(r+c)^2+d_1^2}\Big)\mathscr{L}. \tag{4.16}$$

It also follows from $\gamma_0 \in (\frac{d_1}{\mathscr{L}}, \frac{d_2}{\mathscr{L}})$ that

$$-\frac{a_0(\sigma-\beta_0)}{(\sigma-\beta_0)^2+\gamma_0^2}\leq-\Big(\frac{a_0(r+c)}{(r+c)^2+d_2^2}\Big)\mathscr{L}. \tag{4.17}$$

Now, the equation in (4.14) is positive for $\gamma_0 \in (\frac{d_1}{\mathscr{L}}, \frac{d_2}{\mathscr{L}})$, so we can combine (4.1), (4.14), (4.15), (4.16), and (4.17) to derive

$$\begin{aligned}0\leq{}&\frac{a_0}{r}-\frac{a_1}{r+c}+\frac{a_1r}{r^2+d_1^2}-\frac{a_0(r+c)}{(r+c)^2+d_1^2}-\frac{a_0(r+c)}{(r+c)^2+d_2^2}+\frac{1-\kappa}{2}\sum_{k=0}^{n}a_k\\&+\sum_{k=2}^{n}a_k\Big(\frac{r}{r^2+k^2d_1^2}-\frac{r+c}{(r+c)^2+(k-1)^2d_2^2}-\frac{r+c}{(r+c)^2+(k+1)^2d_2^2}\Big)+\frac{\mathcal{U}(B_{n_0})B_{n_0}}{\mathscr{L}},\end{aligned}$$

where $\mathcal{U}(x)$ is the unit step function defined as in (4.5) and

$$B_{n_0}=2\alpha_1a_0+2\alpha_1\sum_{k=2}^{n}a_k+\alpha_2a_0+\alpha_3\sum_{k=1}^{n}a_k+\sum_{k=1}^{n}a_k\Big(d(k)-\frac{1-\kappa}{2}\log\pi\Big)n_0. \tag{4.18}$$

It follows from $\mathscr{L} \geq \mathscr{L}_0 > 0$ that

$$\begin{aligned}0\leq{}&\frac{a_0}{r}-\frac{a_1}{r+c}+\frac{a_1r}{r^2+d_1^2}-\frac{a_0(r+c)}{(r+c)^2+d_1^2}-\frac{a_0(r+c)}{(r+c)^2+d_2^2}\\&+\sum_{k=2}^{n}a_k\Big(\frac{r}{r^2+k^2d_1^2}-\frac{r+c}{(r+c)^2+(k-1)^2d_2^2}-\frac{r+c}{(r+c)^2+(k+1)^2d_2^2}\Big)+M_2,\end{aligned} \tag{4.19}$$

where

$$M_2 = \frac{1-\kappa}{2}\sum_{k=0}^{n} a_k + \frac{\mathcal{U}(B_{n_0})B_{n_0}}{\mathscr{L}_0}. \tag{4.20}$$

Similar to Case 2, as directly computing the precise values of r, d_1, d_2 and c to satisfy (4.19) is complicated, we provide a sufficient condition such that (4.19) holds for the values of r, d_1, d_2, and c obtained in (4.23) as follows. By the condition $r + c \leq \frac{a_1}{a_0} r$, we must have

$$\frac{1}{1+\frac{z_0}{(r+c)^2}} \leq \frac{1}{1+\frac{z_0 a_0^2}{a_1^2 r^2}}. \tag{4.21}$$

Therefore, the inequality (4.19) holds if the following is true:

$$a_1 + \frac{a_0}{1+\frac{d_1^2 a_0^2}{r^2 a_1^2}} + \frac{a_0}{1+\frac{d_2^2 a_0^2}{r^2 a_1^2}} + \sum_{k=2}^{n} a_k Q(k, d_2, r; a_0, a_1) \leq (r+c)\left(r\sum_{k=0}^{n}\frac{a_k}{r^2+k^2 d_1^2} + M_2\right),$$

where

$$Q(k, d, r; a_0, a_1) = \frac{1}{1+\frac{(k-1)^2 d^2 a_0^2}{a_1^2 r^2}} + \frac{1}{1+\frac{(k+1)^2 d^2 a_0^2}{a_1^2 r^2}}. \tag{4.22}$$

Clearly, the preceding inequality is satisfied if and only if

$$\begin{aligned} c &\geq \frac{a_1 + \frac{a_0}{1+\frac{d_1^2 a_0^2}{r^2 a_1^2}} + \frac{a_0}{1+\frac{d_2^2 a_0^2}{r^2 a_1^2}} + \sum_{k=2}^{n} a_k Q(k, d_2, r; a_0, a_1) - r\left(r\sum_{k=0}^{n}\frac{a_k}{r^2+k^2 d_1^2} + M_2\right)}{r\sum_{k=0}^{n}\frac{a_k}{r^2+k^2 d_1^2} + M_2} \\ &=: \tau_2(r, d_1, d_2). \end{aligned} \tag{4.23}$$

Hence, for such an instance, as $1 - \beta_0 = \frac{c}{\mathscr{L}}$, it follows that

$$\beta_0 \leq 1 - \frac{\tau_2(r, d_1, d_2)}{\mathscr{L}}. \tag{4.24}$$

We defer the computations for this case until Section 4.4, as our eventual choice of d_1 will also have an effect on the outcome in Case 4.

4.3. Case 4

We utilise a different argument in this case. To begin, we note that

$$\begin{aligned} f_L(\sigma, 0) = &-\sum_{\substack{\varrho \in Z_L \\ \beta \geq 1/2}}{}' \left(F(\sigma, \varrho) - \kappa F(\sigma_1, \varrho)\right) + \frac{1-\kappa}{2}\mathscr{L} \\ &+ F(\sigma, 1) - \kappa F(\sigma_1, 1) + \operatorname{Re}\left(\frac{\gamma_L'}{\gamma_L}(\sigma) - \kappa\frac{\gamma_L'}{\gamma_L}(\sigma_1)\right). \end{aligned} \tag{4.25}$$

It follows from Lemmas 2.3, 2.4, and 2.5 that

$$f_L(\sigma, 0) \leq 2\left(\alpha_1 - \frac{\sigma-\beta_0}{(\sigma-\beta_0)^2+\gamma_0^2}\right) + \frac{1-\kappa}{2}\mathscr{L} + \frac{1}{\sigma-1} + \alpha_2 + \left(d(0) - \frac{(1-\kappa)\log\pi}{2}\right)n_L.$$

Recall that $\sigma - 1 = \frac{r}{\mathscr{L}}$, $1 - \beta_0 = \frac{c}{\mathscr{L}}$, and note that the coefficient of n_L in the above equation is negative. Therefore, $f_L(\sigma, 0) \geq 0$, $n_L \geq n_0$, and $\gamma_0 \leq \frac{d_1}{\mathscr{L}}$ imply

$$0 \leq \mathscr{L}\left(\frac{1}{r} - 2\frac{r+c}{(r+c)^2 + d_1^2} + \frac{1-\kappa}{2} + \frac{\mathcal{U}(C_{n_0})C_{n_0}}{\mathscr{L}}\right), \tag{4.26}$$

where $\mathcal{U}(x)$ is defined as in (4.5) and

$$C_{n_0} = 2\alpha_1 + \alpha_2 + n_0\Big(d(0) - \frac{(1-\kappa)\log\pi}{2}\Big). \tag{4.27}$$

Since $\mathscr{L} \geq \mathscr{L}_0 > 0$, the preceding discussion tells us

$$0 \leq \frac{1}{r} - 2\frac{r+c}{(r+c)^2 + d_1^2} + M_3, \tag{4.28}$$

in which

$$M_3 = \frac{1-\kappa}{2} + \frac{\mathcal{U}(C_{n_0})C_{n_0}}{\mathscr{L}_0}. \tag{4.29}$$

Observe that (4.28) is equivalent to

$$c^2 + \Big(\frac{2M_3r^2}{1+M_3r}\Big)c + \Big(d_1^2 + \frac{M_3r^3 - r^2}{1+M_3r}\Big) \geq 0.$$

As the roots of the above quadratic equation occur at

$$c = \frac{-M_3r^2 \pm \sqrt{r^2 - d_1^2(1+M_3r)^2}}{1+M_3r},$$

and the last inequality above holds when

$$c \geq \frac{\sqrt{r^2 - d_1^2(1+M_3r)^2} - M_3r^2}{1+M_3r} =: \tau_3(r, d_1), \tag{4.30}$$

which yields

$$\beta_0 \leq 1 - \frac{\tau_3(r, d_1)}{\mathscr{L}}. \tag{4.31}$$

Note that the values of r, d_1, and c satisfying (4.30) will automatically satisfy (4.28).

4.4. Proof of Theorem 1.2

Recall that the coefficients of $\mathscr{L}^{-1}$ in (4.12), (4.24), and (4.31) respectively, which we aim to maximise, are

$$\tau_1(r, d_2) = \frac{a_1 + a_0\Big(1 + \Big(\frac{a_0d_2}{a_1r}\Big)^2\Big)^{-1} - r\left(\frac{a_0}{r} + \frac{a_1r}{r^2+d_2^2} + M_1\right)}{\frac{a_0}{r} + \frac{a_1/r}{1+(d_2/r)^2} + M_1},$$

$$\tau_2(r, d_1, d_2) = \frac{a_1 + \frac{a_0}{1+\frac{d_1^2a_0^2}{r^2a_1^2}} + \frac{a_0}{1+\frac{d_2^2a_0^2}{r^2a_1^2}} + \sum\limits_{k=2}^{n} a_kQ(k, d_2, r; a_0, a_1) - r\left(r\sum_{k=0}^{n}\frac{a_k}{r^2+k^2d_1^2} + M_2\right)}{r\sum_{k=0}^{n}\frac{a_k}{r^2+k^2d_1^2} + M_2},$$

Table 3
Optimised parameter choices and computations for τ_2 and τ_3.

n_L	polynomial	r_A	r_B	d_1	d_2	$\tau_2(r_A, d_1, d_2)^{-1}$	$\tau_3(r_B, d_1)^{-1}$
2	p_{46}	0.4827	1.2294	0.5353	0.70445	6.099311	6.099299
3	p_{16}	0.3348	0.8220	0.4140	0.56927	9.968097	9.963504
4	p_8	0.3590	0.8656	0.4768	0.68000	9.890467	9.890536
5	p_8	0.4522	1.1072	0.6004	0.83500	7.981275	7.976893
6	p_8	0.5545	1.3776	0.7122	0.91803	6.029385	6.026121
7	p_{46}	0.6533	1.8187	0.8173	0.98024	4.595864	4.595872
≥ 8	p_{46}	0.7190	1.9524	0.8492	0.98024	3.884018	3.884011

$$\tau_3(r, d_1) = \frac{\sqrt{r^2 - d_1^2(1 + M_3 r)^2} - M_3 r^2}{1 + M_3 r}.$$

Equivalently, we are seeking r, d_1, d_2 such that the largest value of $\tau_1(r, d_2)^{-1}$, $\tau_2(r, d_1, d_2)^{-1}$, and $\tau_3(r, d_1)^{-1}$ is minimised. We have already presented optimised computations for $\tau_1(r, d_2)^{-1}$ in Table 2, which we expect to be the worst case, so all that remains is to fix the values of d_2 presented in Table 2, then choose r and d_1 such that $\max\{\tau_2(r, d_1, d_2)^{-1}, \tau_3(r, d_1)^{-1}\}$ is minimal. For these computations, we choose the polynomials based on numerical experiments, and we implement the lower bound from (4.13) for d_L. Furthermore, we note that our choice of r does not need to be the same when computing $\tau_1(r, d_2)^{-1}$, $\tau_2(r, d_1, d_2)^{-1}$, and $\tau_3(r, d_1)^{-1}$; the only parameters that need to be consistent across each case are d_1 and d_2. Moreover, to ensure that $\tau_3(r, d_1)$ is real, d_1 must be chosen so that $r^2 \geq d_1^2(1 + M_3 r)^2$.

Finally, fixing the choices of d_2 presented in Table 2 and one of the polynomials p_n of degree $n \in \{8, 16, 40, 46\}$ that were discussed in Section 2.4, we have determined that the choices of r and d_1 presented in Table 3 give the best outcomes (up to four decimal places). Theorem 1.2 mainly follows from these resulting computations; the only remaining part of the proof is to deal with the real zeros that will be handled in Section 5.1.

Remark 13. The computations in Table 3 also demonstrate how much improvement would be available in Theorem 1.2 if future advancements enabled us to reduce the size of $\tau_1(r, d_2)^{-1}$.

4.5. Proof of Theorem 1.3

Suppose that an exceptional zero, denoted β_1, exists such that $\beta_1 \geq 1 - \frac{\nu}{\mathscr{L}}$, where $\nu > 0$. Note that if $k = 0$ and $\varrho = \beta + i\gamma \in Z_L$ such that $\gamma = 0$, then

$$\begin{aligned} F(s_k, \varrho) - \kappa F(s_k', \varrho) &= \frac{\sigma - \beta}{(\sigma - \beta)^2 + (kt - \gamma)^2} + \frac{\sigma - 1 + \beta}{(\sigma - 1 + \beta)^2 + (kt - \gamma)^2} \\ &\quad - \kappa\Big(\frac{\sigma_1(\sigma) - \beta}{(\sigma_1(\sigma) - \beta)^2 + (kt - \gamma)^2} + \frac{\sigma_1(\sigma) - 1 + \beta}{(\sigma_1(\sigma) - 1 + \beta)^2 + (kt - \gamma)^2}\Big) \\ &= \frac{1}{\sigma - \beta} + \frac{1}{\sigma - 1 + \beta} - \kappa\Big(\frac{1}{\sigma_1(\sigma) - \beta} + \frac{1}{\sigma_1(\sigma) - 1 + \beta}\Big). \end{aligned}$$

Hence, it follows from [6, Eqn. (2.16)] that if $k = 0$ and $\varrho = \beta + i\gamma \in Z_L$ such that $\gamma = 0$, then

$$F(s_k, \varrho) - \kappa F(s_k', \varrho) \geq \frac{1}{\sigma - \beta}. \tag{4.32}$$

Repeating the arguments in Sections 4.1-4.3, while using the bound (4.32) at the opportune moment to account for the extra negative contribution given by the exceptional zero β_1, we see that an isolated zero $\varrho_0 = \beta_0 + i\gamma_0 \in Z_L$ such that $0 < |\gamma_0| \leq 1$ will satisfy

Table 4
Optimised parameter choices and computations for $\hat{\tau}_1, \hat{\tau}_2$ and $\hat{\tau}_3$ with $\nu = 0.5$ and polynomial $p_{4,3}$.

n_L	r_A	r_B	r_C	d_1	d_2	$\hat{\tau}_1(r_A, d_2, 0.5)^{-1}$	$\hat{\tau}_2(r_B, d_1, d_2, 0.5)^{-1}$	$\hat{\tau}_3(r_C, d_1, 0.5)^{-1}$
2	0.38	1.19	1.93	1.69	1.69	6.033995	6.032340	1.583250
3	0.30	0.47	0.82	0.95	1.22	8.252293	8.250563	5.056856
4	0.36	0.57	0.86	1.06	1.27	6.422064	6.414193	2.951337
5	0.45	0.72	1.10	1.44	1.56	4.668165	4.658875	2.192861
6	0.53	0.80	1.37	1.85	1.85	3.659414	3.658877	1.458647
7	0.54	0.88	1.81	2.40	2.46	3.581287	3.579828	1.309188
8	0.54	0.94	2.10	2.71	2.93	3.576704	3.570913	1.217929
9	0.54	1.50	2.54	2.93	2.93	3.576704	3.549946	2.090838
10	0.54	1.78	2.82	2.93	2.93	3.576704	3.561554	1.007469
11	0.54	2.00	3.31	2.93	2.93	3.576704	3.542126	0.953569
≥ 12	0.54	2.08	3.60	2.93	2.93	3.576704	3.545183	1.035471

$$\beta_0 < 1 - \begin{cases} \frac{\widehat{\tau_1}(r,d_2,\nu)}{\mathscr{L}} & \text{if } \frac{d_2}{\mathscr{L}} < |\gamma_0| \leq 1; \\ \frac{\widehat{\tau_2}(r,d_1,d_2,\nu)}{\mathscr{L}} & \text{if } \frac{d_1}{\mathscr{L}} < |\gamma_0| \leq \frac{d_2}{\mathscr{L}}; \\ \frac{\widehat{\tau_3}(r,d_1,\nu)}{\mathscr{L}} & \text{if } 0 < |\gamma_0| \leq \frac{d_1}{\mathscr{L}}, \end{cases}$$

in which

$$\widehat{\tau_1}(r, d_2, \nu) = \frac{a_1 + a_0\left(1 + \left(\frac{a_0 d_2}{a_1 r}\right)^2\right)^{-1} - r\left(\frac{a_0}{r} - \frac{a_0}{r+\nu} + \frac{a_1 r}{r^2+d_2^2} + M_1\right)}{\frac{a_0}{r} - \frac{a_0}{r+\nu} + \frac{a_1/r}{1+(d_2/r)^2} + M_1},$$

$$\widehat{\tau_2}(r, d_1, d_2, \nu) = \frac{a_1 + \frac{a_0}{1+\frac{d_1^2 a_0^2}{r^2 a_1^2}} + \frac{a_0}{1+\frac{d_2^2 a_0^2}{r^2 a_1^2}} + \sum_{k=2}^{n} a_k Q(k, d_2, r; a_0, a_1)}{r\sum_{k=0}^{n} \frac{a_k}{r^2+k^2 d_1^2} - \frac{a_0}{r+\nu} + M_2} - r,$$

$$\widehat{\tau_3}(r, d_1, \nu) = \frac{-\left(M_3 r^2 + r\left(\frac{\nu}{r+\nu} - 1\right)\right) + \sqrt{r^2 - d_1^2\left(M_3 r + \frac{\nu}{r+\nu}\right)^2}}{M_3 r + \frac{\nu}{r+\nu}}.$$

Again, we need to choose the parameters r, d_1, and d_2 such that the largest value of $\widehat{\tau_1}(r, d_2)^{-1}$, $\widehat{\tau_2}(r, d_1, d_2)^{-1}$, and $\widehat{\tau_3}(r, d_1)^{-1}$ is minimised.

For the optimisation carried out in Section 4.4, we observed that the best $\tau_1(r, d_2)^{-1}$ was greater than the optimal $\max\{\tau_2(r, d_1, d_2)^{-1}, \tau_3(r, d_1)^{-1}\}$ values, when computing with the same d_2. Thus, we optimised $\tau_1(r, d_2)^{-1}$ first and used the d_2 obtained in the process to compute optimal $\max\{\tau_2(r, d_1, d_2)^{-1}, \tau_3(r, d_1)^{-1}\}$.

Here, the situation is different. We notice that the d_2 obtained from the best $\widehat{\tau_1}(r, d_2)^{-1}$ produces values of $\max\{\widehat{\tau_2}(r, d_1, d_2, \nu)^{-1}, \widehat{\tau_3}(r, d_1^{-1}, \nu)\}$ which are greater than $\widehat{\tau_1}(r, d_2)^{-1}$. Therefore, we adopt a different strategy to address this case. We optimise the parameters r, d_1, and d_2 together such that $\max\{\widehat{\tau_1}(r, d_2, \nu)^{-1}, \widehat{\tau_2}(r, d_1, d_2, \nu)^{-1}, \widehat{\tau_3}(r, d_1, \nu)^{-1}\}$ is as small as possible. The case $\nu = 0.5$ is given in Table 4 and the case $\nu = 0.05$ is given in Table 5. We remark that in all these computations, the degree 4 polynomials $p_{4,2}$ (from [6, Sec. 3.2]) and $p_{4,3}$ (from [6, Sec. 3.3]) produced the best values. In addition, as may be noticed, most calculations are optimised with $d_1 = d_2$, which means Case 3 does not occur, and Cases 2&4 give the entire region $(0, 1]$. Finally, we note that Theorem 1.3 is a natural consequence of the resulting computations, except for the case of zeros on the real line, which will be handled in Section 5.2.

5. Case 5: bounds for the real zero

Bring forward all of the notations and set-up from Section 2. In this section, we will complete the proof of Theorems 1.2-1.5. In particular, results from Section 5.1 complete proofs of Theorems 1.2 and 1.4, and results from Section 5.2 complete proofs of Theorems 1.3 and 1.5.

Table 5
Optimised parameter choices and computations for $\hat{\tau}_1, \hat{\tau}_2$ and $\hat{\tau}_3$ with $\nu = 0.05$.

n_L	Poly.	r_A	r_B	r_C	d_1	d_2	$\hat{\tau}_1(r_A, d_2, 0.05)^{-1}$	$\hat{\tau}_2(r_B, d_1, d_2, 0.05)^{-1}$	$\hat{\tau}_3(r_C, d_1, 0.05)^{-1}$
2	$p_{4,2}$	0.54	0.54	1.93	2.14	2.22	2.574226	2.574109	1.135791
3	$p_{4,3}$	0.42	0.45	0.82	1.12	1.12	3.315911	3.311491	0.579590
4	$p_{4,3}$	0.49	0.50	0.86	1.16	1.16	2.851208	2.847171	0.456349
5	$p_{4,3}$	0.61	0.61	1.10	1.41	1.41	2.312215	2.310168	0.349058
6	$p_{4,3}$	0.71	0.75	1.37	1.61	1.61	1.960091	1.954528	0.282884
7	$p_{4,2}$	0.75	0.76	1.81	2.30	2.30	1.853457	1.852388	0.273632
8	$p_{4,2}$	0.76	0.88	2.10	2.75	2.75	1.829453	1.829104	0.284979
9	$p_{4,3}$	0.77	0.78	2.54	3.52	3.89	1.806402	1.801663	0.931655
10	$p_{4,2}$	0.77	0.88	2.82	3.53	4.20	1.805190	1.799424	1.030697
11	$p_{4,2}$	0.77	1.03	3.23	3.55	4.20	1.805190	1.793811	1.801953
≥ 12	$p_{4,2}$	0.77	1.10	3.23	3.55	4.20	1.805190	1.803936	1.801953

5.1. Real zeros (unconditional)

To begin, suppose that $\sigma - 1 = \frac{r}{\mathscr{L}}$, $1 - \beta_0 = \frac{c}{\mathscr{L}}$, and $\zeta_L(s)$ admits two real zeros β_0 and β_1 such that $\beta_0 \leq \beta_1$. Similar to Section 4.3, by (4.25), Lemmas 2.4-2.5, (4.32) with $\varrho = \beta_0$ and $\varrho = \beta_1$, and the fact that $\frac{1}{\sigma-\beta_0} + \frac{1}{\sigma-\beta_1} \geq \frac{2}{\sigma-\beta_0}$, we derive

$$f_L(\sigma, 0) \leq -\frac{2}{\sigma - \beta_0} + \frac{1-\kappa}{2}\mathscr{L} + \frac{1}{\sigma - 1} + \alpha_2 + \Big(d(0) - \frac{(1-\kappa)\log \pi}{2}\Big)n_L. \tag{5.1}$$

Therefore, $f_L(\sigma, 0) \geq 0$ and $\mathscr{L} \geq \mathscr{L}_0 > 0$ imply

$$\begin{aligned} 0 &\leq -\frac{2}{\sigma - \beta_0} + \mathscr{L}\Big(\frac{1-\kappa}{2} + \frac{1}{r}\Big) + \alpha_2 + \Big(d(0) - \frac{(1-\kappa)\log\pi}{2}\Big)n_L \\ &= \mathscr{L}\Big(\frac{1-\kappa}{2} + \frac{1}{r} - \frac{2}{r+c}\Big) + \alpha_2 + \Big(d(0) - \frac{(1-\kappa)\log\pi}{2}\Big)n_L \\ &\leq \mathscr{L}\Big(\frac{1-\kappa}{2} + \frac{1}{r} - \frac{2}{r+c} + \frac{\mathcal{U}(D_{n_L})D_{n_L}}{\mathscr{L}_0}\Big), \end{aligned} \tag{5.2}$$

where $\mathcal{U}(x)$ was defined in (4.5) and

$$D_{n_0} = \alpha_2 + n_0\Big(d(0) - \frac{(1-\kappa)\log\pi}{2}\Big). \tag{5.3}$$

Rearranging (5.2), we see that

$$c \geq \eta(n_L, r) := \frac{2}{\frac{1-\kappa}{2} + \frac{1}{r} + \frac{\mathcal{U}(D_{n_L})D_{n_L}}{\mathscr{L}_0}} - r. \tag{5.4}$$

For each $n_L \in \{2, 3, 4, 5, 6, 7\}$, optimised choices for r and the resulting values of $\eta(n_L, r)$ are presented in Table 6. Similar to our earlier computations, we asserted $d_L \geq 400\,001$ when $n_L = 2$, $d_L \geq 240$ when $n_L = 3$, $d_L \geq 321$ when $n_L = 4$, and $d_L \geq d_{\min}(n_L)$ when $n_L > 4$. However, we have been careful to choose r such that $r \leq 0.15\mathscr{L}_0$, as most of our computations require $\sigma \leq 1.15$.

The lower bound in (5.4) remains constant for $n_L \geq 7$, so $n_L \geq 7$ implies $c \geq \eta(7, r)$. These computations yield Table 6, complete the proof of Theorem 1.2, and prove Theorem 1.4.

5.2. Real zeros (assuming an exceptional zero exists)

Assume that $\sigma - 1 = \frac{r}{\mathscr{L}}$, $1 - \beta_0 = \frac{c}{\mathscr{L}}$, and $\zeta_L(s)$ admits two real zeros β_0 and β_1 such that $\beta_0 \leq \beta_1$. It follows from (4.25), Lemmas 2.4- 2.5, and (4.32) with $\varrho = \beta_0$ and $\varrho = \beta_1$ that

Table 6
Optimised parameter choices and computations for $\eta(n_L, r)$.

n_L	r	$\eta(n_L, r)^{-1}$
2	1.49859	1.61094
3	0.822093	1.93173
4	0.865713	1.88178
5	1.10750	1.69958
6	1.37771	1.61857
≥ 7	1.49859	1.61094

Table 7
Optimised parameter choices and computations for $\widehat{\eta}(n_L, r, 0.5)$ and $\widehat{\eta}(n_L, r, 0.05)$.

n_L	r_1	$\hat{\eta}(n_L, r_1, 0.5)^{-1}$	r_2	$\hat{\eta}(n_L, r_2, 0.05)^{-1}$
2	1.48920	1.32086	0.949128	0.478632
3	0.822090	1.86631	0.822090	0.483802
4	0.865709	1.77210	0.865709	0.480747
5	1.10750	1.45550	0.949128	0.478632
6	1.37770	1.33079	0.949128	0.478632
≥ 7	1.48920	1.32086	0.949128	0.478632

$$f_L(\sigma, 0) \leq -\Big(\frac{1}{\sigma - \beta_1} + \frac{1}{\sigma - \beta_0}\Big) + \frac{1-\kappa}{2}\log d_L + \frac{1}{\sigma - 1} + \alpha_2 + \Big(-\frac{(1-\kappa)\log \pi}{2} + d(0)\Big)n_L.$$

Therefore, $f_L(\sigma, 0) \geq 0$ and $\mathscr{L} \geq \mathscr{L}_0 > 0$ imply

$$\begin{aligned} 0 &\leq -\Big(\frac{1}{\sigma - \beta_1} + \frac{1}{\sigma - \beta_0}\Big) + \mathscr{L}\Big(\frac{1-\kappa}{2} + \frac{1}{r}\Big) + \alpha_2 + \Big(d(0) - \frac{(1-\kappa)\log \pi}{2}\Big)n_L \\ &= \mathscr{L}\Big(\frac{1}{r} - \frac{1}{r+\nu} - \frac{1}{r+c} + \frac{1-\kappa}{2}\Big) + \alpha_2 + \Big(d(0) - \frac{(1-\kappa)\log \pi}{2}\Big)n_L \\ &\leq \mathscr{L}\Big(\frac{1-\kappa}{2} + \frac{1}{r} - \frac{1}{r+\nu} - \frac{1}{r+c} + \frac{\mathcal{U}(D_{n_L})D_{n_L}}{\mathscr{L}_0}\Big), \end{aligned} \tag{5.5}$$

where $\mathcal{U}(x)$ and D_{n_0} are defined as in (4.5) and (5.3), respectively. Rearranging (5.5) then yields

$$c \geq \widehat{\eta}(n_L, r, \nu) := \frac{1}{\frac{1-\kappa}{2} + \frac{1}{r} - \frac{1}{r+\nu} + \frac{\mathcal{U}(D_{n_L})D_{n_L}}{\mathscr{L}_0}} - r. \tag{5.6}$$

Finally, it remains to compute $\hat{\eta}(n_L, r, \nu)$. For $\nu = 0.5$ and 0.05, we complete the computations with the same optimisation strategy mentioned beneath (5.4); the results are given in Table 7. These computations complete the proof of Theorem 1.3 and prove Theorem 1.5.

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Appendix A. Tables

Table 8 provides the admissible values of $\mathfrak{d}(\delta, k)$ for Lemma 2.2. Tables 9, 10, and 11 contain computations toward Theorem 1.1.

Table 8
Values of $\mathfrak{d}(\delta, k)$.

k	$\delta = 0$	$\delta = 1$
1	-0.199351128738030570	-0.0813946816693186803
2	-0.00124241978210990096	0.0273794535972986669
3	0.114084432427194876	0.122300543665333397
4	0.193622937944706114	0.196875436840859980
5	0.254905404299203220	0.256570417484008384
6	0.304939896564586199	0.305953879673129092
7	0.347272949065725700	0.347963044467412053
8	0.383977593776409942	0.384482381069023826
9	0.416381018169095007	0.416768771337744870
10	0.445387629210662161	0.445696071325147825
11	0.471642652575186894	0.471894504563624251
12	0.495623005500675062	0.495832877025740915
13	0.517691340292122715	0.517869110940852861
14	0.538129924980734864	0.538282546011518703
15	0.557162798876846832	0.557295319763611574
16	0.574970772272345387	0.575086958395945369
17	0.591701881908478833	0.591804603327346634
18	0.607478862713224155	0.607570348191724730
19	0.622404603717508165	0.622486612036975084
20	0.636566207972667386	0.636640146614778746
21	0.650038064524389947	0.650105073828502023
22	0.662884207683184790	0.662945221743776680
23	0.675160153278594466	0.675215944805046542
24	0.686914345156188544	0.686965559046277541
25	0.698189307163083250	0.698236485988400823
26	0.709022569768260946	0.709066173368575803
27	0.719447422233358336	0.719487842956925361
28	0.729493528314488215	0.729531102997373782
29	0.739187434165484158	0.739222453638918475
30	0.748552990323224754	0.748585707026276914
31	0.757611704643699757	0.757642338768938384
32	0.766383039316073544	0.766411783808659042
33	0.774884662259901380	0.774911686912753805
34	0.783132661061269442	0.783158115891840700
35	0.79114172595177534	0.791165744003572424
36	0.798925307054014988	0.798948006734258587
37	0.806495750117107169	0.806517237157871714
38	0.813864414178669282	0.813884783290518610
39	0.821041773965830557	0.821061110238844649
40	0.828037509350176948	0.828055889446331683
41	0.834860583772007780	0.834878076944323810
42	0.841519313226652188	0.841535982193814758
43	0.848021427143618833	0.848037328843449179
44	0.854374122275479042	0.854389308516414592
45	0.860584110537916169	0.860598628564266499
46	0.866657661597714180	0.866671554581723291

Table 9
Admissible computations for C_1, C_2, C_3, C_4 upon choosing the polynomial p_{16} such that (1.2) holds with $T = 1$.

n_0	ε	C_1	C_2	C_3	C_4
3	0.1295	12.24107	9.53466	-11.79351	4.7255
4	0.0735	12.24107	9.53466	-12.06914	3.60145
5	0.0311	12.24107	9.53466	-12.18761	2.7179
6	0.0240	12.24107	9.53466	-12.20789	2.56754
7	0.0154	12.24107	9.53466	-12.23261	2.38456
8	0.0131	12.24107	9.53466	-12.23926	2.33546
9	0.0101	12.24107	9.53466	-12.24795	2.27133
10	0.0091	12.24107	9.53466	-12.25085	2.24992
11	0.0073	12.24107	9.53466	-12.25608	2.21137
12	0.0067	12.24107	9.53466	-12.25782	2.19851
13	0.0057	12.24107	9.53466	-12.26073	2.17707
14	0.0053	12.24107	9.53466	-12.26190	2.16848
15	0.0046	12.24107	9.53466	-12.26394	2.15346
16	0.0044	12.24107	9.53466	-12.26452	2.14917
17	0.0039	12.24107	9.53466	-12.26598	2.13844
18	0.0037	12.24107	9.53466	-12.26656	2.13414
19	0.0033	12.24107	9.53466	-12.26773	2.12555
20	0.0032	12.24107	9.53466	-12.26802	2.1234
21	0.0030	12.24107	9.53466	-12.2686	2.11911

Table 10
Admissible computations for C_1, C_2, C_3, C_4 upon choosing the polynomial p_{40} such that (1.2) holds with $T = 1$.

n_0	ε	C_1	C_2	C_3	C_4
3	0.1250	12.21608	9.53979	-11.63414	4.60646
4	0.0708	12.21608	9.53979	-11.89009	3.52232
5	0.0305	12.21608	9.53979	-12.0016	2.68709
6	0.0235	12.21608	9.53979	-12.02138	2.53978
7	0.0152	12.21608	9.53979	-12.04498	2.3643
8	0.0130	12.21608	9.53979	-12.05127	2.31765
9	0.0100	12.21608	9.53979	-12.05986	2.25394
10	0.0090	12.21608	9.53979	-12.06273	2.23267
11	0.0072	12.21608	9.53979	-12.0679	2.19437
12	0.0067	12.21608	9.53979	-12.06934	2.18373
13	0.0056	12.21608	9.53979	-12.07251	2.16029
14	0.0053	12.21608	9.53979	-12.07337	2.1539
15	0.0046	12.21608	9.53979	-12.07539	2.13898
16	0.0043	12.21608	9.53979	-12.07625	2.13258
17	0.0039	12.21608	9.53979	-12.07741	2.12405
18	0.0037	12.21608	9.53979	-12.07798	2.11979
19	0.0033	12.21608	9.53979	-12.07914	2.11125
20	0.0032	12.21608	9.53979	-12.07943	2.10912
21	0.0030	12.21608	9.53979	-12.0800	2.10485

Table 11
Admissible computations for C_1, C_2, C_3, C_4 upon choosing the polynomial p_{46} such that (1.2) holds with $T = 1$.

n_0	ε	C_1	C_2	C_3	C_4
3	0.1239	12.21124	9.54177	-11.59548	4.57803
4	0.0701	12.21124	9.54177	-11.84681	3.50267
5	0.0303	12.21124	9.54177	-11.9567	2.67879
6	0.0234	12.21124	9.54177	-11.97615	2.53379
7	0.0151	12.21124	9.54177	-11.9997	2.35857
8	0.0129	12.21124	9.54177	-12.00597	2.31198
9	0.0099	12.21124	9.54177	-12.01454	2.24836
10	0.0089	12.21124	9.54177	-12.0174	2.22713
11	0.0072	12.21124	9.54177	-12.02227	2.19101
12	0.0067	12.21124	9.54177	-12.02371	2.18038
13	0.0056	12.21124	9.54177	-12.02686	2.15698
14	0.0053	12.21124	9.54177	-12.02773	2.1506
15	0.0046	12.21124	9.54177	-12.02974	2.1357
16	0.0043	12.21124	9.54177	-12.0306	2.12931
17	0.0038	12.21124	9.54177	-12.03204	2.11866
18	0.0037	12.21124	9.54177	-12.03233	2.11653
19	0.0033	12.21124	9.54177	-12.03348	2.10801
20	0.0032	12.21124	9.54177	-12.03377	2.10588
21	0.0030	12.21124	9.54177	-12.03434	2.10162

Table 12
Table of coefficients for Mossinghoff–Trudgian–Yang's polynomial $p_{40}(\varphi) \in P_{40}$.

k	a_k	k	a_k
0	1	21	$4.66702819061453 \cdot 10^{-7}$
1	1.74600190914994	22	$8.88183754657211 \cdot 10^{-7}$
2	1.14055431833244	23	$6.61799442215331 \cdot 10^{-5}$
3	0.518966962914028	24	$3.70153227317542 \cdot 10^{-5}$
4	0.130885859164882	25	$6.2332255794641 \cdot 10^{-8}$
5	$8.86418531143308 \cdot 10^{-8}$	26	$3.29243016002061 \cdot 10^{-5}$
6	$1.79787121328335 \cdot 10^{-6}$	27	$4.89938220699415 \cdot 10^{-5}$
7	0.0137716529944408	28	$1.50988491954013 \cdot 10^{-5}$
8	0.00825900683475376	29	$1.13051732969427 \cdot 10^{-7}$
9	$4.91544374578637 \cdot 10^{-6}$	30	$2.11823533257304 \cdot 10^{-5}$
10	$2.20263007866541 \cdot 10^{-6}$	31	$2.13859401551174 \cdot 10^{-5}$
11	0.00243120523137902	32	$1.55071932288034 \cdot 10^{-6}$
12	0.00172926530269636	33	$1.51812185041036 \cdot 10^{-6}$
13	$1.35500078722447 \cdot 10^{-6}$	34	$1.67615806595912 \cdot 10^{-5}$
14	$2.20879127662495 \cdot 10^{-6}$	35	$1.60031224178442 \cdot 10^{-5}$
15	0.00069712400164774	36	$3.94634065729451 \cdot 10^{-6}$
16	0.000530583559753362	37	$4.08859029078879 \cdot 10^{-7}$
17	$6.3973072524226 \cdot 10^{-7}$	38	$1.77819241241605 \cdot 10^{-6}$
18	$5.37323136636712 \cdot 10^{-7}$	39	$5.06885733758335 \cdot 10^{-8}$
19	0.000234320877800568	40	$7.50406436813653 \cdot 10^{-9}$
20	0.000177364641910045		

Table 13
Table of coefficients for Mossinghoff–Trudgian–Yang's polynomial $p_{46}(\varphi) \in P_{46}$.

k	a_k	k	a_k
0	1	24	0.000127104592072581
1	1.74708744081848	25	$1.74058423843506 \cdot 10^{-7}$
2	1.14338015090023	26	$6.156980223188 \cdot 10^{-9}$
3	0.521864216745001	27	$7.4923012998548 \cdot 10^{-5}$
4	0.132187571762225	28	$6.29610657045172 \cdot 10^{-5}$
5	$1.44250682908725 \cdot 10^{-7}$	29	$4.51492091998615 \cdot 10^{-7}$
6	$4.69075278525482 \cdot 10^{-9}$	30	$1.76696516341167 \cdot 10^{-8}$
7	0.0141904926848435	31	$3.57616762286565 \cdot 10^{-5}$
8	0.00859097729886965	32	$2.9356535048273 \cdot 10^{-5}$
9	$5.05758761820625 \cdot 10^{-7}$	33	$2.6547976338407 \cdot 10^{-7}$
10	$4.42284301054098 \cdot 10^{-10}$	34	$7.39578841754684 \cdot 10^{-7}$
11	0.00262452919575262	35	$1.5703528751761 \cdot 10^{-5}$
12	0.0018969952017721	36	$1.16349907747152 \cdot 10^{-5}$
13	$4.69472495111911 \cdot 10^{-10}$	37	$1.01423339047177 \cdot 10^{-7}$
14	$2.18058618368512 \cdot 10^{-7}$	38	$1.71248131672039 \cdot 10^{-6}$
15	0.000818384876659817	39	$7.84636117271159 \cdot 10^{-6}$
16	0.000639651965532567	40	$5.93829512034697 \cdot 10^{-6}$
17	$3.11262094946825 \cdot 10^{-8}$	41	$9.47232309558493 \cdot 10^{-7}$
18	$7.74994211145798 \cdot 10^{-7}$	42	$4.84440446543232 \cdot 10^{-8}$
19	0.000329183630974004	43	$9.72548049252508 \cdot 10^{-7}$
20	0.000268358318561904	44	$8.45180184576162 \cdot 10^{-7}$
21	$4.43747297378809 \cdot 10^{-7}$	45	$2.25111200007826 \cdot 10^{-7}$
22	$1.87358718910571 \cdot 10^{-7}$	46	$6.56678999833493 \cdot 10^{-10}$
23	0.000151428354073652		

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