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The eighth moment of the Riemann zeta function

In memory of Aleksandar Ivić

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Abstract. We establish an asymptotic formula for the eighth moment of the Riemann zeta function, assuming the Riemann hypothesis and a quaternary additive divisor conjecture. This builds on the work of the first author on the sixth moment of the Riemann zeta function and the works of Conrey–Gonek and Ivić. A key input is a sharp bound for a certain shifted moment of the Riemann zeta function, assuming the Riemann hypothesis.

Keywords: moments of the Riemann zeta function, additive divisor sums.

1. Introduction

This article concerns the eighth moment $I_4(T)$ of the Riemann zeta function $\zeta(s)$, where

$$I_k(T) = \int_0^T \left|\zeta\left(\tfrac{1}{2} + \mathrm{i}t\right)\right|^{2k} dt \tag{1.1}$$

denotes the $2k$-th moment of the Riemann zeta function. There is a long and extensive history of research on the moments (1.1). Fundamental results may be found in the books [21, 22, 24, 35]. Regarding its size, it was proven by Heath-Brown [17] and Ramachandra [30, 31] that under the Riemann hypothesis, for any real $k \geq 0$,

$$I_k(T) \gg T(\log T)^{k^2}. \tag{1.2}$$

Recent work of Radziwiłł and Soundararajan [29] and Heap and Soundararajan [15] shows that (1.2) holds unconditionally for $k > 0$.

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Under the Riemann hypothesis, Harper [14] established that

$$I_k(T) \ll T(\log T)^{k^2}. \tag{1.3}$$

This improved an earlier result of Soundararajan [33] in which the right of (1.3) was larger by a factor of $(\log T)^{\varepsilon}$. Unconditionally (see [22, Theorem 8.3]), it is known that

$$I_k(T) \ll T^{M+\varepsilon} \quad \text{with} \quad M \le \begin{cases} 1+\frac{k-2}{4} & \text{if } 2 \le k \le 6, \\ 2+\frac{3(k-6)}{11} & \text{if } 6 \le k \le \frac{178}{26}, \\ 1+\frac{35(k-3)}{108} & \text{if } k \ge \frac{178}{26}. \end{cases}$$

Keating and Snaith [23], using a random matrix model, conjectured that for $k \in \mathbb{N}$,

$$I_k(T) \sim \frac{g_k a_k}{(k^2)!} T(\log T)^{k^2} \tag{1.4}$$

as $T \to \infty$, where

$$g_k = k^2! \prod_{j=0}^{k-1} \frac{j!}{(k+j)!} \quad \text{and} \quad a_k = \prod_p \left(1-\frac{1}{p}\right)^{k^2} \sum_{m=0}^{\infty} \binom{k+m-1}{m}^2 p^{-m}. \tag{1.5}$$

In 2005, Conrey et al. [5], using a heuristic argument with the approximate functional equation, conjectured that for $k \in \mathbb{N}$,

$$I_k(T) = T\mathscr{P}_{k^2}(\log T) + o(T), \tag{1.6}$$

where $\mathscr{P}_{k^2}$ is a certain polynomial of degree k^2. In 1918, Hardy and Littlewood [13] established the asymptotic (1.4) for $k = 1$, and in 1926, Ingham [19] established the case $k = 2$. The asymptotic for $k = 3$ was first conjectured by Conrey and Ghosh [6]. Conrey and Gonek [7] provided a heuristic argument, which suggests that (1.4) holds for $k = 3, 4$. Their work is based on conjectural asymptotics for additive divisor sums. Recently, building on the works of Conrey–Gonek [7] and Hughes–Young [18], the first author [26] showed that a certain conjecture for smoothed ternary additive divisor sums implies that (1.4) and (1.6) are true in the case $k = 3$. In this article, we extend the ideas in [26] to the case $k = 4$.

A general approach in evaluating (1.1) is to write $|\zeta(\frac{1}{2}+it)|^{2k} = \zeta(\frac{1}{2}+it)^k \zeta(\frac{1}{2}-it)^k$. This leads naturally to the Dirichlet series

$$\zeta(s)^k = \sum_{n=1}^{\infty} \tau_k(n) n^{-s}$$

where the k-th divisor function τ_k is given precisely by

$$\tau_k(n) = \#\{(d_1,\ldots,d_k) \in \mathbb{N}^k \mid d_1 \cdots d_k = n\} \quad \text{for } n \in \mathbb{N}.$$

The moments $I_k(T)$, for $k \in \mathbb{N}$, are intimately related to the additive divisor sums

$$D_k(x;r) = \sum_{n \le x} \tau_k(n)\tau_k(n+r) \tag{1.7}$$

for $x > 0$. In [19], Ingham required an upper bound for $D_2(x;r)$ in order to establish the asymptotic (1.4) for $k = 2$. Heath-Brown [16] developed this further and established (1.6) for $k = 2$ by using a more precise asymptotic formula for $D_2(x;r)$. Deshouillers and Iwaniec [8], and then Motohashi [24], improved the error term in (1.7), for $k = 2$, by making use of the spectral theory of automorphic forms (Kuznetsov's formula). For $k > 2$, Conrey and Gonek [7] and Ivić [22] studied the relationship between $D_k(x;r)$ and asymptotics and bounds for $I_k(T)$. In practice it is more convenient to consider more general sums of the shape

$$D_{f;k,\ell}(r) = \sum_{m-n=r} \tau_k(m)\tau_\ell(n) f(m,n), \tag{1.8}$$

where $k, \ell \in \mathbb{N}$, $r \in \mathbb{Z} \setminus \{0\}$, and f is an arbitrary function of two variables. Observe that the sum in (1.7) is a special case of (1.8). In our work, we assume f is a smooth function that satisfies conditions (1.9) and (1.10) below. Duke, Friedlander, and Iwaniec [9] introduced these sums when $k = \ell = 2$ (with the more general summation condition $am - bn = r$) to study the subconvexity problem for GL_2. The sums (1.8) provide much greater flexibility and are more useful in moment problems than the classical unsmoothed sums (1.7). This is due to the presence of the smooth function f which allows for the application of the Poisson and Voronoi summation formulae.

The function f in (1.8) satisfies the following properties. There exist positive X, Y, and P such that

$$\mathrm{supp}(f) \subset [X, 2X] \times [Y, 2Y], \tag{1.9}$$

and the partial derivatives of f satisfy the growth conditions

$$x^m y^n f^{(m,n)}(x,y) \ll_{m,n} P^{m+n}. \tag{1.10}$$

We now describe a conjectural formula for the sums $D_{f;k,\ell}(r)$. We first need to introduce an arithmetic function that appears in the conjecture.

Definition 1. Let $k \in \mathbb{N}$. The multiplicative function $n \mapsto g_k(s,n)$ is given by

$$g_k(s,n) = \prod_{p^\alpha || n} \frac{\sum_{j=0}^{\infty} \frac{\tau_k(p^{j+\alpha})}{p^{js}}}{\sum_{j=0}^{\infty} \frac{\tau_k(p^j)}{p^{js}}}.$$

In other words, for $n \in \mathbb{N}$, we have $\sum_{m=1}^{\infty} \frac{\tau_k(nm)}{m^s} = g_k(s,n)\zeta(s)^k$.

The multiplicative function $n \mapsto G_k(s,n)$ is given by

$$G_k(s,n) = \sum_{d|n} \frac{\mu(d) d^s}{\phi(d)} \sum_{e|d} \frac{\mu(e)}{e^s} g_k\left(s, \frac{ne}{d}\right). \tag{1.11}$$

It can be shown that for $s \approx 1$, $G_k(s, p^j) \approx \tau_k(p^j) - p^{s-1}\tau_k(p^{j-1})$ (see Lemma 2.3 below and [25, Lemma 5.4, p. 521] for precise statements).

Conjecture 1 (Quaternary additive divisor conjecture). *There exists $C > 0$ for which the following holds. Let ε_0 and ε' be arbitrarily small positive constants. Let $P > 1$, and let $X, Y > 1/2$ satisfy $Y \asymp X$. Let f be a smooth function satisfying* (1.9) *and* (1.10)*. Then, if X is sufficiently large* (*in absolute terms*)*, one has*

$$D_{f;4,4}(r) = \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathcal{B}_1} \int_{\mathcal{B}_2} \zeta(z_1)^4 \zeta(z_2)^4 \sum_{q=1}^{\infty} \frac{c_q(r) G_4(z_1, q) G_4(z_2, q)}{q^{z_1+z_2}} \times \int_0^{\infty} f(x, x-r) x^{z_1-1} (x-r)^{z_2-1} \, dx \, dz_2 \, dz_1 + O(P^C X^{1/2+\varepsilon_0}), \tag{1.12}$$

uniformly for $1 \le |r| \ll X^{1-\varepsilon'}$, where for $i = 1, 2$, $\mathcal{B}_i = \{z_i \in \mathbb{C} \mid |z_i - 1| = r_i\} \subset \mathbb{C}$ are circles, centred at 1, of radii $r_i \in (\frac{1}{100}, \frac{1}{10})$, and $c_q(r) = \sum^*_{d \ (\mathrm{mod}\, q)} e(-dr/q)$ is the Ramanujan sum.

This form of the additive divisor conjecture is worked out in Appendix 1 (Section 9) below; see also [7, pp. 589–591]. The derivation is based on Duke, Friedlander, and Iwaniec's δ-method (a form of the circle method). The Ramanujan sums $c_q(r)$ appear in the detection of the additive condition $m = n + r$. The size of the error term is expected based on an analogy to Aryan's work [1] in the case $k = \ell = 2$ and to Blomer's work [2] on shifted convolutions of modular forms. Furthermore, numerical work of Nguyen [28] also suggests this is the correct size.

We now state our main result.

Theorem 1.1. *Assume the Riemann hypothesis and Conjecture 1* (*the quaternary additive divisor conjecture*) *are true. Then*

$$I_4(T) \sim \frac{24024 a_4}{16!} T (\log T)^{16} \tag{1.13}$$

as $T \to \infty$, where a_4 is defined in (1.5)*.*

Our proof of Theorem 1.1 relies in a fundamental way on the following result on shifted moments of the Riemann zeta function established by the authors in the companion article [27].

Theorem 1.2. *Let $k \ge 1/2$ and assume the Riemann hypothesis. For $T > 1$ and $t_0 \in \mathbb{R}$, define*

$$\mathcal{G}(T, t_0) = \begin{cases} \min\left(\frac{1}{|2t_0|}, \log T\right) & \text{if } |t_0| \le \frac{1}{200}, \\ \log(2 + |2t_0|) & \text{if } |t_0| > \frac{1}{200}. \end{cases}$$

Then for T sufficiently large and $|t_0| \le 0.5T$, we have[1]

$$\int_0^T \left|\zeta\left(\tfrac{1}{2} + \mathrm{i}t + \mathrm{i}t_0\right)\right|^k \left|\zeta\left(\tfrac{1}{2} + \mathrm{i}t - \mathrm{i}t_0\right)\right|^k dt \ll_k T (\log T)^{k^2/2} \mathcal{G}(T, t_0)^{k^2/2}. \tag{1.14}$$

[1] Note $\mathcal{G}(T, t_0) = \mathcal{F}(T, t_0, -t_0)$ where $\mathcal{F}(T, \alpha_1, \alpha_2)$ is defined in [27].

There are a number of key differences between the proof of Theorem 1.1 and that of the main theorem in [26], which deals with the sixth moment $I_3(T)$. These are outlined in the remarks below.

Remarks. (1) In our main theorem, we only "win by an ε" and no lower order terms in the main term are obtained. By contrast, in [26], the full main term with a power savings error term is obtained for $I_3(T)$ assuming the ternary additive divisor conjecture.

(2) In [20], Ivić showed that an averaged form of the quaternary additive divisor conjecture implies that $I_4(T) \ll T^{1+\varepsilon}$ for any $\varepsilon > 0$ (see [20, Theorem 1, Corollaries 1, 2]). Conrey and Gonek [7] provided a heuristic argument which shows that the unsmoothed quaternary additive divisor conjecture implies $I_4(T) \sim \frac{24024a_4}{16!} T(\log T)^{16}$. (Numerous error terms are not bounded and no smoothing functions are used in their argument.) Note that we can establish Theorem 1 assuming an averaged version of the quaternary additive divisor conjecture similar to the arguments in [11,20].

(3) The approximate functional equation for $|\zeta(\frac{1}{2} + it)|^8$ expresses this in terms of a product of two Dirichlet polynomials, each of length essentially T^2. We prove a technical lemma that reduces the length of the Dirichlet polynomials involved by a factor of T^{ε} for an arbitrarily small ε. This allows us to work with a polynomial of length $T^{2-\varepsilon}$ and the key point is that square root cancellation in the additive divisor conjecture then leads to an acceptable error of size $O((T^{2-\varepsilon})^{1/2}) = O(T^{1-0.5\varepsilon})$. In order to make this work, we need a bound on shifted moments of the Riemann zeta function of the type in (1.14). (The upper bound method of Soundararajan [33] as applied in [3] does not allow us to obtain the asymptotic (1.13).) This idea of reducing the length of a Dirichlet polynomial was used first in an article of Soundararajan and Young [34] and later in work of Chandee and Li [4] and Shen [32]. In these articles, the length of the Dirichlet polynomial is reduced by a factor of $(\log T)^A$ for an arbitrarily large A. By contrast, we reduce the length by a factor of T^{ε} for an arbitrarily small ε. This is one of the key new inputs in this paper.

(4) It seems likely that the methods of this article would allow one to prove that the quaternary additive divisor conjecture implies

$$I_4(T) \geq \frac{24024a_4}{16!} T(\log T)^{16}(1 + o(1)).$$

The proof of such a result would follow ideas from [34, Theorem 1.1] and also [32, Theorem 1.2], where sharp lower bounds for the second moment of quadratic twists of a cusp form and for the fourth moment of quadratic Dirichlet L-functions are obtained unconditionally.

(5) Shifted moments of the Riemann zeta function were studied by Chandee, and she formulated a conjecture regarding them in [3, Conjecture 1]. In a companion article [27], assuming the Riemann hypothesis, we prove Chandee's conjecture for two shifts. Theorem 1.2 above is based on important ideas of Harper [14]. We show that his method still works with a shifting parameter t_0. Note that Theorem 1.2 includes Harper's theorem (the bound (1.3)) as a special case.

(6) The other main difference with [26] is that we deal directly with the eighth moment instead of the shifted eighth moment. The reason for this is that we encountered a technical

difficulty when we tried to generalize the argument of [26] from six to eight shifts. We hope to revisit this in the future. The approach using shifted moments can be advantageous since the main term arises from many polar terms which arise from simple poles of the Dirichlet series under consideration. By [5, Lemma 2.5.1] these polar terms can be shown to cancel. When no shifts are introduced, the main term arises from many multiple poles of the Dirichlet series. In this case, the computation of the multiple residues makes the calculation very involved and it is convenient to use Maple in the computation.

Conventions and notation. Given two functions $f(x)$ and $g(x)$, we shall interchangeably use the notation $f(x) = O(g(x))$, $f(x) \ll g(x)$, and $g(x) \gg f(x)$ to mean there exists $M > 0$ such that $|f(x)| \le M|g(x)|$ for all sufficiently large x. We write $f(x) \asymp g(x)$ to mean that the estimates $f(x) \ll g(x)$ and $g(x) \ll f(x)$ simultaneously hold. If we write $f(x) = O_{a_1,\ldots,a_\ell}(g(x))$, $f(x) \ll_{a_1,\ldots,a_\ell} g(x)$, or $f(x) \asymp_{a_1,\ldots,a_\ell} g(x)$ for real numbers $a_1, \ldots, a_\ell$, then we mean that the corresponding implied constants depend on $a_1, \ldots, a_\ell$. In addition, B shall denote a positive constant, which may be taken arbitrarily large and which may change from line to line. The letter p will always be used to denote a prime number. For a function $\varphi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{C}$, $\varphi^{(m,n)}(x, y) = \frac{\partial^m}{\partial x^m} \frac{\partial^n}{\partial y^n} \varphi(x, y)$. The integral notation $\int_{(c)} f(s)\, ds$ for a complex function $f(s)$ and $c \in \mathbb{R}$ will be used frequently and is defined by the contour integral

$$\int_{(c)} f(s)\, ds = \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} f(s)\, ds.$$

In this article, we shall consider $s \in \mathbb{C}$ and usually write its real part as $\sigma = \Re e(s)$. We will often use the fact that $\omega(t)$ has support in $[c_1 T, c_2 T]$ so that $t \asymp T$. In addition, we use the notation i to denote the imaginary number which satisfies $\mathrm{i}^2 = -1$, and i shall be used as an integer variable.

2. Preliminary lemmas

The following proposition is a straightforward generalization of [18, Proposition 2.1, p. 209] and [26], which gives approximate functional equations for $|\zeta(\frac{1}{2} + \mathrm{i}t)|^k$ with $k = 4, 6$ (see also [16, Lemma 1]).

Proposition 2.1. *Let $G(s)$ be an even entire function, with $G(0) = 1$, satisfying*

$$|G(s)| \ll \exp(-c|t|) \tag{2.1}$$

for some (explicit) c and $\Re e(s)$ in an appropriate interval.[2] *Define*

$$V_t(x) = \frac{1}{2\pi \mathrm{i}} \int_{(1)} \frac{G(s)}{s} g(s, t) x^{-s}\, ds,$$

[2] For instance, one may choose $G(s) = \exp(s^2)$.

where

$$g(s,t) = \left(\frac{\Gamma\big(\frac{1}{2}(\frac{1}{2}+s+\mathrm{i}t)\big)}{\Gamma\big(\frac{1}{2}(\frac{1}{2}+\mathrm{i}t)\big)} \frac{\Gamma\big(\frac{1}{2}(\frac{1}{2}+s-\mathrm{i}t)\big)}{\Gamma\big(\frac{1}{2}(\frac{1}{2}-\mathrm{i}t)\big)} \right)^4.$$

Then for any constant $B > 0$, we have

$$\left|\zeta\big(\tfrac{1}{2}+\mathrm{i}t\big)\right|^8 = 2\sum_{m,n=1}^{\infty} \frac{\tau_4(m)\tau_4(n)}{(mn)^{1/2}}\left(\frac{m}{n}\right)^{-\mathrm{i}t} V_t(\pi^4 mn) + O((1+|t|)^{-B}). \tag{2.2}$$

For $t \asymp T$, one can see that the terms (m,n) that contribute to this sum satisfy $mn \ll T^{4+\varepsilon}$. In this article we shall use various facts about $g(s,t)$. In addition, we shall encounter the function $\widetilde{V}_t(x)$ defined by

$$\widetilde{V}_t(x) = \frac{1}{2\pi\mathrm{i}}\int_{(1)} \frac{G(s)}{s} g(s,t)\left(\frac{U}{t}\right)^{4s} x^{-s}\,ds, \tag{2.3}$$

where

$$U = T^{1-\varepsilon}. \tag{2.4}$$

We have the following lemma providing bounds for $g(s,t)$, its partial derivatives, and $\widetilde{V}_t(x)$.

Lemma 2.2. *Let A be a positive constant, and assume $|t| \gg 1$.*

(i) *For $0 \le \Re\mathrm{e}(s) \le A$, we have*

$$g(s,t) = \left(\frac{t}{2}\right)^{4s}\left(1 + O\left(\frac{|s|^2+1}{t}\right)\right).$$

(ii) *Let $\varepsilon_0 > 0$ and $i \ge 0$. For $\Re\mathrm{e}(s) = \varepsilon_0$ and $|\Im\mathrm{m}(s)| \le \sqrt{T}$, and $c_1 T \le t \le c_2 T$ with $0 < c_1 < c_2$,*

$$\frac{d^i}{dt^i} g(s,t) \ll_{i,\varepsilon_0} |s|^i T^{4\varepsilon_0 - i}. \tag{2.5}$$

(iii) *For $x > U^4$, $\widetilde{V}_t(x) = O((U^4/x)^A)$.*

Proof. As the proofs of Lemma 2.2 (i, ii) are very similar to those of [26, Lemma 2.2 (ii, iii)], we shall omit them. Now, we prove Lemma 2.2 (iii). Moving the line of integration in $\widetilde{V}_t(x)$ to $\Re\mathrm{e}(s) = A$, we see that $\widetilde{V}_t(x)$ becomes

$$\frac{1}{2\pi\mathrm{i}}\int_{(A)} \frac{G(s)}{s} g(s,t)\left(\frac{U}{t}\right)^{4s} x^{-s}\,ds \ll \int_{A-\mathrm{i}\infty}^{A+\mathrm{i}\infty} \frac{|G(s)|}{|s|}\left(\frac{U^4}{16x}\right)^A \left(1+\frac{|s|^2+1}{t}\right)|ds|$$
$$\ll \left(\frac{U^4}{x}\right)^A$$

as desired. ∎

Last but not least, to use Conjecture 1, it is crucial to know the behaviour of $G_k(s,n)$ defined in (1.11). We shall therefore prove the following lemma.

Lemma 2.3. *Let $z \in \mathbb{C}$ and $q \in \mathbb{N}$. Then*

$$|G_4(z,q)| \leq 2^{\omega(q)}\tau_4(q)\prod_{p|q}(1+p^{-\Re e(z)})^3(1+p^{\Re e(z)-1}),$$

where $\omega(q)$ is the number of distinct prime divisors of q. In addition, if $1-r \leq \Re e(z) \leq 1+r$ for some $r \in (0,1)$, then

$$|G_4(z,q)| \leq 32^{\omega(q)}\tau_4(q)q^r. \tag{2.6}$$

Moreover, for any prime p and $j \in \mathbb{N}$, we have

$$G_4(z,p^j) = \frac{p}{p-1}\big(\tau_4(p^j)\mathcal{Q}_j(p^{-z}) - p^{z-1}\tau_4(p^{j-1})\mathcal{Q}_{j-1}(p^{-z})\big), \tag{2.7}$$

where $\mathcal{Q}_0(x) = 1$ and

$$\mathcal{Q}_j(x) = 1 - \frac{j}{j+1}3x + \frac{j}{j+2}3x^2 - \frac{j}{j+3}x^3. \tag{2.8}$$

For $j = 1,2$, (2.7) and (2.8) give

$$G_4(z,p) = \frac{p}{p-1}(4 - p^{z-1} - 6p^{-z} + 4p^{-2z} - p^{-3z}) \tag{2.9}$$

and

$$G_4(z,p^2) = \frac{p}{p-1}\big((10-4p^{z-1}) + (-20+6p^{z-1})p^{-z} + (15-4p^{z-1})p^{-2z} + (-4+p^{z-1})p^{-3z}\big). \tag{2.10}$$

Proof. We begin with some facts about $g_4(z,p^j)$. In [26], it was proven that

$$g_4(z,p^j) = \tau_4(p^j)\mathcal{Q}_j(p^{-z}), \tag{2.11}$$

where $\mathcal{Q}_j$ is defined in (2.8). Furthermore, for $j \geq 1$, we have the identity

$$\mathcal{Q}_j(x) = jx^{-j}\int_0^x t^{j-1}(1-t)^3\,dt.$$

Performing the integration gives (2.8), so $\mathcal{Q}_j(x)$ is a degree-3 polynomial such that $\mathcal{Q}_j(0) = 1$. By the triangle inequality, we have

$$|\mathcal{Q}_j(x)| \leq 1 + 3|x| + 3|x|^2 + |x|^3 = (1+|x|)^3.$$

We now compute $G_4(z,p^j)$. By definition (1.11), we know

$$\begin{aligned} G_4(z,p^j) &= \sum_{d|p^j}\frac{\mu(d)d^z}{\phi(d)}\sum_{e|d}\frac{\mu(e)}{e^z}g_4\left(z,\frac{p^je}{d}\right) \\ &= g_4(z,p^j) - \frac{p^z}{p-1}g_4(z,p^{j-1}) + \frac{1}{\phi(p)}g_4(z,p^j) \\ &= \frac{p}{p-1}g_4(z,p^j) - \frac{p^z}{p-1}g_4(z,p^{j-1}). \end{aligned}$$

Inserting (2.11) in the last expression, we establish (2.7). Hence for $x = \Re e(z)$,

$$|G_4(z,p^j)| \le \frac{p}{p-1}\tau_4(p^j)\left((1+p^{-x})^3 + p^{x-1}\frac{\tau_4(p^{j-1})}{\tau_4(p^j)}(1+p^{-x})^3\right)$$
$$= \frac{p}{p-1}\tau_4(p^j)\left((1+p^{-x})^3 + p^{x-1}\frac{j}{j+3}(1+p^{-x})^3\right)$$
$$\le \frac{p}{p-1}\tau_4(p^j)(1+p^{-x})^3(1+p^{x-1}).$$

By multiplicativity, it follows that

$$|G_4(z,q)| \le 2^{\omega(q)}\tau_4(q)\prod_{p|q}(1+p^{-\Re e(z)})^3(1+p^{\Re e(z)-1}).$$

Note that if $1-r \le \Re e(z) \le 1+r$ for some $r \in (0, \frac{1}{10})$, then $|G_4(z,q)|$ is bounded by

$$2^{\omega(q)}\tau_4(q)\prod_{p|q}(1+p^{-\Re e(z)})^3(1+p^{\Re e(z)-1}) \le 2^{\omega(q)}\tau_4(q)(2^{\omega(q)})^3\prod_{p|q}(2p^r)$$
$$= 32^{\omega(q)}\tau_4(q)q^r,$$

as required. ∎

3. Truncating the sum

As there are many small parameters involved in our argument, for the convenience of the reader, we track them here.

- ε is used in the definition $U = T^{1-\varepsilon}$ in (2.4).
- ε_0 is an arbitrarily small parameter that differs from line to line. It appears in Conjecture 1, (2.5), Lemma 4.1, and the integral above (4.6).
- ε_1 appears in (6.1), the definition of f^*, where we integrate over $\Re e(s) = \varepsilon_1$. Furthermore, ε_1 is used in the condition $|\log(m/n)| \ll T_0^{-1+\varepsilon_1}$ (see (6.4)). In addition, we choose $P = T^{1+\varepsilon_1}T_0^{-1}$ (to apply Conjecture 1) in Section 6.2.
- ε_2 is introduced to cut the sum at $mn \ll U^{1+\varepsilon_2}$ in Section 6.1.
- ε_3 is for the line $\Re e(s) = \varepsilon_3$. In Section 6.3, we move the integral in $\tilde{I}_{M,N}$ from $\Re e(s) = \varepsilon_1$ to $\Re e(s) = \varepsilon_3$. In addition, we move the integral in (6.24) from $\Re e(s) = 1$ to $\Re e(s) = \varepsilon_3$, in Section 6.9. We assume $0 < \delta < \varepsilon_3 < 0.15$, where δ is a parameter described below.
- ε' is related to the range of r in Conjecture 1; more precisely, $|r| \le M^{1-\varepsilon'}$.
- η is a parameter used to define $\omega^+(t)$ and $\omega^-(t)$ in the proof of Theorem 1.1, which starts below (3.8).
- η_0 is a parameter introduced below (3.9) and (3.10) in the proof of Theorem 1.1 to define j.

- δ appears in Lemma 6.4 for the condition $|a_{i_1}|, |b_{i_2}| \le \delta$ and in the proof of Lemma 7.2 as the radius of the circle $C(0,\delta)$ centred at 0. We assume $0 < \delta < \frac{1}{10}$.
- δ' is used for $|u_i| < \delta'$ in Proposition 6.5.
- r_1, r_2 are the radii for $\mathcal{B}_1$, $\mathcal{B}_2$, respectively. We assume $\frac{1}{100} < r_1, r_2 \le \delta < \frac{1}{10}$ (cf. Conjecture 1).

We begin by evaluating the smoothed eighth moment

$$I_\omega := \int_{-\infty}^{\infty} \omega(t)\big|\zeta\big(\tfrac{1}{2}+\mathrm{i}t\big)\big|^8\,dt,$$

where $\omega : \mathbb{R} \to \mathbb{C}$ has the following three properties:

(i) ω is smooth;

(ii) we have

$$\operatorname{supp}\omega \subseteq [c_1T, c_2T] \tag{3.1}$$

where $c_1 < c_2$ are positive absolute constants;

(iii) there exists T_0 such that

$$T^{1/2} \le T_0 \ll T \quad \text{and} \quad \omega^{(j)}(t) \ll T_0^{-j} \quad \text{for all } j \in \mathbb{Z}^+ \text{ and } t \in \mathbb{R}. \tag{3.2}$$

From (2.2) with $B = 2$, it follows that

$$\begin{aligned}
I_\omega &= 2\int_{-\infty}^{\infty} \omega(t)\Bigg(\sum_{m,n=1}^{\infty} \frac{\tau_4(m)\tau_4(n)}{(mn)^{1/2}}\left(\frac{m}{n}\right)^{-it} V_t(\pi^4 mn)\Bigg)\,dt + O(T^{-1})\\
&= 2\int_{-\infty}^{\infty} \omega(t)\sum_{m,n=1}^{\infty} \frac{\tau_4(m)\tau_4(n)}{(mn)^{1/2}}\left(\frac{m}{n}\right)^{-it} \frac{1}{2\pi\mathrm{i}}\int_{(1)} \frac{G(s)}{s} g(s,t)t^{-4s}\left(\frac{t^4}{\pi^4 mn}\right)^{s} ds\,dt\\
&\quad + O(T^{-1}).
\end{aligned}$$

One problem with this expression is that the Dirichlet polynomials are too long and they need to be made slightly shorter. This will be done by introducing the expression

$$\begin{aligned}
\tilde{I}_\omega &:= \tilde{I}_\omega(U)\\
&= 2\int_{-\infty}^{\infty} \omega(t)\sum_{m,n=1}^{\infty} \frac{\tau_4(m)\tau_4(n)}{(mn)^{1/2}}\left(\frac{m}{n}\right)^{-it} \frac{1}{2\pi\mathrm{i}}\int_{(1)} \frac{G(s)}{s} g(s,t)t^{-4s}\left(\frac{U^4}{\pi^4 mn}\right)^{s} ds\,dt
\end{aligned}$$

with $U = T^{1-\varepsilon}$ chosen as in (2.4). In the following key proposition we show I_ω is closely approximated by $\tilde{I}_\omega$ for $U = T^{1-\varepsilon}$. This truncation argument has previously been used in the works of Soundararajan–Young [34] on the second moment of modular L-functions, Chandee–Li [4] on the eighth moment of Dirichlet L-functions, and Shen [32] on the fourth moment of quadratic Dirichlet L-functions.

Proposition 3.1. *Under the Riemann hypothesis, for any $\varepsilon > 0$, there exists T_ε such that*

$$I_\omega - \tilde{I}_\omega = O(\varepsilon T(\log T)^{16})$$

for $T \ge T_\varepsilon$, where the implied constant is independent of ε.

Proof. We start with the observation

$$I_\omega - \tilde{I}_\omega = 2\int_{-\infty}^{\infty} \omega(t) \sum_{m,n=1}^{\infty} \frac{\tau_4(m)\tau_4(n)}{(mn)^{1/2}}\left(\frac{m}{n}\right)^{-\mathrm{i}t} \times \frac{1}{2\pi \mathrm{i}}\int_{(1)} G(s)g(s,t)t^{-4s}\frac{1}{(\pi^4 mn)^s}\left(\frac{t^{4s}-U^{4s}}{s}\right) ds\, dt + O(1)$$
$$= \int_{-\infty}^{\infty} \omega(t)\frac{1}{\pi \mathrm{i}}\int_{(1)} \zeta\big(\tfrac12+s+\mathrm{i}t\big)^4\zeta\big(\tfrac12+s-\mathrm{i}t\big)^4 G(s)g(s,t)t^{-4s}\pi^{-4s}\left(\frac{t^{4s}-U^{4s}}{s}\right) \times ds\, dt + O(1).$$

As $\frac{t^{4s}-U^{4s}}{s}$ is entire, if we move the last integration to the line $\Re e(s)=0$, we will encounter poles at $1/2 \pm \mathrm{i}t$. Since $G^{(j)}(1/2\pm \mathrm{i}t)$ decays rapidly, the contribution of these poles is $O(1)$. Hence, moving the integration, by Lemma 2.2 (i), we deduce

$$I_\omega - \tilde{I}_\omega = \int_{-\infty}^{\infty} \omega(t)\frac{1}{\pi \mathrm{i}}\int_{(0)} \zeta\big(\tfrac12+s+\mathrm{i}t\big)^4\zeta\big(\tfrac12+s-\mathrm{i}t\big)^4 G(s)(2\pi)^{-4s}\left(\frac{t^{4s}-U^{4s}}{s}\right) ds\, dt$$
$$+ O\bigg(\int_{-\infty}^{\infty} \omega(t)\int_{(0)} \bigg|\zeta\big(\tfrac12+s+\mathrm{i}t\big)^4\zeta\big(\tfrac12+s-\mathrm{i}t\big)^4 G(s)\frac{|s|^2+1}{t}\left(\frac{t^{4s}-U^{4s}}{s}\right)\bigg| \times d|s|\, dt\bigg) + O(1).$$

Recalling that $U=T^{1-\varepsilon}$, we have $\frac{t^{4\mathrm{i}t_0}-U^{4\mathrm{i}t_0}}{\mathrm{i}t_0} \ll \log\frac{t^4}{U^4} \ll \varepsilon\log T$. Thus, the first big-$O$ term in the above expression is

$$\ll \varepsilon\log T\int_{-\infty}^{\infty}\omega(t)\int_{-\infty}^{\infty}\big|\zeta\big(\tfrac12+\mathrm{i}t_0+\mathrm{i}t\big)\big|^4\big|\zeta\big(\tfrac12+\mathrm{i}t_0-\mathrm{i}t\big)\big|^4|G(\mathrm{i}t_0)|\frac{|t_0|^2+1}{t}\,dt_0\,dt$$
$$\ll \frac{\varepsilon\log T}{T}\int_{-\infty}^{\infty}|G(\mathrm{i}t_0)|(|t_0|^2+1)\int_{-\infty}^{\infty}\omega(t)\big|\zeta\big(\tfrac12+\mathrm{i}t+\mathrm{i}t_0\big)\big|^4\big|\zeta\big(\tfrac12+\mathrm{i}t-\mathrm{i}t_0\big)\big|^4\,dt\,dt_0. \tag{3.3}$$

Since $|\zeta(\frac12+\mathrm{i}t\pm\mathrm{i}t_0)| \ll |t_0|+|t|$ and (2.1) holds, the contribution of $|t_0|>\sqrt{T}$ to (3.3) is $\ll 1$. For $|t_0|\le\sqrt{T}$, applying Theorem 1.2, we derive

$$\int_0^T \big|\zeta\big(\tfrac12+\mathrm{i}t+\mathrm{i}t_0\big)\big|^4\big|\zeta\big(\tfrac12+\mathrm{i}t-\mathrm{i}t_0\big)\big|^4\,dt \ll T(\log T)^{16}. \tag{3.4}$$

Hence, we obtain

$$\int_{|t_0|\le\sqrt{T}}|G(\mathrm{i}t_0)|(|t_0|^2+1)\int_{-\infty}^{\infty}\omega(t)\big|\zeta\big(\tfrac12+\mathrm{i}t+\mathrm{i}t_0\big)\big|^4\big|\zeta\big(\tfrac12+\mathrm{i}t-\mathrm{i}t_0\big)\big|^4\,dt\,dt_0 \ll T(\log T)^{16}.$$

Therefore, (3.3) is $\ll \varepsilon(\log T)^{17}$, and we arrive at

$$I_\omega - \tilde{I}_\omega = \int_{-\infty}^{\infty} \omega(t)\, dt \frac{1}{\pi \mathrm{i}} \int_{(0)} \zeta\big(\tfrac{1}{2}+s+\mathrm{i}t\big)^4 \zeta\big(\tfrac{1}{2}+s-\mathrm{i}t\big)^4 G(s)(2\pi)^{-4s} \frac{t^{4s}-U^{4s}}{s}\, ds \\ + O(\varepsilon(\log T)^{17}).$$

The double integral above is

$$\ll \varepsilon \log T \int_{-\infty}^{\infty} |G(\mathrm{i}t_0)| \int_{-\infty}^{\infty} \omega(t) \big|\zeta\big(\tfrac{1}{2}+\mathrm{i}t+\mathrm{i}t_0\big)\big|^4 \big|\zeta\big(\tfrac{1}{2}+\mathrm{i}t-\mathrm{i}t_0\big)\big|^4\, dt\, dt_0. \tag{3.5}$$

As argued above, the contribution of $|t_0| > \sqrt{T}$ to (3.5) is $\ll 1$. So, it remains to consider the contribution of $|t_0| \le \sqrt{T}$. It follows from (3.4) that

$$\varepsilon \log T \int_{\log T < |t_0| \le \sqrt{T}} |G(\mathrm{i}t_0)| \int_{-\infty}^{\infty} \omega(t) \big|\zeta\big(\tfrac{1}{2}+\mathrm{i}t+\mathrm{i}t_0\big)\big|^4 \big|\zeta\big(\tfrac{1}{2}+\mathrm{i}t-\mathrm{i}t_0\big)\big|^4\, dt\, dt_0 \ll \varepsilon T.$$

For $\frac{1}{200} < |t_0| \le \log T$, it follows from Theorem 1.2 that

$$\int_0^T \big|\zeta\big(\tfrac{1}{2}+\mathrm{i}t+\mathrm{i}t_0\big)\big|^4 \big|\zeta\big(\tfrac{1}{2}+\mathrm{i}t-\mathrm{i}t_0\big)\big|^4\, dt \ll T(\log T)^8 (\log\log T)^8 \ll T(\log T)^9$$

and so

$$\varepsilon \log T \int_{\frac{1}{200} < |t_0| \le \log T} |G(\mathrm{i}t_0)| \int_{-\infty}^{\infty} \omega(t) \big|\zeta\big(\tfrac{1}{2}+\mathrm{i}t+\mathrm{i}t_0\big)\big|^4 \big|\zeta\big(\tfrac{1}{2}+\mathrm{i}t-\mathrm{i}t_0\big)\big|^4\, dt\, dt_0 \\ \ll \varepsilon T(\log T)^{10}.$$

Finally, by Theorem 1.2, for $|t_0| \le \frac{1}{200}$, we have

$$\int_{-\infty}^{\infty} \omega(t) \big|\zeta\big(\tfrac{1}{2}+\mathrm{i}t+\mathrm{i}t_0\big)\big|^4 \big|\zeta\big(\tfrac{1}{2}+\mathrm{i}t-\mathrm{i}t_0\big)\big|^4\, dt \ll T(\log T)^8 \min\Big(\frac{1}{|2t_0|^8}, (\log T)^8\Big).$$

Thus, the contribution of $|t_0| \le \frac{1}{200}$ to (3.5) is

$$\begin{aligned} &\ll \varepsilon \log T \int_{|t_0| \le \frac{1}{\log T}} |G(\mathrm{i}t_0)| T(\log T)^{16}\, dt_0 \\ &\quad + \varepsilon \log T \int_{\frac{1}{\log T} \le |t_0| \le \frac{1}{200}} |G(\mathrm{i}t_0)| T(\log T)^8 \frac{1}{|t_0|^8}\, dt_0 \\ &\ll \varepsilon \log T \cdot T(\log T)^{15} + \varepsilon \log T \cdot T(\log T)^8 \cdot (\log T)^7 \\ &\ll \varepsilon T(\log T)^{16}. \end{aligned}$$

Hence, gathering everything together, we obtain

$$I_\omega - \tilde{I}_\omega \ll \varepsilon T(\log T)^{16}$$

as desired. ■

We now have $I_\omega = \tilde{I}_\omega + O(\varepsilon T(\log T)^{16})$. We shall divide $\tilde{I}_\omega$ into diagonal parts $m = n$ and off-diagonal parts $m \neq n$ as follows. Setting

$$I_D := 2\int_{-\infty}^{\infty} \omega(t) \sum_{m=1}^{\infty} \frac{\tau_4(m)^2}{m} \frac{1}{2\pi \mathrm{i}} \int_{(1)} \frac{G(s)}{s} g(s,t) t^{-4s} \left(\frac{U^4}{\pi^4 m^2}\right)^s ds\, dt, \tag{3.6}$$

$$I_O := 2\int_{-\infty}^{\infty} \omega(t) \sum_{\substack{m,n=1\\ m\neq n}}^{\infty} \frac{\tau_4(m)\tau_4(n)}{(mn)^{1/2}} \left(\frac{m}{n}\right)^{-it} \frac{1}{2\pi \mathrm{i}} \int_{(1)} \frac{G(s)}{s} g(s,t) t^{-4s} \left(\frac{U^4}{\pi^4 mn}\right)^s ds\, dt, \tag{3.7}$$

we obtain

$$I_\omega = I_D + I_O + O(\varepsilon T(\log T)^{16}).$$

We shall evaluate I_D and I_O in the following propositions, to be proved in the later sections.

Proposition 3.2. *Unconditionally, we have*

$$I_D = \frac{4a_4(\log U)^{16}}{638512875} \int_{-\infty}^{\infty} \omega(t)\, dt + O(T(\log T)^{15}).$$

Proposition 3.3. *Assuming Conjecture 1, we have*

$$I_O = -\frac{13381 a_4}{2615348736000} \int_{-\infty}^{\infty} \omega(t)(\log T)^{16}\, dt + O(\varepsilon T(\log T)^{16}) + O\left(T\left(\frac{T}{T_0}\right)^{1+C}\right),$$

where C is defined as in the statement of Conjecture 1.

Combining Propositions 3.2 and 3.3, we find

$$I_\omega = a_4 \int_{-\infty}^{\infty} \omega(t) \frac{24024}{16!} (\log T)^{16}\, dt + O(\varepsilon T(\log T)^{16}) + O\left(T\left(\frac{T}{T_0}\right)^{1+C}\right). \tag{3.8}$$

From this formula, we shall deduce Theorem 1.1 by removing the smooth weight ω.

Proof of Theorem 1.1. Let $\eta \in (0,1)$. Let $\omega^+(t)$ be a majorant for the indicator function $\mathbb{1}_{[T,2T]}(t)$ supported on $[(1-\eta)T, (2+\eta)T]$, and $\omega^+(t) \equiv 1$ when $t \in [T, 2T]$. Then

$$I_4(2T) - I_4(T) \leq I_{\omega^+}.$$

Let $\varepsilon \in (0,1)$. It follows from (3.8) and the choice $T_0 = \eta T$ that there is T_ε such that for $T \geq \max(1/\eta^2, T_\varepsilon)$,

$$\begin{aligned} I_4(2T) - I_4(T) &\leq I_{\omega^+} \\ &= \frac{g_4 a_4}{16!} \int_{-\infty}^{\infty} \omega^+(t)(\log T)^{16}\, dt + O(\varepsilon T(\log T)^{16}) + O(T\eta^{-1-C}) \\ &= \frac{g_4 a_4}{16!} T(\log T)^{16} + O(\varepsilon T(\log T)^{16}) + O(\eta T(\log T)^{16}) + O(T\eta^{-1-C}). \end{aligned} \tag{3.9}$$

Let $\eta_0 \in (0,1)$. Set[3] $j = \lceil \log_2(1/\eta_0) \rceil$ so that $\eta_0 T/2 \le T/2^j \le \eta_0 T$. Replacing T in (3.9) by $T/2, \ldots, T/2^j$ and then adding, we obtain

$$\begin{aligned} &I_4(T) - I_4(T/2^j) \\ &\quad \le \frac{g_4 a_4}{16!} T(\log T)^{16} + O(\varepsilon T(\log T)^{16}) + O(\eta T(\log T)^{16}) + O(T\eta^{-1-C}) \end{aligned}$$

whenever $\eta_0 T/2 \ge \max(1/\eta^2, T_\varepsilon)$. (Here, we use the bound $T/2 + \cdots + T/2^j \le T$.) As $T/2^j \le \eta_0 T$, by Harper's result [14], on the Riemann hypothesis, there is $\widetilde{T}$ such that for $\eta_0 T \ge \widetilde{T}$, we have

$$|I_4(T/2^j)| \le I_4(\eta_0 T) \ll (\eta_0 T)(\log(\eta_0 T))^{16} \ll \eta_0 T(\log T)^{16}.$$

Combining everything, we find that for $T \ge \frac{1}{\eta_0}\max(2/\eta^2, 2T_\varepsilon, \widetilde{T})$,

$$\begin{aligned} I_4(T) \le \frac{g_4 a_4}{16!} T(\log T)^{16} &+ C_1 \varepsilon T(\log T)^{16} + C_2 \eta T(\log T)^{16} \\ &+ C_3 T\eta^{-1-C} + C_4 \eta_0 T(\log T)^{16} \end{aligned}$$

for certain absolute explicit constants C_1, C_2, C_3, C_4. Now, dividing both sides by $T(\log T)^{16}$ and taking lim sup, we obtain

$$\limsup_{T\to\infty} \frac{I_4(T)}{T(\log T)^{16}} \le \frac{g_4 a_4}{16!} + C_1\varepsilon + C_2\eta + C_4\eta_0.$$

As $\varepsilon, \eta, \eta_0 \in (0,1)$ are arbitrary, we arrive at

$$\limsup_{T\to\infty} \frac{I_4(T)}{T(\log T)^{16}} \le \frac{g_4 a_4}{16!}.$$

The lower bound is proved similarly and is based on the inequality $I_4(2T) - I_4(T) \ge I_{\omega^-}$, where $\omega^-(t)$ is a minorant for $\mathbb{1}_{[T,2T]}(t)$ supported on $[(1+\eta)T, (2-\eta)T]$, giving

$$\begin{aligned} &I_4(2T) - I_4(T) \\ &\quad \ge \frac{g_4 a_4}{16!} T(\log T)^{16} + O(\varepsilon T(\log T)^{16}) + O(\eta T(\log T)^{16}) + O(T\eta^{-1-C}). \end{aligned} \tag{3.10}$$

For $\eta_0 \in (0,1)$, we again set $j = \lceil \log_2(1/\eta_0) \rceil$ and replacing T in (3.10) by $T/2, \ldots,$ $T/2^j$, we obtain

$$\begin{aligned} &I_4(T) - I_4(T/2^j) \\ &\quad \ge \frac{g_4 a_4}{16!} \sum_{i=1}^{j} \frac{T}{2^i}\left(\log \frac{T}{2^i}\right)^{16} + O(\varepsilon T(\log T)^{16}) + O(\eta T(\log T)^{16}) + O(T\eta^{-1-C}). \end{aligned} \tag{3.11}$$

[3] Here $\log_2$ denotes the base 2 logarithm.

Note that

$$\begin{aligned}\sum_{i=1}^{j}\frac{T}{2^i}\left(\log\frac{T}{2^i}\right)^{16} &= \sum_{i=1}^{j}\frac{T}{2^i}\sum_{u+v=16}\binom{16}{u}(\log T)^u(-i\log 2)^v\\ &= T\sum_{u+v=16}\binom{16}{u}(\log T)^u(-\log 2)^v\sum_{i=1}^{j}\frac{i^v}{2^i}. \end{aligned}\tag{3.12}$$

Now, observe that

$$\sum_{i=1}^{j}\frac{i^v}{2^i} = C'_v + O\bigg(\sum_{i=j+1}^{\infty}\frac{i^v}{2^i}\bigg) = C'_v + O(j^v 2^{-j}) = C'_v + O\bigg(\eta_0\left(\log_2\frac{1}{\eta_0}\right)^v\bigg)$$

for some constants C'_v such that $C'_0 = 1$. Plugging this last estimate in (3.12), we find that

$$\sum_{i=1}^{j}\frac{T}{2^i}\left(\log\frac{T}{2^i}\right)^{16}$$

$$= T(\log T)^{16} + O(\eta_0 T(\log T)^{16}) + O\bigg(\bigg(1+\eta_0\left(\log\frac{1}{\eta_0}\right)^{15}\bigg)T(\log T)^{15}\bigg).$$

From this, together with (3.11), we can establish

$$\liminf_{T\to\infty}\frac{I_4(T)}{T(\log T)^{16}} \geq \frac{g_4 a_4}{16!},$$

which completes the proof of the theorem. ∎

4. The diagonal terms: Proof of Proposition 3.2

From the definition (3.6) of I_D, we see

$$I_D = 2\int_{-\infty}^{\infty}\omega(t)\frac{1}{2\pi \mathrm{i}}\int_{(1)}\sum_{m=1}^{\infty}\frac{\tau_4(m)^2}{m^{1+2s}}\pi^{-4s}\frac{G(s)}{s}g(s,t)t^{-4s}U^{4s}\,ds\,dt. \tag{4.1}$$

Lemma 4.1. *For $\Re\mathrm{e}(s) > \varepsilon_0 > 0$, we have*

$$\sum_{m=1}^{\infty}\frac{\tau_4(m)^2}{m^{1+2s}} = \zeta(1+2s)^{16}Z_1(s), \tag{4.2}$$

where

$$Z_1(s) = \prod_p\left(1-\frac{1}{p^{1+2s}}\right)^{16}\sum_{m=1}^{\infty}\frac{\tau_4(m)^2}{m^{1+2s}}. \tag{4.3}$$

In addition, $Z_1(s)$ is holomorphic and absolutely convergent on $\Re\mathrm{e}(s) > -1/4+\varepsilon_0$. Also, $Z_1(0) = a_4$, where a_4 is defined in (1.5).

Proof. For $\Re e(s) > \varepsilon_0$, the identity (4.2) is trivial. Now we shall show that $Z_1(s)$ is holomorphic and absolutely convergent on $\Re e(s) > -1/4 + \varepsilon_0$. Note that for $\Re e(s) > \varepsilon_0$,

$$\sum_{m=1}^{\infty} \frac{\tau_4(m)^2}{m^{1+2s}} = \prod_p \sum_{r=0}^{\infty} \frac{\tau_4(p^r)^2}{p^{r(1+2s)}} = \prod_p \sum_{r=0}^{\infty} \frac{\binom{r+3}{r}^2}{p^{r(1+2s)}} = \prod_p \left(1 + \frac{16}{p^{1+2s}} + B_p(s)\right), \tag{4.4}$$

where $B_p(s) := \sum_{r=2}^{\infty} \frac{\binom{r+3}{r}^2}{p^{r(1+2s)}}$. By (4.3),

$$\begin{aligned} Z_1(s) &= \prod_p \left(\sum_{r=0}^{16} \binom{16}{r} (-1)^r \frac{1}{p^{r+2rs}}\right) \prod_p \left(1 + \frac{16}{p^{1+2s}} + B_p(s)\right) \\ &= \prod_p \left(1 - \frac{16}{p^{1+2s}} + \sum_{r=2}^{16} \binom{16}{r} (-1)^r \frac{1}{p^{r+2rs}}\right) \prod_p \left(1 + \frac{16}{p^{1+2s}} + B_p(s)\right) \\ &= \prod_p \left(1 + A_p(s) + \frac{16}{p^{1+2s}} A_p(s) + A_p(s)B_p(s) + B_p(s) - \frac{16^2}{p^{2+4s}} - \frac{16}{p^{1+2s}} B_p(s)\right), \end{aligned} \tag{4.5}$$

where $A_p(s) := \sum_{r=2}^{16} \binom{16}{r} (-1)^r \frac{1}{p^{r+2rs}}$. Clearly, for $\Re e(s) > -1/4 + \varepsilon_0$,

$$|A_p(s)| \ll \sum_{r=2}^{16} \frac{1}{p^{(1/2+2\varepsilon_0)r}} \ll \frac{1}{p^{1+4\varepsilon_0}}.$$

In addition, for $\Re e(s) > -1/4 + \varepsilon_0$, we have

$$|B_p(s)| \ll \sum_{r=2}^{\infty} \frac{\binom{r+3}{r}^2}{p^{(1/2+2\varepsilon_0)r}} \ll \sum_{r=2}^{\infty} \frac{r^6}{p^{(1/2+2\varepsilon_0)r}} \ll \sum_{r=2}^{\infty} \frac{1}{p^{(1/2+\varepsilon_0)r}} \ll \frac{1}{p^{1+2\varepsilon_0}}.$$

The last two bounds imply that (4.5) is absolutely and uniformly convergent when $\Re e(s) > -1/4 + \varepsilon_0$, and thus is holomorphic in this region. Finally, $Z_1(0) = a_4$ follows from (4.3) and (4.4) by setting $s = 0$. ■

By (4.1) and Lemma 4.1, together with Lemma 2.2 (i), we get

$$\begin{aligned} I_D &= 2 \int_{-\infty}^{\infty} \omega(t) \frac{1}{2\pi i} \int_{(1)} \zeta(1+2s)^{16} Z_1(s) \pi^{-4s} \frac{G(s)}{s} g(s,t) t^{-4s} U^{4s}\, ds\, dt \\ &= 2 \int_{-\infty}^{\infty} \omega(t) \frac{1}{2\pi i} \int_{(1)} \zeta(1+2s)^{16} Z_1(s) (2\pi)^{-4s} \frac{G(s)}{s} U^{4s} \left(1 + O\left(\frac{|s|^2+1}{t}\right)\right) ds\, dt. \end{aligned}$$

One can prove that the big-O term contributes an at most $O(T^{4\varepsilon_0})$ error for any small $\varepsilon_0 > 0$. Indeed, moving the line of integration to $\Re e(s) = \varepsilon_0$, without encountering any poles, gives

$$2\int_{-\infty}^{\infty}\omega(t)\frac{1}{2\pi\mathrm{i}}\int_{(\varepsilon_0)}\zeta(1+2s)^{16}Z_1(s)(2\pi)^{-4s}\frac{G(s)}{s}U^{4s}O\Big(\frac{|s|^2+1}{t}\Big)\,ds\,dt \ll \int_{-\infty}^{\infty}\omega(t)\frac{U^{4\varepsilon_0}}{t}\,dt,$$

which is $\ll T^{4\varepsilon_0}$. Thus, we have

$$I_D = 2\int_{-\infty}^{\infty}\omega(t)\frac{1}{2\pi\mathrm{i}}\int_{(1)}\zeta(1+2s)^{16}Z_1(s)(2\pi)^{-4s}\frac{G(s)}{s}U^{4s}\,ds\,dt + O(T^{4\varepsilon_0}). \tag{4.6}$$

Moving the line of integration to $\Re e(s) = -1/4+\varepsilon_0$, we encounter a pole of order 17 at $s = 0$ and a new integral

$$2\int_{-\infty}^{\infty}\omega(t)\frac{1}{2\pi\mathrm{i}}\int_{(-1/4+\varepsilon_0)}\zeta(1+2s)^{16}Z_1(s)(2\pi)^{-4s}\frac{G(s)}{s}U^{4s}\,ds\,dt \ll TU^{-1+4\varepsilon_0} \ll T^{4\varepsilon_0+\varepsilon}. \tag{4.7}$$

With the help of Maple, we see the residue of this pole is

$$\frac{2Z_1(0)(\log U)^{16}}{638512875} + O((\log U)^{15}). \tag{4.8}$$

Substituting (4.7) and (4.8) in (4.6), we obtain

$$I_D = 2\int_{-\infty}^{\infty}\omega(t)\,dt\,\frac{2Z_1(0)(\log U)^{16}}{638512875} + O(T(\log T)^{15}).$$

This, together with the fact that $Z_1(0) = a_4$ (see Lemma 4.1), completes the proof of Proposition 3.2.

5. The off-diagonal terms: Sketch of proof

The most difficult part in evaluating $I_4(T)$ is the off-diagonal term I_O. As the asymptotic evaluation of I_O is very involved, we provide a high level summary of the key steps. This argument can be viewed as a descendant of the arguments in [10, 16, 18, 19, 26]. The key idea is that the off-diagonal terms can be evaluated by rewriting them in terms of additive divisor sums and by inserting the main term from the additive divisor conjecture. The one difference between this article and the earlier articles [10, 16, 19] is that the off-diagonals are related to the smoothed sums (1.8) rather than the classical unsmoothed sums (1.7). This approach was first used in [18] and then in [26]. Let $\varepsilon_1 > 0$. We move the s-contour in (3.7) to $\Re e(s) = \varepsilon_1$ to obtain

$$I_O = 2T\sum_{m\neq n}\frac{\tau_4(m)\tau_4(n)}{\sqrt{mn}}f^*(m,n),$$

where

$$f^*(x,y) := \frac{1}{2\pi \mathrm{i}} \int_{(\varepsilon_1)} \frac{G(s)}{s} \left(\frac{1}{\pi^4 xy}\right)^s \frac{1}{T} \int_{-\infty}^{\infty} \left(\frac{x}{y}\right)^{-it} g(s,t) \left(\frac{U}{t}\right)^{4s} \omega(t)\, dt\, ds.$$

First, a smooth partition of unity is introduced so that

$$I_O = \sum_{M,N} I_{M,N},$$

where the sum is over $M, N \in \{2^{k/2} \mid k \geq -1\}$, and

$$I_{M,N} = \frac{2T}{\sqrt{MN}} \sum_{m \neq n} \tau_4(m)\tau_4(n) W\left(\frac{m}{M}\right) W\left(\frac{n}{N}\right) f^*(m,n).$$

The point of introducing the smooth partition of unity is so that we can apply Conjecture 1 (the quaternary additive divisor conjecture). The parameters M, N can be truncated to the region $MN \ll U^{4+\varepsilon_2}$. Note that since $U = T^{1-\varepsilon}$, the function $f^*(m,n)$ is very small if $mn \gg U^{4+\varepsilon_2}$ and thus the contribution of $I_{M,N}$ when $MN \gg U^{4+\varepsilon_2}$ is negligible. Thus,

$$I_O \sim \sum_{\substack{M,N \\ MN \ll U^{4+\varepsilon_2}}} I_{M,N}.$$

Note that

$$MN \ll U^{4+\varepsilon_2} \leq T^{4-3\varepsilon} \quad \text{for } \varepsilon_2 \leq \varepsilon. \tag{5.1}$$

Next, we claim that m and n must be close together, i.e., if $|\log(m/n)| \gg T_0^{-1+\varepsilon_1}$, then $f^*(m,n)$ is very small. Thus, we may insert the condition

$$|\log(m/n)| \ll T_0^{-1+\varepsilon_1} \tag{5.2}$$

with a negligible error. Note that this last condition forces M and N to be within a constant multiple of each other, namely,

$$N/3 \leq M \leq 3N.$$

Combining this with (5.1), it follows that

$$M, N \leq T^{2-3\varepsilon/2}.$$

The idea now is to manipulate $I_{M,N}$ so that instead of summing over m, n, the sum is over r, n where the change of variable $m = n + r$ is made. Thus

$$I_{M,N} = \frac{2T}{\sqrt{MN}} \sum_{r \neq 0} \left(\sum_{m-n=r} \tau_4(m)\tau_4(n) W\left(\frac{m}{M}\right) W\left(\frac{n}{N}\right) f^*(m,n) \right) + O(T^{-10}).$$

Further, conditions (5.1) and (5.2) imply we can insert conditions on r and n so that

$$I_{M,N} = \frac{2T}{\sqrt{MN}} \sum_{\substack{r\neq 0 \\ |r| \ll \frac{M}{T_0} T_0^{\varepsilon_1}}} \Bigg(\sum_{\substack{m-n=r \\ |\log(m/n)| \ll T_0^{-1+\varepsilon_1}}} \tau_4(m)\tau_4(n) W\left(\frac{m}{M}\right) W\left(\frac{n}{N}\right) f^*(m,n) \Bigg) + O(T^{-10}).$$

Applying the quaternary additive divisor conjecture we obtain

$$I_{M,N} = \frac{2T}{\sqrt{MN}} \sum_{0<|r|\ll \frac{M}{T_0}T_0^{\varepsilon_1}} D_{f_r}(r) + O\left(\left(\frac{T}{T_0}\right)^{1+C} T^{1-\varepsilon/2} \right) + O(T^{-10})$$

for a function f_r defined in (6.7) below and $D_{f_r}(r) := D_{f_r;4,4}(r)$ where we recall the definition (1.8). Observe that since $M \le T^{2-3\varepsilon/2}$, square root cancellation gives $\sqrt{M} \ll T^{1-3\varepsilon/4}$ and thus we obtain an error term $O((T/T_0)^{1+C} T^{1-\varepsilon/2})$ from the quaternary additive divisor conjecture (for full details see Lemma 6.3 below). It follows that

$$I_O = \sum_{M,N} \frac{2T}{\sqrt{MN}} \sum_{0<|r|\le R_0} \tilde{D}_{f_r}(r) + O\left(\left(\frac{T}{T_0}\right)^{1+C} T^{1-\varepsilon/2} \right),$$

where $\tilde{D}_{f_r}(r)$ is the main term in the additive divisor conjecture, given by the main term in (1.12) associated to f_r. The next step is to sum back over the partition of unity. Doing this, we arrive at

$$I_O = 2T \sum_{1\le |r| \le R_0} \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathscr{B}_1} \int_{\mathscr{B}_2} \zeta(z_1)^4 \zeta(z_2)^4 \sum_{q=1}^{\infty} \frac{c_q(r) G_4(z_1,q) G_4(z_2,q)}{q^{z_1+z_2}} \times \int_{\max(0,r)}^{\infty} f^*(x,x-r) x^{z_1-3/2} (x-r)^{z_2-3/2}\, dx\, dz_2\, dz_1 + O\left(\left(\frac{T}{T_0}\right)^{1+C} T^{1-\varepsilon/2} \right),$$

where

$$f^*(x,x-r) = \frac{1}{2\pi \mathrm{i}} \int_{(\varepsilon_1)} \frac{G(s)}{s} \left(\frac{1}{\pi^4 x(x-r)} \right)^s \frac{1}{T} \int_{-\infty}^{\infty} \left(1 + \frac{r}{x-r}\right)^{-it} g(s,t) \left(\frac{U}{t}\right)^{4s} \omega(t)\, dt\, ds.$$

Observe that the expression on the right hand side depends on quantities that arise from the additive divisor conjecture, such as the Ramanujan sum $c_q(r)$ and the divisor-type functions $G_4(z,r)$. These quantities will then become part of the Dirichlet series

$$\mathscr{H}_s(z_1,z_2) = \sum_{r=1}^{\infty} \sum_{q=1}^{\infty} \frac{c_q(r) G_4(z_1,q) G_4(z_2,q)}{q^{z_1+z_2} r^{2s+2-z_1-z_2}}.$$

After summing separately over $0 < r \leq R_0$ and $-R_0 \geq r < 0$ and then recombining terms by using the symmetry $c_q(r) = c_q(-r)$, we obtain

$$I_O = \frac{2}{(2\pi \mathrm{i})^3} \int_{\mathscr{B}_1} \int_{\mathscr{B}_2} \int_{(\varepsilon_3)} \zeta(z_1)^4 \zeta(z_2)^4 \sum_{1 \leq r \leq R_0} \sum_{q=1}^{\infty} \frac{c_q(r) G_4(z_1, q) G_4(z_2, q)}{q^{z_1+z_2} r^{2s+2-z_1-z_2}} \frac{G(s)}{s\pi^{4s}}$$
$$\times \int_{-\infty}^{\infty} g(s,t) \left(\frac{U}{t}\right)^{4s} \omega(t)$$
$$\times \Gamma(-z_1 - z_2 + 2s + 2) \left(\frac{\Gamma\left(z_2 - s - \frac{1}{2} + \mathrm{i}t\right)}{\Gamma\left(s - z_1 + \frac{3}{2} + \mathrm{i}t\right)} + \frac{\Gamma\left(z_1 - s - \frac{1}{2} - \mathrm{i}t\right)}{\Gamma\left(s - z_2 + \frac{3}{2} - \mathrm{i}t\right)} \right) dt\, ds\, dz_2\, dz_1$$
$$+ O\left(\left(\frac{T}{T_0}\right)^{1+C} T^{1-\varepsilon/2} \right).$$

Note that the x-integrals have disappeared as they can be computed exactly in terms of the B function, which then leads to the five Gamma factors in the last equation. The next step is to move the s-integral to the right to $\Re e(s) = 1$ and to let $R_0 \to \infty$. The point of this is that the double sums $\sum_{r,q}$ are absolutely convergent in this region and by extending $R_0 \to \infty$ we obtain $\mathscr{H}_s(z_1, z_2)$. We then make use of the meromorphic continuation (see Proposition 6.5 below):

$$\mathscr{H}_s(z_1, z_2) = \zeta(2s + 2 - z_1 - z_2) \frac{\zeta(1+2s)^{16} \zeta(1 + 2s - z_1 - z_2 + 2)}{\zeta(1 + 2s - z_1 + 1)^4 \zeta(1 + 2s - z_2 + 1)^4} \tilde{\mathscr{I}}(z_1, z_2, s)$$

where $\tilde{\mathscr{I}}(z_1, z_2, s)$ is holomorphic for $\Re e(s) > -1/4 + 2\delta$ for some $\delta > 0$. Furthermore, we know from Lemma 2.2 that $g(s,t)$ can be approximated by $(t/2)^{4s}$, and thus

$$I_O = \int_{-\infty}^{\infty} \omega(t) \left(\frac{2}{(2\pi \mathrm{i})^3} \int_{\mathscr{B}_1} \int_{\mathscr{B}_2} \int_{(\varepsilon_3)} \zeta(z_1)^4 \zeta(z_2)^4 \right.$$
$$\times \zeta(2s+2-z_1-z_2) \frac{\zeta(1+2s)^{16} \zeta(1+2s-z_1-z_2+2)}{\zeta(1+2s-z_1+1)^4 \zeta(1+2s-z_2+1)^4} \tilde{\mathscr{I}}(z_1, z_2, s) \frac{G(s)}{s\pi^{4s}} \left(\frac{U}{2}\right)^{4s}$$
$$\left. \times \Gamma(-z_1-z_2+2s+2) \left(\frac{\Gamma\left(z_2-s-\frac{1}{2}+\mathrm{i}t\right)}{\Gamma\left(s-z_1+\frac{3}{2}+\mathrm{i}t\right)} + \frac{\Gamma\left(z_1-s-\frac{1}{2}-\mathrm{i}t\right)}{\Gamma\left(s-z_2+\frac{3}{2}-\mathrm{i}t\right)} \right) ds\, dz_2\, dz_1 \right) dt$$
$$+ O\left(\left(\frac{T}{T_0}\right)^{1+C} T^{1-\varepsilon/2} \right).$$

A calculation with Stirling's formula (see (8.13) and (8.16) below) shows that for $|\Im m(s)| \leq t + 1$,

$$\frac{\Gamma\left(z_2 - s - \frac{1}{2} + \mathrm{i}t\right)}{\Gamma\left(s - z_1 + \frac{3}{2} + \mathrm{i}t\right)} + \frac{\Gamma\left(z_1 - s - \frac{1}{2} - \mathrm{i}t\right)}{\Gamma\left(s - z_2 + \frac{3}{2} - \mathrm{i}t\right)}$$
$$\sim 2t^{-(2s+2-z_1-z_2)} \cos\left(\frac{\pi}{2}(2s + 2 - z_1 - z_2) \right),$$

and it follows that

$$I_O \sim \int_{-\infty}^{\infty} \omega(t) \bigg(\frac{4}{(2\pi i)^3} \int_{\mathscr{B}_1} \int_{\mathscr{B}_2} \int_{(\varepsilon_3)} \zeta(z_1)^4 \zeta(z_2)^4 \\ \times \zeta(2s+2-z_1-z_2) \frac{\zeta(1+2s)^{16} \zeta(1+2s-z_1-z_2+2)}{\zeta(1+2s-z_1+1)^4 \zeta(1+2s-z_2+1)^4} \tilde{\mathscr{I}}(z_1,z_2,s) \frac{G(s)}{s\pi^{4s}} \left(\frac{U}{2}\right)^{4s} \\ \times \Gamma(-z_1-z_2+2s+2) t^{-(2s+2-z_1-z_2)} \cos\left(\frac{\pi}{2}(2s+2-z_1-z_2)\right) ds\, dz_2\, dz_1 \bigg) dt$$

up to an error term $O((T/T_0)^{1+C} T^{1-\varepsilon/2})$. These integrals are standard and can be evaluated by shifting the contour of $\Re e(s)$ past $s = 0$ and then applying the residue theorem. Note that $\zeta(1+u) = 1/u + O(1)$ as $u \to 1$ and $\tilde{\mathscr{I}}(0,0,0) = a_4$ so that

$$I_O \sim a_4 \int_{-\infty}^{\infty} \omega(t) \bigg(\frac{4}{(2\pi i)^3} \int_{\mathscr{B}_1} \int_{\mathscr{B}_2} \int_{(\varepsilon_3)} \\ \times \frac{(2s-z_1+1)^4 (2s-z_2+1)^4}{(z_1-1)^4 (z_2-1)^4 (2s+1-z_1-z_2)(2s)^{16}(2s-z_1-z_2+2)} \frac{G(s)}{s\pi^{4s}} \left(\frac{U}{2}\right)^{4s} \\ \times \Gamma(-z_1-z_2+2s+2) t^{-(2s+2-z_1-z_2)} \cos\left(\frac{\pi}{2}(2s+2-z_1-z_2)\right) ds\, dz_2\, dz_1 \bigg) dt.$$

up to an error term $O(T(\log T)^{15} + (\frac{T}{T_0})^{1+C} T^{1-\varepsilon/2})$. For full details of the calculation, see Section 7 below. It should be noted that the last multiple integral is a multivariable version of the types of integrals that appear in standard applications of Perron's formula.

6. The off-diagonal terms: The full details

6.1. Using a smooth partition of unity and restricting M and N

Let $\varepsilon_1 > 0$ and

$$f^*(x,y) := \frac{1}{2\pi i} \int_{(\varepsilon_1)} \frac{G(s)}{s} \left(\frac{1}{\pi^4 xy}\right)^s \frac{1}{T} \int_{-\infty}^{\infty} \left(\frac{x}{y}\right)^{-it} g(s,t) \left(\frac{U}{t}\right)^{4s} \omega(t)\, dt\, ds. \tag{6.1}$$

We move the s-contour in (3.7) to $\Re e(s) = \varepsilon_1$ to obtain

$$I_O = 2T \sum_{m \neq n} \frac{\tau_4(m)\tau_4(n)}{\sqrt{mn}} f^*(m,n).$$

The next step is to introduce a smooth partition of unity so that the variables m and n lie in dyadic boxes of the shape $[M, 2M]$ and $[N, 2N]$. This will allow us to apply the smoothed additive divisor conjecture for τ_4.

We now recall that there exists a smooth function W_0 supported in $[1,2]$ such that $\sum_{k\in\mathbb{Z}} W_0(x/2^{k/2}) = 1$ for $x > 0$. (For an explicit construction of a smooth partition of unity, see [12, p. 360].) Note that for $x \geq 1$, we have

$$\sum_{\substack{M=2^{k/2} \\ k \geq -1}} W_0\left(\frac{x}{M}\right) = 1.$$

Hence, setting $W(x) = x^{-1/2} W_0(x)$, we can write

$$I_O = \sum_{M,N} I_{M,N},$$

where the sum is over $M, N \in \{2^{k/2} \mid k \geq -1\}$, and

$$I_{M,N} = \frac{2T}{\sqrt{MN}} \sum_{m \neq n} \tau_4(m)\tau_4(n) W\left(\frac{m}{M}\right) W\left(\frac{n}{N}\right) f^*(m,n).$$

We now restrict the size of M and N so that $MN \ll U^{4+\varepsilon_2}$. To do so, we shall show that the contribution of M, N with $MN \gg U^{4+\varepsilon_2}$ is negligibly small. We first write

$$f^*(x,y) = \frac{1}{T} \int_{-\infty}^{\infty} \left(\frac{x}{y}\right)^{-it} \omega(t) \widetilde{V}_t(\pi^4 xy)\, dt, \tag{6.2}$$

where $\widetilde{V}_t(u)$ is defined in (2.3). By Lemma 2.2 (iii), when $mn \gg U^{4+\varepsilon_2}$ and $c_1 T \leq t \leq c_2 T$, we have

$$\widetilde{V}_t(\pi^4 mn) \ll \left(\frac{U^4}{mn}\right)^A$$

for any $A > 0$. Then

$$f^*(m,n) \ll \frac{1}{T} \int_{c_1 T}^{c_2 T} \omega(t) \left(\frac{U^4}{mn}\right)^A dt \ll \left(\frac{U^4}{mn}\right)^A.$$

From this, it follows that

$$\begin{aligned}
\sum_{\substack{M,N \\ MN \gg U^{4+\varepsilon_2}}} I_{M,N} &\ll TU^{4A} \sum_{\substack{M,N \\ MN \gg U^{4+\varepsilon_2}}} \frac{1}{\sqrt{MN}} \sum_{m \neq n} \frac{\tau_4(m)\tau_4(n)}{(mn)^A} W\left(\frac{m}{M}\right) W\left(\frac{n}{N}\right) \\
&\ll TU^{4A} \sum_{\substack{M,N \\ MN \gg U^{4+\varepsilon_2}}} \sum_{m \neq n} \frac{\tau_4(m)\tau_4(n)}{(mn)^{A+1/2}} W_0\left(\frac{m}{M}\right) W_0\left(\frac{n}{N}\right) \\
&\ll TU^{4A} \sum_{\substack{M,N \\ MN \gg U^{4+\varepsilon_2}}} \sum_{\substack{M \leq m \leq 2M \\ N \leq n \leq 2N}} \frac{1}{(mn)^A} \\
&\ll TU^{4A} \sum_{\substack{M,N \\ MN \gg U^{4+\varepsilon_2}}} \frac{1}{(MN)^{A-1}} \\
&\ll TU^{4A} U^{-(A-2)(4+\varepsilon_2)}.
\end{aligned}$$

By taking $A \geq \frac{(1-\varepsilon)(8+2\varepsilon_2)+1+B}{\varepsilon_2(1-\varepsilon)}$, we have

$$\sum_{\substack{M,N \\ MN \gg U^{4+\varepsilon_2}}} I_{M,N} \ll T^{-B}$$

for any large constant $B > 0$, which allows us to assume $MN \ll U^{4+\varepsilon_2}$ for $I_{M,N}$ in the remaining discussion.

Our next step is to show that m and n must be close together. This is because $f^*(x, y)$ is small unless x and y are close together, which we shall prove rigorously as follows. We first control the inner integral in (6.1) for s such that $\Re e(s) = \varepsilon_1$ and $|s| \leq \sqrt{T}$. Using integration by parts j times, we derive

$$\int_{-\infty}^{\infty} \left(\frac{x}{y}\right)^{-it} g(s,t) \left(\frac{U}{t}\right)^{4s} \omega(t)\,dt \ll \frac{U^{4\varepsilon_1}}{|\log(x/y)|^j} \int_{c_1 T}^{c_2 T} \left| \frac{\partial^j}{\partial t^j} (g(s,t) t^{-4s} \omega(t)) \right| dt.$$

To bound the above j-th partial derivative, we apply the generalized product rule to deduce

$$\frac{\partial^j}{\partial t^j}(g(s,t)t^{-4s}\omega(t)) = \sum_{a+b+c=j} \binom{j}{a,b,c} \frac{\partial^a}{\partial t^a} g(s,t) \frac{\partial^b}{\partial t^b} t^{-4s} \omega^{(c)}(t),$$

which by (iii) and Lemma 2.2 (ii) is

$$\ll \sum_{a+b+c=j} \binom{j}{a,b,c} |s|^a T^{4\varepsilon_1 - a} \frac{(|s|+j)^b}{T^{4\varepsilon_1 + b}} T_0^{-c} \ll \left(\frac{|s|}{T} + \frac{|s|+j}{T} + \frac{1}{T_0} \right)^j$$
$$\ll \frac{\max(|s|,1)^j}{T_0^j}.$$

Therefore, we arrive at

$$\int_{-\infty}^{\infty} \left(\frac{x}{y}\right)^{-it} g(s,t) \left(\frac{U}{t}\right)^{4s} \omega(t)\,dt \ll T^{1+4\varepsilon_1} \frac{\max(|s|,1)^j}{|\log(x/y)|^j T_0^j}$$

whenever $|s| \leq \sqrt{T}$.

Secondly, we study the inner integral in (6.1) for s such that $\Re e(s) = \varepsilon_1$ and $|s| > \sqrt{T}$. By Lemma 2.2 (i), as $\Re e(s) = \varepsilon_1 > 0$, we know

$$\int_{-\infty}^{\infty} \left(\frac{x}{y}\right)^{-it} g(s,t) \left(\frac{U}{t}\right)^{4s} \omega(t)\,dt \ll \int_{c_1 T}^{c_2 T} |g(s,t)|\,dt$$
$$\ll \int_{c_1 T}^{c_2 T} \left(\frac{t}{2}\right)^{4\varepsilon_1} \left(1 + O\left(\frac{|s|^2}{t}\right)\right) dt,$$

which is $\ll (T + |s|^2)T^{4\varepsilon_1}$. As $|s| > \sqrt{T}$, it is clear that $(T + |s|^2)T^{4\varepsilon_1} \ll T^{4\varepsilon_1}|s|^2 = T^{4\varepsilon_1} \max(|s|, 1)^2$ and thus

$$(T + |s|^2)T^{4\varepsilon_1} \ll T^{4\varepsilon_1} \max(|s|,1)^2 \left(\frac{\max(|s|,1)}{\sqrt{T}}\right)^{2j} = T^{1+4\varepsilon_1} \frac{\max(|s|,1)^{2j+2}}{T^{j+1}}.$$

Assuming $x, y \gg 1$ and $xy \ll U^{4+\varepsilon_2}$, we see

$$|\log(x/y)| \le |\log x| + |\log y| \ll \log U \le \log T \ll_j T^{1/j},$$

which implies that $|\log(x/y)|^j T_0^j \ll_j T T_0^j \le T^{1+j}$.

To summarise, we have shown that for any s with $\Re e(s) = \varepsilon_1$,

$$\int_{-\infty}^{\infty} \left(\frac{x}{y}\right)^{-it} g(s,t) \left(\frac{U}{t}\right)^{4s} \omega(t)\, dt \ll T^{1+4\varepsilon_1} \frac{\max(|s|,1)^{2j+2}}{|\log(x/y)|^j T_0^j}.$$

Plugging this bound into (6.1) then yields

$$\begin{aligned} f^*(x,y) &\ll \frac{T^{4\varepsilon_1}}{(xy)^{\varepsilon_1} |\log(x/y)|^j T_0^j} \int_{(\varepsilon_1)} \frac{|G(s)|}{|s|} \max(|s|,1)^{2j+2} |ds| \\ &\ll_j \frac{T^{4\varepsilon_1}}{|\log(x/y)|^j T_0^j} \end{aligned} \tag{6.3}$$

if $x, y \gg 1$ and $xy \ll U^{4+\varepsilon_2}$. Now, assume

$$|\log(x/y)| \gg T_0^{-1+\varepsilon_1}.$$

By (iii) and (6.3), for any constant $B > 0$, we have

$$f^*(x,y) \ll \frac{T^{4\varepsilon_1}}{T_0^{j\varepsilon_1}} \le T^{\varepsilon_1(4-j/2)} \ll T^{-B}$$

if we choose $j \ge 2B/\varepsilon_1 + 8$.

Now, setting $m - n = r$, we have obtained

$$I_{M,N} = \frac{2T}{\sqrt{MN}} \sum_{r \neq 0} \sum_{\substack{m-n=r \\ |\log(m/n)| \ll T_0^{-1+\varepsilon_1}}} \tau_4(m)\tau_4(n) W\left(\frac{m}{M}\right) W\left(\frac{n}{N}\right) f^*(m,n) + O(T^{-B}) \tag{6.4}$$

for $MN \ll U^{4+\varepsilon_2}$ and any constant $B > 0$. We shall further impose several other conditions on M and N. First of all, observe that the condition $|\log(m/n)| \ll T_0^{-1+\varepsilon_1}$ forces the above sum to be empty unless $N/3 \le M \le 3N$. Indeed, if $M < N/3$ or $M > 3N$, then for m, n satisfying $W(m/M)W(n/N) \neq 0$, we have $|\log(m/n)| \ge \log(3/2)$. In addition, as the conditions in the sum tell us $0 \neq r/N \ll T_0^{\varepsilon_1 - 1}$, we know that either the sum is empty, or $3N \ge M \ge N/3 \gg |r| T_0^{1-\varepsilon_1} \gg T_0^{1-\varepsilon_1}$. In light of these, we only need to consider the case $3N \ge M \ge N/3$ in this section. For the sake of convenience, we write $M \asymp N$ to mean

$$N/3 \le M \le 3N. \tag{6.5}$$

(We will only use this restricted meaning for the notation $\asymp$ with M and N.) Note that the contribution of M, N that do not satisfy $M \asymp N \gg T_0^{1-\varepsilon_1}$ to the sum $\sum_{MN \ll U^{4+\varepsilon_2}} I_{M,N}$

is $\ll (\log T)^2 T^{-B}$. From this, we arrive at

$$I_O = \sum_{\substack{M,N \\ M \asymp N,\, MN \ll U^{4+\varepsilon_2} \\ M,N \gg T_0^{1-\varepsilon_1}}} I_{M,N} + O(1), \tag{6.6}$$

and it remains to evaluate $I_{M,N}$ for M, N such that $M \asymp N$ and $M, N \gg T_0^{1-\varepsilon_1}$. Now, fixing $x - y = r$ and using (6.1), we have the following proposition that summarises what we have done thus far in this section.

Proposition 6.1. *Let $B > 0$ be arbitrary and fixed. For $0 \neq r \in \mathbb{Z}$, set*

$$f_r(x, y) = f_{r;M,N}(x, y) := W\left(\frac{x}{M}\right) W\left(\frac{y}{N}\right) \frac{1}{2\pi i} \int_{(\varepsilon_1)} \frac{G(s)}{s} \left(\frac{1}{\pi^4 xy}\right)^s \times \frac{1}{T} \int_{-\infty}^{\infty} \left(1 + \frac{r}{y}\right)^{-it} g(s, t) \left(\frac{U}{t}\right)^{4s} \omega(t)\, dt\, ds \tag{6.7}$$

if $x, y > 0$; otherwise, set $f_r(x, y) = 0$. Let $D_{f_r}(r) := D_{f_r,4,4}(r)$ be the quaternary additive divisor sum given by (1.8) with $f = f_r$ and $k = \ell = 4$. Then for $M, N \gg T_0^{1-\varepsilon_1}$ such that $M \asymp N$ and $MN \ll U^{4+\varepsilon_2}$, we have

$$I_{M,N} = \frac{2T}{\sqrt{MN}} \sum_{0 < |r| \ll \frac{M}{T_0} T_0^{\varepsilon_1}} D_{f_r}(r) + O(T^{-B}). \tag{6.8}$$

In addition, for those $M, N \gg 1$ satisfying $MN \ll U^{4+\varepsilon_2}$, such that either $M \not\asymp N$ or $\min(M, N) \ll T_0^{1-\varepsilon_1}$, we have the bound $I_{M,N} \ll T^{-B}$.

6.2. *Applying the quaternary additive divisor conjecture*

Observe that for $\varepsilon_2 \leq \varepsilon$, we have

$$U^{4+\varepsilon_2} \leq (T^{1-\varepsilon})^{4+\varepsilon_2} = T^{4+\varepsilon_2 - 4\varepsilon - \varepsilon\varepsilon_2} \leq T^{4-3\varepsilon}.$$

In this section, we shall apply the additive divisor conjecture, Conjecture 1, to handle $I_{M,N}$ in (6.8). This first leads to the following lemma telling us that $f(x, y) := T^{-4\varepsilon_1} f_r(x, y)$ satisfies conditions (1.9) and (1.10) with $X = M$, $Y = N$, and $P = T^{1+\varepsilon_1} T_0^{-1}$, where $f_r(x, y)$ is defined as in (6.7). (The proof of this technical lemma will be given in Section 10.)

Lemma 6.2. *Let $0 < \varepsilon_1, \varepsilon_2 \leq 1/2$. Then*

$$\operatorname{supp}(f_{r;M,N}) \subseteq [M, 2M] \times [N, 2N].$$

In addition, for $M \ll U^{2+\varepsilon_2}$, $M \asymp N$, and $1 \leq |r| \ll \frac{M}{T_0} T_0^{\varepsilon_1}$, we have

$$x^m y^n f_{r;M,N}^{(m,n)}(x, y) \ll T^{4\varepsilon_1} P^n, \quad \text{where} \quad P = T^{1+\varepsilon_1} T_0^{-1}.$$

By Lemma 6.2, the support of $f(x,y) = T^{-4\varepsilon_1} f_r(x,y)$ is in $[M,2M]\times[N,2N]$ with $M \asymp N$, and $f(x,y)$ satisfies condition (1.10). In addition, we claim $|r| \ll M^{1-\varepsilon_1} \le M^{1-\varepsilon'}$ for $\varepsilon' \le \varepsilon_1 < 1/6$. Indeed, if $\varepsilon_1 < 1/6$ and $\varepsilon_2 \le \varepsilon$, we have $(MT_0)^{\varepsilon_1} \ll T^{3\varepsilon_1} \le T^{1/2} \le T_0$, which is equivalent to

$$\frac{M}{T_0} T_0^{\varepsilon_1} \ll M^{1-\varepsilon_1}.$$

Now, by (6.6) and Proposition 6.1, applying Conjecture 1 (with $X = M$, $Y = N$, $P = T^{1+\varepsilon_1} T_0^{-1}$, and $f(x,y) = T^{-4\varepsilon_1} f_r(x,y)$), we derive

$$I_O = \sum_{\substack{M,N \\ M\asymp N,\, MN \ll U^{4+\varepsilon_2} \\ M,N \gg T_0^{1-\varepsilon_1}}} \tilde{I}_{M,N} + \sum_{\substack{M,N \\ M\asymp N,\, MN \ll U^{4+\varepsilon_2} \\ M,N \gg T_0^{1-\varepsilon_1}}} O(\mathcal{E}_{M,N}) + O(1), \tag{6.9}$$

where I_O is defined in (3.7) and

$$\tilde{I}_{M,N} = \frac{2T}{\sqrt{MN}} \sum_{0<|r|\ll \frac{M}{T_0} T_0^{\varepsilon_1}} \tilde{D}_{f_r}(r), \tag{6.10}$$

$$\tilde{D}_{f_r}(r) = \frac{1}{(2\pi i)^2} \int_{\mathcal{B}_1} \int_{\mathcal{B}_2} \zeta(z_1)^4 \zeta(z_2)^4 \sum_{q=1}^{\infty} \frac{c_q(r) G_4(z_1,q) G_4(z_2,q)}{q^{z_1+z_2}}$$
$$\times \int_{\max(0,r)}^{\infty} f_r(x, x-r) x^{z_1-1} (x-r)^{z_2-1}\, dx\, dz_2\, dz_1, \tag{6.11}$$

$$\mathcal{E}_{M,N} = \frac{T^{1+4\varepsilon_1} T^{C\varepsilon_1}}{\sqrt{MN}} \sum_{0<|r|\ll \frac{M}{T_0} T_0^{\varepsilon_1}} \left(\frac{T}{T_0}\right)^{C} M^{1/2+\varepsilon_0}. \tag{6.12}$$

Therefore, assuming Conjecture 1, to study I_O we shall estimate $\tilde{I}_{M,N}$ and $\mathcal{E}_{M,N}$ for the sums in (6.9). To end this section, we bound the contribution of $\mathcal{E}_{M,N}$ to I_O in (6.9) by proving the following lemma.

Lemma 6.3. *Let $\varepsilon, \varepsilon_0, \varepsilon_1, \varepsilon_2 > 0$ be constants such that*

$$\max(\varepsilon_0, \varepsilon_1, C\varepsilon_1) \le \frac{\varepsilon}{40} \quad \textit{and} \quad \varepsilon_2 \le \varepsilon.$$

Let $\mathcal{E}_{M,N}$ be defined as in (6.12). Then for T sufficiently large with respect to these parameters, we have

$$\sum_{\substack{M,N \\ M\asymp N,\, MN \ll U^{4+\varepsilon_2}}} \mathcal{E}_{M,N} \ll \left(\frac{T}{T_0}\right)^{1+C} T^{1-\varepsilon/2}.$$

Proof. As $\varepsilon_1 \max(C,1) \le \frac{\varepsilon}{40}$ and $M \asymp N$, a straightforward calculation shows that $\mathcal{E}_{M,N}$ is

$$\ll \frac{T^{1+4\varepsilon_1+C\varepsilon_1}}{M} \sum_{0<|r|\ll \frac{M}{T_0} T^{\varepsilon_1}} \left(\frac{T}{T_0}\right)^{C} M^{1/2+\varepsilon_0} \ll T^{\frac{\varepsilon}{8}} \left(\frac{T}{M}\right) \left(\frac{T}{T_0}\right)^{C} M^{1/2+\varepsilon_0} \left(\frac{M}{T_0} T^{\varepsilon_1}\right),$$

which is $\ll T^{\varepsilon/4}(T/T_0)^{1+C}M^{1/2}$. Hence,

$$\sum_{\substack{M\asymp N\\ MN\ll U^{4+\varepsilon_2}}} \mathcal{E}_{M,N} \ll \sum_{\substack{M\asymp N\\ MN\ll U^{4+\varepsilon_2}}} T^{\varepsilon/4}\left(\frac{T}{T_0}\right)^{1+C} M^{1/2} \ll T^{\varepsilon/4}\left(\frac{T}{T_0}\right)^{1+C} \sum_{\substack{M\ll U^{2+\varepsilon_2/2}\\ N\asymp M}} M^{1/2},$$

which is

$$\ll T^{\varepsilon/4}\left(\frac{T}{T_0}\right)^{1+C}(U^{2+\varepsilon_2/2})^{1/2} = T^{\varepsilon/4}\left(\frac{T}{T_0}\right)^{1+C} U^{1+\varepsilon_2/4}.$$

Finally, as $\varepsilon_2 \le \varepsilon$, it is clear that

$$U^{1+\varepsilon_2/4} = (T^{1-\varepsilon})^{1+\varepsilon_2/4} = T^{1-\varepsilon+\varepsilon_2/4-\varepsilon\varepsilon_2/4} \le T^{1-3\varepsilon/4},$$

and thus

$$\sum_{\substack{M,N\\ M\asymp N,\, MN\ll U^{4+\varepsilon_2}}} \mathcal{E}_{M,N} \ll T^{\varepsilon/4}\left(\frac{T}{T_0}\right)^{1+C} T^{1-3\varepsilon/4} \le \left(\frac{T}{T_0}\right)^{1+C} T^{1-\varepsilon/2}$$

as desired. ∎

6.3. *Expanding the range of* $|r|$

Recall that by Lemma 6.3, equation (6.9) yields

$$I_O = \sum_{\substack{M,N\\ M\asymp N,\, MN\ll U^{4+\varepsilon_2}\\ M,N\gg T_0^{1-\varepsilon_1}}} \tilde{I}_{M,N} + O\left(\left(\frac{T}{T_0}\right)^{1+C} T^{1-\varepsilon/2}\right),$$

where $\tilde{I}_{M,N}$ is defined in (6.10). Note that for the s-integral of $\tilde{I}_{M,N}$ (see (6.7)), we can move the integration from the line $\Re e(s) = \varepsilon_1$ to $\Re e(s) = \varepsilon_3$, where $0 < \delta < \varepsilon_3 < 0.15$, without encountering any poles. (We remind the reader that $\delta \in (0, \frac{1}{10})$ satisfies $\frac{1}{100} < r_1, r_2 \le \delta$, where r_1, r_2 are the same as in Conjecture 1.) To handle the sum above, we first control $\tilde{I}_{M,N}$, defined in (6.10), by expanding the range of the sum in (6.10) to $0 < |r| \le R_0$ with

$$R_0 = T^5.$$

Observe that from (6.1) and (6.7), it follows that the x-integral in (6.11) can be written as

$$i_{M,N,r} = \int_{\max(0,r)}^{\infty} W\left(\frac{x}{M}\right) W\left(\frac{x-r}{N}\right) f^*(x, x-r) x^{z_1-1}(x-r)^{z_2-1}\, dx. \tag{6.13}$$

Since W is supported in $[1, 2]$, we only need to consider the situation that the variable x in (6.13) lies in $(M, 2M) \cap (N + r, 2N + r)$. It is clear that if such an open interval is empty, then $i_{M,N,r}=0$. Without loss of generality, we consider the case $r\geq 1$. Note that the condition $x>N+r$ gives $x-r\geq N\geq 2^{-1/2}$, which allows us to have $x\geq r+2^{-1/2}$. Also, $N+r\leq 2M$ (because $(M,2M)\cap(N+r,2N+r)$ is empty otherwise). This and assumption (6.5) give $r\leq 2M-N\leq 2M-M/3=5M/3$. In light of these observations, applying (6.3), we conclude that for $r\gg \frac{M}{T_0}T_0^{\varepsilon_1}$ and j sufficiently large,

$$f^*(x,x-r)\ll \frac{T^{4\varepsilon_3}}{\left|\log(\frac{x}{x-r})T_0\right|^j}\leq \frac{T^{4\varepsilon_3}}{\left|\log(\frac{2M}{2M-r})T_0\right|^j}$$

as $x\in[r+2^{-1/2},2M]$. Thus, since $|\log(1-x)|\gg x$ for $x\in(0,5/6]$, we arrive at

$$f^*(x,x-r)\ll \frac{T^{4\varepsilon_3}}{\left|\log(1-\frac{r}{2M})T_0\right|^j}\ll \frac{T^{4\varepsilon_3}}{\left|\frac{r}{2M}T_0\right|^j}\ll T^{4\varepsilon_3-j\varepsilon_1}\ll T^{-B}. \tag{6.14}$$

Using (6.13) and (6.14), we then deduce $i_{M,N,r}\ll M^{1+2\delta}T^{-B}$ for $r\gg\frac{M}{T_0}T^{\varepsilon_1}$. Arguing similarly, we also have $i_{M,N,r}\ll M^{1+2\delta}T^{-B}$ for r such that $-r\gg\frac{M}{T_0}T^{\varepsilon_1}$.

Note that we are trying to bound

$$\sum_{\substack{M,N\\ M\asymp N,\, MN\ll U^{4+\varepsilon_2}\\ M,N\gg T_0^{1-\varepsilon_1}}}\ \sum_{\frac{M}{T_0}T_0^{\varepsilon_1}\ll|r|\leq R_0}\frac{2T}{\sqrt{MN}}|\tilde{D}_{f_r}(r)|.$$

From (6.11), (6.14), and Lemma 2.3, it follows that

$$\begin{aligned}|\tilde{D}_{f_r}(r)|&\ll\int_{\mathcal{B}_1}\int_{\mathcal{B}_2}\frac{1}{(r_1r_2)^4}\sum_{q=1}^{\infty}\frac{(q,r)(32^{\omega(q)}\tau_4(q))^2q^{r_1+r_2}}{q^{2-r_1-r_2}}\\&\quad\times\int_{\max(0,r)}^{2M}T^{-B}|x|^{\Re e(z_1)-1}|x-r|^{\Re e(z_2)-1}\,dx\,|dz_2|\,|dz_1|\\&\ll\frac{1}{(r_1r_2)^4}(r_1r_2)\sum_{q=1}^{\infty}\frac{(q,r)32^{2\omega(q)}\tau_4(q)^2}{q^{2-2(r_1+r_2)}}T^{-B}M^{1+r_1+r_2}\\&\ll\tau_2(r)T^{-B}M^{1+r_1+r_2}\end{aligned}\tag{6.15}$$

as long as $r_1+r_2\leq 1/4$ and r_1,r_2 are bounded away from zero. (Here we use the bounds $\sum_{q=1}^{\infty}\frac{(q,r)}{q^c}\leq\tau_2(r)\zeta(c)$ for $c>1$, and $\omega(q)\ll\frac{\log q}{\log\log q}+1$.) Thus, we have

$$\begin{aligned}&\sum_{\substack{M,N\\ M\asymp N,\, MN\ll U^{4+\varepsilon_2}}}\ \sum_{\frac{M}{T_0}T_0^{\varepsilon_1}\ll|r|\leq R_0}\frac{2T}{\sqrt{MN}}\tau_2(r)T^{-B}M^{1+r_1+r_2}\\&\qquad\ll T^{1-B}\sum_{\substack{M,N\\ M\asymp N,\, MN\ll U^{4+\varepsilon_2}}}M^{1+r_1+r_2}(R_0\log R_0)\\&\qquad\ll T^{1-B}(R_0\log R_0)(U^{2+\varepsilon_2/2})^{1+r_1+r_2}\\&\qquad\ll T^{1-B}R_0T^{(2+\varepsilon_2/2)(1+r_1+r_2)}\log T\ll T^{1+5+(2+\varepsilon_2/2)(1+r_1+r_2)-B}\log T.\end{aligned}$$

So, if we choose $B \geq 2 + 5 + (2 + \varepsilon_2/2)(1 + r_1 + r_2)$, then we have an error term that is $O((\log T)/T)$. Hence, we have established

$$I_O = \sum_{\substack{M,N \\ M \asymp N,\, MN \ll U^{4+\varepsilon_2} \\ M,N \gg T_0^{1-\varepsilon_1}}} \frac{2T}{\sqrt{MN}} \sum_{0<|r|\leq R_0} \tilde{D}_{f_r}(r) + O\left(\left(\frac{T}{T_0}\right)^{1+C} T^{1-\varepsilon/2}\right). \quad (6.16)$$

6.4. *Making M and N unrestricted*

In this section, we shall remove the conditions on M and N from the sum in (6.16) by showing that $i_{M,N,r}$ is small for M and N not satisfying those conditions. Firstly, for $M \not\asymp N$, we know $N > 3M$ or $N < M/3$. Without loss of generality, we consider the case that $N > 3M$. From the support of W, in (6.13), we only need to consider x satisfying $M \leq x \leq 2M$ and $N \leq x - r \leq 2N$, which tell us $|\log(\frac{x}{x-r})| = \log(\frac{x-r}{x}) \geq \log(\frac{N}{2M}) \geq \log(\frac{3}{2})$. Hence, by (6.3), (6.13) and the bound $|z_i - 1| < \frac{1}{10}$, we deduce

$$\begin{aligned} i_{M,N,r} &\ll_j \int_M^{2M} \frac{T^{4\varepsilon_3}}{T_0^j} (MN)^{\frac{1}{10}}\, dx \ll \frac{T^{4\varepsilon_3}}{T_0^j} M(MN)^{\frac{1}{10}} \ll \frac{T^{4\varepsilon_3}}{T_0^j} \frac{MN}{T_0^{1-\varepsilon_1}} (MN)^{\frac{1}{10}} \\ &\ll \frac{T^{4\varepsilon_3 + \frac{11}{10}(1-\varepsilon)(4+\varepsilon_2)}}{T_0^{j+1-\varepsilon_1}}, \end{aligned}$$

where we have used the condition $N \gg T_0^{1-\varepsilon_1}$ (resp. $MN \ll U^{4+\varepsilon_2} = T^{(1-\varepsilon)(4+\varepsilon_2)}$) for the third (resp. last) estimate. Using (iii), we can choose j large enough that $i_{M,N,r} \ll T^{-B}$. Therefore, for M and N with $M \not\asymp N$, arguing as in the previous section we obtain $\frac{2T}{\sqrt{MN}} \sum_{0<|r|\leq R_0} \tilde{D}_{f_r}(r) \ll (\log T)^{-2}$.

Secondly, we deal with the situation that either M or $N \ll T_0^{1-\varepsilon_1}$. We may suppose, without loss of generality, that $M, N \geq 2^{-1/2}$, $M \ll T_0^{1-\varepsilon_1}$, and $r \geq 1$. By the support of W, in (6.13), we can require that $x \geq N + r > r \geq 1$, which implies $|\log(\frac{x}{x-r})| \geq \log(\frac{x}{x-1}) > x^{-1} \geq (2M)^{-1} \gg T_0^{\varepsilon_1 - 1}$. Thus, by (6.3), for any $B > 0$, we derive

$$f^*(x, x-r) \ll T^{4\varepsilon_3} T_0^{-j(\varepsilon_1-1)-j} \ll T^{-B}$$

when j is sufficiently large. With this bound in hand, we again have

$$\frac{2T}{\sqrt{MN}} \sum_{0<|r|\leq R_0} \tilde{D}_{f_r}(r) \ll (\log T)^{-2}.$$

From the above discussion, if we include in the sum in (6.16) all M and N with $MN \ll U^{4+\varepsilon_2}$ such that either $M \not\asymp N$ or $\min(M, N) \ll T_0^{1-\varepsilon_1}$, we will introduce an extra error, which is at most $(\log T)(\log T)(\log T)^{-2} \ll 1$ and thus negligible because the choices for each such M and N are $O(\log T)$. In other words, we have removed the conditions $M \asymp N$ and $M, N \gg T_0^{1-\varepsilon_1}$ from the first sum in (6.16) upon a negligible error.

Lastly, we consider the contribution of $\widetilde{\Delta}$ arising from adding those pairs (M, N) with $MN \gg U^{4+\varepsilon_2}$ in (6.16). Using (6.2) and Lemma 2.2 (iii), for any constant $A > 0$, we obtain $f^*(x, y) \ll (\frac{U^4}{xy})^A$ if $xy > U^4$. Similar to (6.15), for any $A > 0$,

$$\widetilde{\Delta} \ll \sum_{r\neq 0} \tau_2(|r|) \sum_{\substack{M,N \\ MN \gg U^{4+\varepsilon_2} \\ M-2N<r<2M-N}} \frac{T}{\sqrt{MN}} \int_{\max(M,N+r,r,0)}^{\min(2M,2N+r)} \left(\frac{U^4}{x(x-r)}\right)^A (MN)^{\frac{1}{10}}\, dx$$
$$\ll T \sum_{r\neq 0} \tau_2(|r|) \sum_{\substack{M,N \\ MN \gg \max(U^{4+\varepsilon_2},|r|)}} (MN)^{-1/2+1/10}\left(\frac{U^4}{MN}\right)^A \min(M, N)$$

provided that $r_1 + r_2 \leq 1/4$ and r_1, r_2 are bounded away from zero. (The condition $MN \gg |r|$ in the last sum above is due to the condition $M - 2N < r < 2M - N$ which implies $|r| \ll |M + N| \ll MN$.) To estimate the last sum over M, N, we may consider $MN = H$, which forces M and N to satisfy $\min(M, N) \leq H^{1/2}$. Clearly, there is an integer $h = h(H) \geq 0$ such that $H = 2^{(h-2)/2} \gg U^{4+\varepsilon_2}$. Also, we observe that for every given H, we can have at most $h(H) + 1 \ll \log H$ pairs M, N satisfying $MN = H$. Hence, we arrive at

$$\widetilde{\Delta} \ll T \sum_{H \gg U^{4+\varepsilon_2}} H^{-1/2+1/10}\left(\frac{U^4}{H}\right)^A H^{1/2}(\log H) \sum_{1\leq r\ll H} \tau_2(r)$$
$$\ll T \sum_{H \gg U^{4+\varepsilon_2}} H^2\left(\frac{U^4}{H}\right)^A.$$

For $A > 2$, the last sum is convergent and is $\ll U^{2(4+\varepsilon_2)-A\varepsilon_2} = T^{(1-\varepsilon)(2(4+\varepsilon_2)-A\varepsilon_2)}$. Therefore, $\widetilde{\Delta} \ll_A 1$ for any $A \geq \frac{1}{(1-\varepsilon)\varepsilon_2} + \frac{8}{\varepsilon_2} + 2$.

To summarise, we have removed all the restrictions on M and N appearing in (6.16). More precisely, we have shown

$$I_O = \sum_{M,N} \frac{2T}{\sqrt{MN}} \sum_{0<|r|\leq R_0} \tilde{D}_{f_r}(r) + O\left(\left(\frac{T}{T_0}\right)^{1+C} T^{1-\varepsilon/2}\right). \tag{6.17}$$

6.5. A smooth partition of unity, again

Recalling $W(x) = x^{-1/2}W_0(x)$, we see that

$$\sum_{M,N} \frac{1}{\sqrt{MN}} W\left(\frac{x}{M}\right) W\left(\frac{x-r}{N}\right)$$
$$= \sum_{M,N} \frac{1}{\sqrt{MN}} \left(\frac{x}{M}\right)^{-1/2}\left(\frac{x-r}{N}\right)^{-1/2} W_0\left(\frac{x}{M}\right) W_0\left(\frac{x-r}{N}\right)$$
$$= x^{-1/2}(x-r)^{-1/2} \sum_{M,N} W_0\left(\frac{x}{M}\right) W_0\left(\frac{x-r}{N}\right),$$

which is equal to $x^{-1/2}(x-r)^{-1/2}$ as $\sum_{k\in\mathbb{Z}} W_0(x/2^{k/2}) = 1$. From (6.7) and the above identity, inserting (6.11) into (6.17), we have

$$I_O = 2T \sum_{1\le |r|\le R_0} \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathcal{B}_1}\int_{\mathcal{B}_2} \zeta(z_1)^4\zeta(z_2)^4 \sum_{q=1}^{\infty} \frac{c_q(r)G_4(z_1,q)G_4(z_2,q)}{q^{z_1+z_2}}$$
$$\times \int_{\max(0,r)}^{\infty} f^*(x,x-r)x^{z_1-3/2}(x-r)^{z_2-3/2}\,dx\,dz_2\,dz_1 + O\left(\left(\frac{T}{T_0}\right)^{1+C} T^{1-\varepsilon/2}\right),$$

where

$$f^*(x,x-r) = \frac{1}{2\pi \mathrm{i}} \int_{(\varepsilon_3)} \frac{G(s)}{s}\left(\frac{1}{\pi^4 x(x-r)}\right)^s \frac{1}{T}\int_{-\infty}^{\infty}\left(1+\frac{r}{x-r}\right)^{-\mathrm{i}t}$$
$$\times g(s,t)\left(\frac{U}{t}\right)^{4s}\omega(t)\,dt\,ds,$$

Hence,

$$I_O = 2 \sum_{1\le |r|\le R_0} \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathcal{B}_1}\int_{\mathcal{B}_2} \zeta(z_1)^4\zeta(z_2)^4 \sum_{q=1}^{\infty} \frac{c_q(r)G_4(z_1,q)G_4(z_2,q)}{q^{z_1+z_2}}$$
$$\times \int_{\max(0,r)}^{\infty} x^{z_1-3/2}(x-r)^{z_2-3/2} \frac{1}{2\pi \mathrm{i}} \int_{(\varepsilon_3)} \frac{G(s)}{s}\left(\frac{1}{\pi^4 x(x-r)}\right)^s \int_{-\infty}^{\infty}\left(1+\frac{r}{(x-r)}\right)^{-\mathrm{i}t}$$
$$\times g(s,t)\left(\frac{U}{t}\right)^{4s}\omega(t)\,dt\,ds\,dx\,dz_2\,dz_1 + O\left(\left(\frac{T}{T_0}\right)^{1+C} T^{1-\varepsilon/2}\right).$$

We now split this sum, according to $r>0$ and $r<0$, to obtain

$$I_O = 2 \sum_{1\le r\le R_0} \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathcal{B}_1}\int_{\mathcal{B}_2} \zeta(z_1)^4\zeta(z_2)^4 \sum_{q=1}^{\infty} \frac{c_q(r)G_4(z_1,q)G_4(z_2,q)}{q^{z_1+z_2}}$$
$$\times \int_{\max(0,r)}^{\infty} x^{z_1-3/2}(x-r)^{z_2-3/2} \frac{1}{2\pi \mathrm{i}} \int_{(\varepsilon_3)} \frac{G(s)}{s}\left(\frac{1}{\pi^4 x(x-r)}\right)^s \int_{-\infty}^{\infty}\left(1+\frac{r}{(x-r)}\right)^{-\mathrm{i}t}$$
$$\times g(s,t)\left(\frac{U}{t}\right)^{4s}\omega(t)\,dt\,ds\,dx\,dz_2\,dz_1$$
$$+ 2 \sum_{1\le r\le R_0} \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathcal{B}_1}\int_{\mathcal{B}_2} \zeta(z_1)^4\zeta(z_2)^4 \sum_{q=1}^{\infty} \frac{c_q(-r)G_4(z_1,q)G_4(z_2,q)}{q^{z_1+z_2}}$$
$$\times \int_{\max(0,-r)}^{\infty} x^{z_1-3/2}(x+r)^{z_2-3/2} \frac{1}{2\pi \mathrm{i}} \int_{(\varepsilon_3)} \frac{G(s)}{s}\left(\frac{1}{\pi^4 x(x+r)}\right)^s \int_{-\infty}^{\infty}\left(1+\frac{-r}{(x+r)}\right)^{-\mathrm{i}t}$$
$$\times g(s,t)\left(\frac{U}{t}\right)^{4s}\omega(t)\,dt\,ds\,dx\,dz_2\,dz_1 + O\left(\left(\frac{T}{T_0}\right)^{1+C} T^{1-\varepsilon/2}\right).$$

In the x-integrals, we make the change of variable $x = ry$ to deduce

$$I_O = 2 \sum_{1\le r\le R_0} \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathscr{B}_1}\int_{\mathscr{B}_2} \zeta(z_1)^4\zeta(z_2)^4 \sum_{q=1}^{\infty} \frac{c_q(r)G_4(z_1,q)G_4(z_2,q)}{q^{z_1+z_2}} r^{z_1+z_2-3}$$
$$\times \int_{\max(0,1)}^{\infty} y^{z_1-3/2}(y-1)^{z_2-3/2} \frac{1}{2\pi \mathrm{i}} \int_{(\varepsilon_3)} \frac{G(s)}{s}\frac{1}{r^{2s}}\left(\frac{1}{\pi^4 y(y-1)}\right)^s \int_{-\infty}^{\infty}\left(\frac{y-1}{y}\right)^{\mathrm{i}t}$$
$$\times g(s,t)\left(\frac{U}{t}\right)^{4s}\omega(t)\,dt\,ds\,(rdy)\,dz_2\,dz_1$$
$$+ 2 \sum_{1\le r\le R_0} \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathscr{B}_1}\int_{\mathscr{B}_2} \zeta(z_1)^4\zeta(z_2)^4 \sum_{q=1}^{\infty} \frac{c_q(r)G_4(z_1,q)G_4(z_2,q)}{q^{z_1+z_2}} r^{z_1+z_2-3}$$
$$\times \int_{\max(0,-1)}^{\infty} y^{z_1-3/2}(y+1)^{z_2-3/2} \frac{1}{2\pi \mathrm{i}} \int_{(\varepsilon_3)} \frac{G(s)}{s}\frac{1}{r^{2s}}\left(\frac{1}{\pi^4 y(y+1)}\right)^s \int_{-\infty}^{\infty}\left(\frac{y+1}{y}\right)^{\mathrm{i}t}$$
$$\times g(s,t)\left(\frac{U}{t}\right)^{4s}\omega(t)\,dt\,ds\,(r\,dy)\,dz_2\,dz_1 + O\left(\left(\frac{T}{T_0}\right)^{1+C} T^{1-\varepsilon/2}\right).$$

It follows that $I_O = I^+ + I^- + O((T/T_0)^{1+C}T^{1-\varepsilon/2})$ where

$$I^{\pm} = \frac{2}{(2\pi \mathrm{i})^2} \sum_{1\le r\le R_0} \int_{\mathscr{B}_1}\int_{\mathscr{B}_2} \zeta(z_1)^4\zeta(z_2)^4 \sum_{q=1}^{\infty} \frac{c_q(r)G_4(z_1,q)G_4(z_2,q)}{q^{z_1+z_2}} r^{z_1+z_2-2}$$
$$\times \int_{\max(0,\pm 1)}^{\infty} y^{z_1-\,3/2}(y\mp 1)^{z_2-3/2} \frac{1}{2\pi \mathrm{i}} \int_{(\varepsilon_3)} \frac{G(s)}{s}\frac{1}{r^{2s}}\left(\frac{1}{\pi^4 y(y\mp 1)}\right)^s \int_{-\infty}^{\infty}\left(\frac{y\mp 1}{y}\right)^{\mathrm{i}t}$$
$$\times g(s,t)\left(\frac{U}{t}\right)^{4s}\omega(t)\,dt\,ds\,dy\,dz_2\,dz_1.$$

We see

$$I^{\pm} = \frac{2}{(2\pi \mathrm{i})^3} \int_{\mathscr{B}_1}\int_{\mathscr{B}_2}\int_{(\varepsilon_3)} \zeta(z_1)^4\zeta(z_2)^4 \sum_{1\le r\le R_0}\sum_{q=1}^{\infty} \frac{c_q(r)G_4(z_1,q)G_4(z_2,q)}{q^{z_1+z_2} r^{2s+2-z_1-z_2}}$$
$$\times \frac{G(s)}{s}\frac{1}{\pi^{4s}} \int_{-\infty}^{\infty} g(s,t)\left(\frac{U}{t}\right)^{4s}\omega(t)\left(\int_{\max(0,\pm 1)}^{\infty} y^{z_1-s-3/2-\mathrm{i}t}(y\mp 1)^{z_2-s-3/2+\mathrm{i}t}\,dy\right)$$
$$\times dt\,ds\,dz_2\,dz_1.$$

For $\eta = \pm 1$, we have

$$\int_{\max(0,\pm 1)}^{\infty} y^{z_1-s-3/2-\mathrm{i}t}(y\mp 1)^{z_2-s-3/2+\mathrm{i}t}\,dy$$
$$= \int_{\max(0,\eta)}^{\infty} y^{z_1-s-3/2-\mathrm{i}t}(y-\eta)^{z_2-s-3/2+\mathrm{i}t}\,dy,$$

which equals

$$\begin{cases} \mathrm{B}\big(z_2 - s - \frac{1}{2} + \mathrm{i}t, -z_1 - z_2 + 2s + 2\big) & \text{if } \eta = 1, \\ \mathrm{B}\big(z_1 - s - \frac{1}{2} - \mathrm{i}t, -z_1 - z_2 + 2s + 2\big) & \text{if } \eta = -1, \end{cases}$$
$$= \begin{cases} \dfrac{\Gamma\big(z_2 - s - \frac{1}{2} + \mathrm{i}t\big)\Gamma\big(-z_1 - z_2 + 2s + 2\big)}{\Gamma\big(s - z_1 + \frac{3}{2} + \mathrm{i}t\big)} & \text{if } \eta = 1, \\ \dfrac{\Gamma\big(z_1 - s - \frac{1}{2} - \mathrm{i}t\big)\Gamma\big(-z_1 - z_2 + 2s + 2\big)}{\Gamma\big(s - z_2 + \frac{3}{2} - \mathrm{i}t\big)} & \text{if } \eta = -1. \end{cases}$$

Here, we have used the following identities for the Beta function B:

$$\int_0^\infty \frac{x^{u-1}}{(1+x)^v}\, dx = \mathrm{B}(u, v-u) = \frac{\Gamma(u)\Gamma(v-u)}{\Gamma(v)}$$

for $\Re e(u) > 0$ and $\Re e(v-u) > 0$. Therefore,

$$\begin{aligned} I_O = {} & \frac{2}{(2\pi \mathrm{i})^3} \int_{\mathscr{B}_1} \int_{\mathscr{B}_2} \int_{(\varepsilon_3)} \zeta(z_1)^4 \zeta(z_2)^4 \sum_{1 \le r \le R_0} \sum_{q=1}^{\infty} \frac{c_q(r) G_4(z_1, q) G_4(z_2, q)}{q^{z_1+z_2} r^{2s+2-z_1-z_2}} \frac{G(s)}{s\pi^{4s}} \\ & \times \int_{-\infty}^{\infty} g(s,t) \Big(\frac{U}{t}\Big)^{4s} \omega(t) \Gamma(-z_1 - z_2 + 2s + 2) \\ & \times \left(\frac{\Gamma\big(z_2 - s - \frac{1}{2} + \mathrm{i}t\big)}{\Gamma\big(s - z_1 + \frac{3}{2} + \mathrm{i}t\big)} + \frac{\Gamma\big(z_1 - s - \frac{1}{2} - \mathrm{i}t\big)}{\Gamma\big(s - z_2 + \frac{3}{2} - \mathrm{i}t\big)} \right) dt\, ds\, dz_2\, dz_1 \\ & + O\bigg(\Big(\frac{T}{T_0}\Big)^{1+C} T^{1-\varepsilon/2} \bigg). \end{aligned} \tag{6.18}$$

6.6. *Moving the integral right to $\Re e(s) = 1$*

We now move the s-integral in (6.18) right to the line $\Re e(s) = 1$. We do this so that we can apply Proposition 6.5 (i) later. Observe that the sum in the round bracket in (6.18) has simple poles at $s = \mathrm{p}_1 = -1/2 + z_1 - it$ and $s = \mathrm{p}_2 = -1/2 + z_2 + it$. Using $\Gamma(z) \sim z^{-1}$ as $z \to 0$, we find that the residue at $s = \mathrm{p}_1$ (for the s-integral) is

$$-\frac{G(\mathrm{p}_1)}{\mathrm{p}_1 \pi^{4\mathrm{p}_1} r^{2\mathrm{p}_1}} g(\mathrm{p}_1, t) \Big(\frac{U}{t}\Big)^{4\mathrm{p}_1} \frac{\Gamma(-z_1 - z_2 + 2\mathrm{p}_1 + 2)}{\Gamma\big(\mathrm{p}_1 - z_2 + \frac{3}{2} - \mathrm{i}t\big)},$$

and the residue at $s = \mathrm{p}_2$ is

$$-\frac{G(\mathrm{p}_2)}{\mathrm{p}_2 \pi^{4\mathrm{p}_2} r^{2\mathrm{p}_2}} g(\mathrm{p}_2, t) \Big(\frac{U}{t}\Big)^{4\mathrm{p}_2} \frac{\Gamma(-z_1 - z_2 + 2\mathrm{p}_2 + 2)}{\Gamma\big(\mathrm{p}_2 - z_1 + \frac{3}{2} + \mathrm{i}t\big)}.$$

Inserting $\mathrm{p}_1 = -1/2 + z_1 - it$ and $\mathrm{p}_2 = -1/2 + z_2 + it$ into the above residues, respectively, we see that the residue at p_i is

$$-\frac{G(\mathrm{p}_i)}{\mathrm{p}_i \pi^{4\mathrm{p}_i} r^{2\mathrm{p}_i}} g(\mathrm{p}_i, t) \Big(\frac{U}{t}\Big)^{4\mathrm{p}_i}.$$

By Lemma 2.2 (i), when $c_1 T \le t \le c_2 T$, the above is

$$\ll \frac{|G(\mathrm{p}_i)|}{\sqrt{|\Im\mathfrak{m}(z_i) + (-1)^i t|^2 + 1}} r^{1-2\Re\mathfrak{e}(z_i)} \left(\frac{U}{2}\right)^{-2+4\Re\mathfrak{e}(z_i)} \frac{(|\Im\mathfrak{m}(z_i) + (-1)^i t|)^2 + 1}{t}.$$

As $G(\mathrm{p}_i) \ll (|\Im\mathfrak{m}(z_i) + (-1)^i t|)^2 + 1)^{-B}$, we conclude that the contribution of each residue is at most

$$\sum_{1 \le r \le R_0} \frac{|\zeta(z_1)\zeta(z_2)|^4}{r^{2-\Re\mathfrak{e}(z_1)-\Re\mathfrak{e}(z_2)}} \sum_{q=1}^{\infty} \frac{|c_q(r) G_4(z_1,q) G_4(z_2,q)|}{q^{\Re\mathfrak{e}(z_1)+\Re\mathfrak{e}(z_2)}}$$
$$\times \int_{c_1 T}^{c_2 T} \left(\frac{T}{2}\right)^{-2+4\Re\mathfrak{e}(z_i)} \frac{((|\Im\mathfrak{m}(z_i) + (-1)^i t|)^2 + 1)^{-B}}{t} r^{1-2\Re\mathfrak{e}(z_i)}\, dt.$$

The sum over q can be bounded in a similar manner to (6.11) and (6.15). Therefore the above is

$$\ll T^{3-B} \sum_{1 \le r \le R_0} \frac{\tau_2(r)}{r^{1-r_1-r_2}} \ll T^{3-B} R_0 = T^{8-B}.$$

Choosing B to be large, we then obtain

$$\begin{aligned} I_O = {} & \frac{2}{(2\pi \mathrm{i})^3} \int_{\mathcal{B}_1} \int_{\mathcal{B}_2} \int_{(1)} \zeta(z_1)^4 \zeta(z_2)^4 \sum_{1 \le r \le R_0} \sum_{q=1}^{\infty} \frac{c_q(r) G_4(z_1,q) G_4(z_2,q)}{q^{z_1+z_2} r^{2s+2-z_1-z_2}} \frac{G(s)}{s\pi^{4s}} \\ & \times \int_{-\infty}^{\infty} g(s,t) \left(\frac{U}{t}\right)^{4s} \omega(t) \Gamma(-z_1 - z_2 + 2s + 2) \\ & \times \left(\frac{\Gamma\big(z_2 - s - \frac{1}{2} + \mathrm{i}t\big)}{\Gamma\big(s - z_1 + \frac{3}{2} + \mathrm{i}t\big)} + \frac{\Gamma\big(z_1 - s - \frac{1}{2} - \mathrm{i}t\big)}{\Gamma\big(s - z_2 + \frac{3}{2} - \mathrm{i}t\big)} \right) dt\, ds\, dz_2\, dz_1 \\ & + O(1) + O\left(\left(\frac{T}{T_0}\right)^{1+C} T^{1-\varepsilon/2} \right). \end{aligned} \tag{6.19}$$

6.7. *Extending $R_0 \to \infty$*

In this section, we shall extend R_0 to ∞ for the sum in (6.19), which requires a delicate analysis of the Gamma factors involved. To proceed, we shall invoke the following technical lemma (whose proof can be found in [11, Lemma 4.6]).

Lemma 6.4. *Let $|a_{i_1}|, |b_{i_2}| \le \delta$. Let $\eta_1 \in (0, 1/2)$ and $A > 0$. Assume*

$$\Re\mathfrak{e}(a_{i_1} + s_1) \in [0, A] \quad \textit{and} \quad \Re\mathfrak{e}(b_{i_2} + s_2) \in \left[0, \tfrac{1}{2} - \eta_1\right] \cup \left[\tfrac{1}{2} + \eta_1, \tfrac{3}{2} - \eta_1\right].$$

(i) *Assume* $\Re\mathfrak{e}(s_1+s_2+a_{i_1}+b_{i_2})\le 1$. *When* $|\Im\mathfrak{m}(s_1)|\le t+1$ *and* $|\Im\mathfrak{m}(s_2)|\le t+1$, *we have*

$$\frac{\Gamma\left(\frac{1}{2}-b_{i_2}-s_2\pm \mathrm{i}t\right)}{\Gamma\left(\frac{1}{2}+a_{i_1}+s_1\pm \mathrm{i}t\right)}=t^{-(s_1+s_2+a_{i_1}+b_{i_2})}\exp\left(\mp\mathrm{i}\frac{\pi}{2}(s_1+s_2+a_{i_1}+b_{i_2})\right)\times\left(1+O\left(\frac{1+|s_1|^2+|s_2|^2}{t}\right)\right).$$

(ii) *When* $|\Im\mathfrak{m}(s_1)|\ge t+1$ *or* $|\Im\mathfrak{m}(s_2)|\ge t+1$, *we have*

$$\frac{\Gamma\left(\frac{1}{2}-b_{i_2}-s_2+\mathrm{i}t\right)}{\Gamma\left(\frac{1}{2}+a_{i_1}+s_1+\mathrm{i}t\right)}\ll\frac{\Im\mathfrak{m}(s_1)^2+\Im\mathfrak{m}(s_2)^2}{t^2}e^{\frac{\pi}{2}|\Im\mathfrak{m}(s_1+s_2)|}.$$

(iii) *Assume* $\Re\mathfrak{e}(s_1+s_2+a_{i_1}+b_{i_2})\ge 1$. *When* $|\Im\mathfrak{m}(s_1)|\le t+1$ *and* $|\Im\mathfrak{m}(s_2)|\le t+1$, *we have*

$$\frac{\Gamma\left(\frac{1}{2}-b_{i_2}-s_2\pm \mathrm{i}t\right)}{\Gamma\left(\frac{1}{2}+a_{i_1}+s_1\pm \mathrm{i}t\right)}\ll\frac{\Im\mathfrak{m}(s_1)^2+\Im\mathfrak{m}(s_2)^2}{t^2}e^{\pm\frac{\pi}{2}(\Im\mathfrak{m}(s_1+s_2))}.$$

Now, we start by recalling that $\frac{1}{100}<r_1,r_2\le\delta<\frac{1}{10}$ (where r_1,r_2 are as in Conjecture 1) and observing $\Re\mathfrak{e}(s-z_i+1)\in[1-\delta,1+\delta]\subseteq[0.9,1.1]$, $i=1,2$, because $|z_i-1|=r_i$. This implies that $\Re\mathfrak{e}(s-z_1+1+s-z_2+1)\ge 2-2\delta\ge 1$. For $|\Im\mathfrak{m}(s-z_i+1)|\le t+1$, $i=1,2$, by Lemma 6.4 (iii), it follows that

$$\frac{\Gamma\left(z_2-s-\frac{1}{2}+\mathrm{i}t\right)}{\Gamma\left(s-z_1+\frac{3}{2}+\mathrm{i}t\right)}=\frac{\Gamma\left(\frac{1}{2}-(s-z_2+1)+\mathrm{i}t\right)}{\Gamma\left(\frac{1}{2}+(s-z_1+1)+\mathrm{i}t\right)}\ll\frac{(\Im\mathfrak{m}(s-z_1+1))^2+(\Im\mathfrak{m}(s-z_2+1))^2}{t^2}e^{\frac{\pi}{2}\Im\mathfrak{m}(2s-z_1-z_2)},$$

and

$$\frac{\Gamma\left(z_1-s-\frac{1}{2}-\mathrm{i}t\right)}{\Gamma\left(s-z_2+\frac{3}{2}-\mathrm{i}t\right)}=\frac{\Gamma\left(\frac{1}{2}-(s-z_1+1)-\mathrm{i}t\right)}{\Gamma\left(\frac{1}{2}+(s-z_2+1)-\mathrm{i}t\right)}\ll\frac{(\Im\mathfrak{m}(s-z_1+1))^2+(\Im\mathfrak{m}(s-z_2+1))^2}{t^2}e^{-\frac{\pi}{2}\Im\mathfrak{m}(2s-z_1-z_2)}.$$

Therefore,

$$\frac{\Gamma\left(z_2-s-\frac{1}{2}+\mathrm{i}t\right)}{\Gamma\left(s-z_1+\frac{3}{2}+\mathrm{i}t\right)}+\frac{\Gamma\left(z_1-s-\frac{1}{2}-\mathrm{i}t\right)}{\Gamma\left(s-z_2+\frac{3}{2}-\mathrm{i}t\right)}\ll\frac{(\Im\mathfrak{m}(s-z_1+1))^2+(\Im\mathfrak{m}(s-z_2+1))^2}{t^2}e^{\frac{\pi}{2}|\Im\mathfrak{m}(2s-z_1-z_2)|}. \tag{6.20}$$

On the other hand, assuming either $|\mathfrak{Im}(s-z_1+1)| > t+1$ or $|\mathfrak{Im}(s-z_2+1)| > t+1$, by Lemma 6.4 (ii) we also have

$$\frac{\Gamma\big(z_2-s-\frac{1}{2}+\mathrm{i}t\big)}{\Gamma\big(s-z_1+\frac{3}{2}+\mathrm{i}t\big)}+\frac{\Gamma\big(z_1-s-\frac{1}{2}-\mathrm{i}t\big)}{\Gamma\big(s-z_2+\frac{3}{2}-\mathrm{i}t\big)} \ll \frac{(\mathfrak{Im}(s-z_1+1))^2+(\mathfrak{Im}(s-z_2+1))^2}{t^2}e^{\frac{\pi}{2}|\mathfrak{Im}(2s-z_1-z_2+2)|}. \tag{6.21}$$

By Stirling's formula, for $-4\le x\le 4$ and $|y|\ge 1/2$, we have

$$|\Gamma(x+\mathrm{i}y)| \asymp |y|^{x-1/2}e^{-\frac{\pi}{2}|y|}, \tag{6.22}$$

which gives

$$|\Gamma(-z_1-z_2+2s+2)| \ll (|2s-z_1-z_2|+1)^{4-\mathfrak{Re}(z_1)-\mathfrak{Re}(z_2)-1/2}e^{-\frac{\pi}{2}|\mathfrak{Im}(2s-z_1-z_2)|}. \tag{6.23}$$

In the above, we have used the trivial bound $|\Gamma(-z_1-z_2+2s+2)| \ll 1$ for $\mathfrak{Im}(2s-z_1-z_2+2)\le 2$. Combining (6.20), (6.21) and (6.23), for $\mathfrak{Re}(s)=1$ we have

$$\Gamma(-z_1-z_2+2s+2)\left(\frac{\Gamma\big(z_2-s-\frac{1}{2}+\mathrm{i}t\big)}{\Gamma\big(s-z_1+\frac{3}{2}+\mathrm{i}t\big)}+\frac{\Gamma\big(z_1-s-\frac{1}{2}-\mathrm{i}t\big)}{\Gamma\big(s-z_2+\frac{3}{2}-\mathrm{i}t\big)}\right) \ll \frac{|\mathfrak{Im}(s-z_1+1)|^2+|\mathfrak{Im}(s-z_2+1)|^2}{t^2}(|2s-z_1-z_2|+1)^{4-\mathfrak{Re}(z_1)-\mathfrak{Re}(z_2)-1/2}.$$

Thus the s-integral in (6.19) is

$$\ll \left(\frac{U}{t}\right)^4\int_{(1)}\frac{|G(s)|}{|s|}\left(\frac{t}{2}\right)^4\left(1+O\left(\frac{|s|^2+1}{t}\right)\right)t^{-2}(|s|+|z_1|+|z_2|+1)^{7/2+2\delta}r^{-2}\,|ds|$$
$$\ll r^{-2}t^2\int_{(1)}|G(s)|\,|s|^{10}\,|ds| \ll r^{-2}t^2.$$

Hence, the removal of the condition $r\le R_0=T^5$ adds to the right of (6.19) an error of at most

$$\sum_{r>R_0}\int_{c_1T}^{c_2T}O(r^{-2+2\delta}t^2)\,dt\left|\sum_{q=1}^{\infty}\frac{c_q(r)G_4(z_1,q)G_4(z_2,q)}{q^{z_1+z_2}}\right| \ll T^3\sum_{r>T^5}r^{-2+2\delta}\tau_2(r) \ll T^3T^{-5+15\delta} \ll 1.$$

This means that on the right of (6.19), we can omit the condition $r\le R_0$. Now, by absolute convergence, we may swap the order of summation and integration there to obtain

$$I_O = \frac{2}{(2\pi\mathrm{i})^3}\int_{\mathcal{B}_1}\int_{\mathcal{B}_2}\zeta(z_1)^4\zeta(z_2)^4\int_{-\infty}^{\infty}\omega(t)\int_{(1)}\frac{G(s)}{s\pi^{4s}}g(s,t)\left(\frac{U}{t}\right)^{4s}\mathcal{H}_s(z_1,z_2)$$
$$\times\Gamma(-z_1-z_2+2s+2)\left(\frac{\Gamma\big(z_2-s-\frac{1}{2}+\mathrm{i}t\big)}{\Gamma\big(s-z_1+\frac{3}{2}+\mathrm{i}t\big)}+\frac{\Gamma\big(z_1-s-\frac{1}{2}-\mathrm{i}t\big)}{\Gamma\big(s-z_2+\frac{3}{2}-\mathrm{i}t\big)}\right)ds\,dt\,dz_2\,dz_1$$
$$+O(1)+O\left(\left(\frac{T}{T_0}\right)^{1+C}T^{1-\varepsilon/2}\right), \tag{6.24}$$

where

$$\mathcal{H}_s(z_1,z_2)=\sum_{r=1}^{\infty}\sum_{q=1}^{\infty}\frac{c_q(r)G_4(z_1,q)G_4(z_2,q)}{q^{z_1+z_2}r^{2s+2-z_1-z_2}}.$$

It is convenient to write

$$\mathcal{H}_s(z_1,z_2)=\mathscr{H}(z_1-1,z_2-1,z_1+z_2-2,s) \tag{6.25}$$

where

$$\mathscr{H}(u_1,u_2,u_3,s):=\sum_{r=1}^{\infty}\sum_{q=1}^{\infty}\frac{c_q(r)G_4(1+u_1,q)G_4(1+u_2,q)}{q^{2+u_3}r^{2s-u_3}}. \tag{6.26}$$

6.8. *Handling $\mathcal{H}_s(z_1,z_2)$*

We begin with the following proposition connecting $\mathscr{H}(u_1,u_2,u_3,s)$ and $\zeta(s)$.

Proposition 6.5. *Let $\delta'\in(0,1/5)$. For $i=1,2,3$, assume*

$$|u_i|<\delta'. \tag{6.27}$$

(i) *For $\Re\mathrm{e}(s)>1/2+2\delta'$, we have*

$$\mathscr{H}(u_1,u_2,u_3,s)=\zeta(2s-u_3)\frac{\zeta(1+2s)^{16}\zeta(1+2s-u_1-u_2)}{\zeta(1+2s-u_1)^4\zeta(1+2s-u_2)^4}\mathscr{I}(u_1,u_2,u_3,s),$$

where

$$\mathscr{I}(u_1,u_2,u_3,s)=\prod_p\mathscr{I}_p(u_1,u_2,u_3,s) \tag{6.28}$$

and

$$\begin{aligned}\mathscr{I}_p&(u_1,u_2,u_3,s)\\ &=\left(1-\frac{1}{p^{1+2s}}\right)^{16}\left(1-\frac{1}{p^{1+2s-u_1-u_2}}\right)\left(1-\frac{1}{p^{1+2s-u_1}}\right)^{-4}\left(1-\frac{1}{p^{1+2s-u_2}}\right)^{-4}\\ &\quad\times\left(1+\sum_{j=1}^{\infty}G_4(1+u_1,p^j)G_4(1+u_2,p^j)\cdot\frac{1-p^{2s-1-u_3}}{(p^j)^{1+2s}}\right).\end{aligned} \tag{6.29}$$

(ii) *For $\Re\mathrm{e}(s)=\sigma\ge-1/2+2\delta'$, we have*

$$\mathscr{I}_p(u_1,u_2,u_3,s)=1-Y_p p^{-2-4s}+O(p^{7\delta'+\vartheta(\sigma)}),$$

where

$$\begin{aligned}Y_p&=-6X_2^{-2}-6X_1^{-2}-16(X_1X_2)^{-1}+4(X_2^{-2}X_1^{-1})\\ &\quad+4(X_1^{-2}X_2^{-1})-(X_1X_2)^{-2}-36+24X_2^{-1}+24X_1^{-1},\\ X_i&=p^{-u_i}\quad \textit{for } i=1,2,3,\end{aligned} \tag{6.30}$$

and

$$\vartheta(\sigma) = \begin{cases} -2 & \text{if } \sigma \geq 0, \\ -2-2\sigma & \text{if } 0 > \sigma \geq -1/4, \\ -3-6\sigma & \text{if } -1/4 > \sigma \geq -1/2. \end{cases}$$

Moreover, $\mathscr{I}(u_1,u_2,u_3,s)$ *is holomorphic in* $\Re e(s) > -1/4+\delta'$*, and* $\mathscr{H}(u_1,u_2,u_3,s)$ *is meromorphic in* $\Re e(s) > -1/4+\delta'$ *with poles at*

$$s = \frac{1+u_3}{2}, 0, \frac{u_1+u_2}{2}, \frac{\rho-1+u_1}{2}, \frac{\rho-1+u_2}{2}, \tag{6.31}$$

where ρ *ranges through the non-trivial zeros of* $\zeta(s)$.

We postpone the proof of this proposition to Section 8.1 and focus on handling $\mathcal{H}_s(z_1,z_2)$. By (6.25) and Proposition 6.5,

$$\mathcal{H}_s(z_1,z_2) = \zeta(2s+2-z_1-z_2)\frac{\zeta(1+2s)^{16}\zeta(1+2s-z_1-z_2+2)}{\zeta(1+2s-z_1+1)^4\zeta(1+2s-z_2+1)^4}\tilde{\mathscr{I}}(z_1,z_2,s), \tag{6.32}$$

where

$$\tilde{\mathscr{I}}(z_1,z_2,s) := \mathscr{I}(z_1-1,z_2-1,z_1+z_2-2,s).$$

We see that $\mathcal{H}_s(z_1,z_2)$ has an analytic continuation to $\Re e(s) \geq -1/4+\eta'$, where $\eta' = \eta'(\delta)$, with the exception of poles at

$$s = \frac{z_1+z_2-1}{2}, \ 0, \ -1+\frac{z_1+z_2}{2}, \frac{\rho+z_1-2}{2}, \frac{\rho+z_2-2}{2},$$

where ρ ranges through non-trivial zeros of $\zeta(s)$. We also see that $\Gamma(-z_1-z_2+2s+2)$ in (6.24) has a pole at $s = \frac{z_1+z_2-2}{2}$, while

$$\frac{\Gamma\big(z_2-s-\frac{1}{2}+it\big)}{\Gamma\big(s-z_1+\frac{3}{2}+it\big)} + \frac{\Gamma\big(z_1-s-\frac{1}{2}-it\big)}{\Gamma\big(s-z_2+\frac{3}{2}-it\big)}$$

has poles at

$$s = \mathrm{p}_1, \mathrm{p}_2.$$

6.9. *Moving the integral in* (6.24) *left to* $\Re e(s) = \varepsilon_3$

Recall (6.24). We first observe that $\mathcal{H}_s(z_1,z_2) = \mathscr{H}(1-z_1,1-z_2,z_1+z_2-2,s)$ and move the contour to $\Re e(s) = \varepsilon_3$ with $\varepsilon_3 > \delta > 0$, which is large enough to miss poles at $s=0$ and $s = -1+\frac{z_1+z_2}{2}$ of $\mathcal{H}_s(z_1,z_2)$ and the pole at $s = \frac{z_1+z_2-2}{2}$ of $\Gamma(-z_1-z_2+2s+2)$. For the pole at $s = \frac{z_1+z_2-1}{2}$ of $\mathcal{H}_s(z_1,z_2)$, we note that $\frac{\Gamma(z_2-s-1/2+it)}{\Gamma(s-z_1+3/2+it)} + \frac{\Gamma(z_1-s-1/2-it)}{\Gamma(s-z_2+3/2-it)}$ has a zero at the same point. Also, the contribution from the pole at $s = \mathrm{p}_i$ in the s-integral of (6.24) is

$$-\frac{G(\mathrm{p}_i)}{\mathrm{p}_i} g(\mathrm{p}_i,t)\bigg(\frac{U}{t}\bigg)^{4\mathrm{p}_i} \mathcal{H}_{\mathrm{p}_i}(z_1,z_2)\pi^{-4\mathrm{p}_i} \ll |\mathrm{p}_i|^{-1-B}U^{4\Re e(\mathrm{p}_i)}\frac{|\mathrm{p}_i|^2}{t} \ll T^{3-B}.$$

It then follows from the above bound and (6.24) that

$$I_O = \frac{2}{(2\pi i)^3} \int_{\mathcal{B}_1} \int_{\mathcal{B}_2} \zeta(z_1)^4 \zeta(z_2)^4 \int_{-\infty}^{\infty} \omega(t) \int_{(\varepsilon_3)} \frac{G(s)}{s\pi^{4s}} g(s,t) \left(\frac{U}{t}\right)^{4s} \mathcal{H}_s(z_1,z_2)$$
$$\times \Gamma(-z_1 - z_2 + 2s + 2) \left(\frac{\Gamma\left(z_2 - s - \frac{1}{2} + it\right)}{\Gamma\left(s - z_1 + \frac{3}{2} + it\right)} + \frac{\Gamma\left(z_1 - s - \frac{1}{2} - it\right)}{\Gamma\left(s - z_2 + \frac{3}{2} - it\right)} \right) ds\, dt\, dz_2\, dz_1$$
$$+ O(1) + O\left(\left(\frac{T}{T_0}\right)^{1+C} T^{1-\varepsilon/2} \right).$$

By Lemma 2.2 (i), we know that

$$I_O = \frac{2}{(2\pi i)^3} \int_{\mathcal{B}_1} \int_{\mathcal{B}_2} \zeta(z_1)^4 \zeta(z_2)^4 \int_{-\infty}^{\infty} \omega(t) \int_{(\varepsilon_3)} \frac{G(s)}{s\pi^{4s}} \left(\frac{t}{2}\right)^{4s} \left(\frac{U}{t}\right)^{4s} \mathcal{H}_s(z_1,z_2)$$
$$\times \Gamma(-z_1 - z_2 + 2s + 2) \left(\frac{\Gamma\left(z_2 - s - \frac{1}{2} + it\right)}{\Gamma\left(s - z_1 + \frac{3}{2} + it\right)} + \frac{\Gamma\left(z_1 - s - \frac{1}{2} - it\right)}{\Gamma\left(s - z_2 + \frac{3}{2} - it\right)} \right) ds\, dt\, dz_2\, dz_1$$
$$+ I_{O,E} + O(1) + O\left(\left(\frac{T}{T_0}\right)^{1+C} T^{1-\varepsilon/2} \right), \quad (6.33)$$

where

$$I_{O,E} = \frac{2}{(2\pi i)^3} \int_{\mathcal{B}_1} \int_{\mathcal{B}_2} \zeta(z_1)^4 \zeta(z_2)^4 \int_{-\infty}^{\infty} \omega(t) \int_{(\varepsilon_3)} \frac{G(s)}{s\pi^{4s}} \left(\frac{t}{2}\right)^{4s} O\left(\frac{|s|^2+1}{t}\right) \left(\frac{U}{t}\right)^{4s}$$
$$\times \mathcal{H}_s(z_1,z_2) \Gamma(-z_1 - z_2 + 2s + 2) \left(\frac{\Gamma\left(z_2 - s - \frac{1}{2} + it\right)}{\Gamma\left(s - z_1 + \frac{3}{2} + it\right)} + \frac{\Gamma\left(z_1 - s - \frac{1}{2} - it\right)}{\Gamma\left(s - z_2 + \frac{3}{2} - it\right)} \right)$$
$$\times ds\, dt\, dz_2\, dz_1. \quad (6.34)$$

We shall give an upper bound for $I_{O,E}$. Observe that for $0 < \delta < \varepsilon_3 < 0.15$, we have $\mathfrak{Re}(s - z_i + 1) \in [\varepsilon_3 - \delta, \varepsilon_3 + \delta] \subseteq [0, 0.4]$, $i = 1, 2$. This implies $\mathfrak{Re}(s - z_1 + 1 + s - z_2 + 1) \leq 0.8$. We remind the reader that in (6.34), t does not denote the imaginary part of s but a variable satisfying $c_1 T \leq t \leq c_2 T$. It follows from Lemma 6.4 (ii) and (6.22) that the inner integral in (6.34) on $|\mathfrak{Im}(s - z_1 + 1)| \geq t + 1$ or $|\mathfrak{Im}(s - z_2 + 1)| \geq t + 1$ is

$$\ll \int_{\substack{\mathfrak{Re}(s)=\varepsilon_3 \\ |\mathfrak{Im}(s-z_1+1)|\geq t+1 \\ \text{or } |\mathfrak{Im}(s-z_2+1)|\geq t+1}} \frac{|G(s)|}{|s|} O\left(U^{4\varepsilon_3} \frac{|s|^2}{t}\right) |\mathcal{H}_s(z_1,z_2)| (|2s - z_1 - z_2| + 1)^2$$
$$\times e^{-\frac{\pi}{2}|\mathfrak{Im}(2s-z_1-z_2)|} \frac{|\mathfrak{Im}(s - z_1 + 1)|^2 + |\mathfrak{Im}(s - z_2 + 1)|^2}{t^2} e^{\frac{\pi}{2}|\mathfrak{Im}(2s-z_1-z_2+2)|}\, d|s|$$
$$\ll U^{4\varepsilon_3} \int_{\substack{\mathfrak{Re}(s)=\varepsilon_3 \\ |\mathfrak{Im}(s-z_1+1)|\geq t+1 \\ \text{or } |\mathfrak{Im}(s-z_2+1)|\geq t+1}} |G(s)|\, |s|^5 |\mathcal{H}_s(z_1,z_2)|\, d|s|$$
$$\ll T^{(1-\varepsilon)4\varepsilon_3} \int_{\substack{\mathfrak{Re}(s)=\varepsilon_3 \\ |\mathfrak{Im}(s-z_1+1)|\geq t+1 \\ \text{or } |\mathfrak{Im}(s-z_2+1)|\geq t+1}} |s|^{-10}\, d|s|$$
$$\ll T^{-5}. \quad (6.35)$$

In the second last inequality above, we have used the fact that for $\mathfrak{Re}(s) = \varepsilon_3$ and $0 < \delta < \varepsilon_3$,

$$|\mathscr{H}_s(z_1, z_2)| \ll |\zeta(2s + 2 - z_1 - z_2)| \frac{1}{\varepsilon_3^{16}} \frac{1}{2\varepsilon_3 - 2\delta} \frac{1}{(2\varepsilon_3 - \delta)^4} \frac{1}{(2\varepsilon_3 - \delta)^4} \ll |s| + 1,$$

which can be deduced from (6.32) and Proposition 6.5. By Lemma 6.4 (i) and the bound $\mathfrak{Re}(s - z_1 + 1 + s - z_2 + 1) \leq 0.8$, the inner integral on $|\mathfrak{Im}(s - z_i + 1)| < t + 1$, $i = 1, 2$, in (6.34) is

$$\begin{aligned}
&\ll \int_{\substack{\mathfrak{Re}(s)=\varepsilon_3 \\ |\mathfrak{Im}(s-z_i+1)|<t+1,\, i=1,2}} \frac{|G(s)|}{|s|} O\Big(U^{4\varepsilon_3} \frac{|s|^2+1}{t}\Big) |\mathscr{H}_s(z_1, z_2)| (|2s - z_1 - z_2| + 1)^2 \\
&\quad \times e^{-\frac{\pi}{2}|\mathfrak{Im}(2s - z_1 - z_2)|} \Big[t^{-2s+z_1+z_2-2} \cos\big(\tfrac{\pi}{2}(2s - z_1 - z_2 + 2)\big) \\
&\qquad + O\big(t^{-2\varepsilon_3+2\delta} e^{\frac{\pi}{2}|\mathfrak{Im}(2s-z_1-z_2)|} \tfrac{1+|s-z_1+1|^2+|s-z_2+1|^2}{t}\big)\Big]\, d|s| \\
&\ll \int_{\substack{\mathfrak{Re}(s)=\varepsilon_3 \\ |\mathfrak{Im}(s-z_i+1)|<t+1,\, i=1,2}} \frac{|G(s)|}{|s|} O\Big(U^{4\varepsilon_3} \frac{|s|^2+1}{t}\Big) |\mathscr{H}_s(z_1, z_2)| (|2s - z_1 - z_2| + 1)^2 \\
&\quad \times e^{-\frac{\pi}{2}|\mathfrak{Im}(2s - z_1 - z_2)|} e^{\frac{\pi}{2}|\mathfrak{Im}(2s - z_1 - z_2)|} t^{-2\varepsilon_3+2\delta} \big(1 + \tfrac{1+|s-z_1+1|^2+|s-z_2+1|^2}{t}\big)\, d|s| \\
&\ll T^{-1+2\varepsilon_3+2\delta-4\varepsilon_3\varepsilon}.
\end{aligned} \tag{6.36}$$

In the second inequality above, we have used the fact that for $z \in \mathbb{C}$,

$$\big|\cos\big(\tfrac{\pi}{2} z\big)\big| \ll e^{\frac{\pi}{2}|\mathfrak{Im}(z)|}.$$

Thus, substituting (6.35) and (6.36) in (6.34), we have $I_{O,E} \ll T^{2\varepsilon_3+2\delta-4\varepsilon_3\varepsilon} \ll T^{0.5}$, provided that $0 < \varepsilon_3 < 0.15$ and $0 < \delta < \frac{1}{10}$, which, together with (6.33), yields

$$\begin{aligned}
I_O &= \frac{2}{(2\pi \mathrm{i})^2} \int_{\mathscr{B}_1} \int_{\mathscr{B}_2} \zeta(z_1)^4 \zeta(z_2)^4 \frac{1}{2\pi \mathrm{i}} \int_{(\varepsilon_3)} \frac{G(s)}{s} \mathscr{H}_s(z_1, z_2) \int_{-\infty}^{\infty} \Big(\frac{U}{2\pi}\Big)^{4s} \omega(t) \\
&\quad \times \left(\frac{\Gamma\big(z_2 - s - \frac{1}{2} + \mathrm{i}t\big)}{\Gamma\big(s - z_1 + \frac{3}{2} + \mathrm{i}t\big)} + \frac{\Gamma\big(z_1 - s - \frac{1}{2} - \mathrm{i}t\big)}{\Gamma\big(s - z_2 + \frac{3}{2} - \mathrm{i}t\big)} \right) \Gamma(-z_1 - z_2 + 2s + 2)\, dt\, ds\, dz_2\, dz_1 \\
&\qquad + O(T^{0.5}) + O\bigg(\Big(\frac{T}{T_0}\Big)^{1+C} T^{1-\varepsilon/2} \bigg).
\end{aligned} \tag{6.37}$$

7. Further evaluation of off-diagonal terms: Proof of Proposition 3.3

In this section, we shall give a further simplification of I_O, and extract one main term from there. By a rearrangement for (6.37), we write

$$\begin{aligned}
I_O &= \int_{-\infty}^{\infty} \omega(t) \frac{2}{2\pi \mathrm{i}} \int_{(\varepsilon_3)} \frac{G(s)}{s} \Big(\frac{U}{2\pi}\Big)^{4s} \mathscr{I}(s, t)\, ds\, dt \\
&\quad + O(T^{0.5}) + O\bigg(\Big(\frac{T}{T_0}\Big)^{1+C} T^{1-\varepsilon/2} \bigg),
\end{aligned} \tag{7.1}$$

where

$$\begin{aligned}
\mathscr{J}(s,t) &:= \frac{1}{(2\pi \mathrm{i})^2}\int_{\mathcal{B}_1}\int_{\mathcal{B}_2}\zeta(z_1)^4\zeta(z_2)^4\mathcal{H}_s(z_1,z_2)\\
&\quad\times\left(\frac{\Gamma\big(z_2-s-\frac{1}{2}+\mathrm{i}t\big)}{\Gamma\big(s-z_1+\frac{3}{2}+\mathrm{i}t\big)}+\frac{\Gamma\big(z_1-s-\frac{1}{2}-\mathrm{i}t\big)}{\Gamma\big(s-z_2+\frac{3}{2}-\mathrm{i}t\big)}\right)\Gamma(-z_1-z_2+2s+2)\,dz_2\,dz_1\\
&= \frac{1}{2\pi\mathrm{i}}\int_{\mathcal{B}_2}\zeta(z_2)^4\times \mathrm{inn}(z_2)\,dz_2,\\
\mathrm{inn}(z_2) &:= \frac{1}{2\pi\mathrm{i}}\int_{\mathcal{B}_1}\zeta(z_1)^4\mathcal{F}_{s,t}(z_1,z_2)\,dz_1\\
&= \frac{1}{2\pi\mathrm{i}}\int_{\mathcal{B}_1}\frac{1}{(z_1-1)^4}((z_1-1)\zeta(z_1))^4\mathcal{F}_{s,t}(z_1,z_2)\,dz_1,
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{F}_{s,t}(z_1,z_2)\\
&\quad := \mathcal{H}_s(z_1,z_2)\left(\frac{\Gamma\big(z_2-s-\frac{1}{2}+\mathrm{i}t\big)}{\Gamma\big(s-z_1+\frac{3}{2}+\mathrm{i}t\big)}+\frac{\Gamma\big(z_1-s-\frac{1}{2}-\mathrm{i}t\big)}{\Gamma\big(s-z_2+\frac{3}{2}-\mathrm{i}t\big)}\right)\Gamma(-z_1-z_2+2s+2).
\end{aligned}$$

Setting $H(z_1)=((z_1-1)\zeta(z_1))^4\mathcal{F}_{s,t}(z_1,z_2)$, we have

$$\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{B}_1}\frac{1}{(z_1-1)^4}H(z_1)\,dz_1=\frac{H^{(3)}(1)}{3!}=\sum_{i'+i=3}\frac{b_{i'}\mathcal{F}_{s,t}^{(i,0)}(1,z_2)}{i!},$$

where $b_{i'}\in\mathbb{R}$ satisfy

$$h(z):=((z-1)\zeta(z))^4=1+b_1(z-1)+b_2(z-1)^2+\cdots.$$

Here

$$\mathcal{F}_{s,t}^{(i,j)}(z_1,z_2):=\frac{\partial^i}{\partial z_1^i}\frac{\partial^j}{\partial z_2^j}\mathcal{F}_{s,t}(z_1,z_2).$$

We now compute $\mathcal{F}_{s,t}^{(i,0)}(z_1,z_2)$ for $i\in\{0,1,2,3\}$. By the generalized product rule, we obtain

$$\begin{aligned}
\mathcal{F}_{s,t}^{(i,0)}(1,z_2) &= \sum_{u+v+w=i}\binom{i}{u,v,w}\mathcal{H}_s^{(u,0)}(z_1,z_2)\Big|_{z_1=1}\\
\times\frac{\partial^v}{\partial z_1^v}&\left(\frac{\Gamma\big(z_2-s-\frac{1}{2}+\mathrm{i}t\big)}{\Gamma\big(s-z_1+\frac{3}{2}+\mathrm{i}t\big)}+\frac{\Gamma\big(z_1-s-\frac{1}{2}-\mathrm{i}t\big)}{\Gamma\big(s-z_2+\frac{3}{2}-\mathrm{i}t\big)}\right)\Bigg|_{z_1=1}(-1)^w\Gamma^{(w)}(-z_2+2s+1).
\end{aligned}\tag{7.2}$$

Observe that by Proposition 6.5 (i), for $\Re\mathrm{e}(s)>1/2+4\delta$ and $u\in\mathbb{Z}_{\geq 0}$, we have

$$\mathcal{H}_s^{(u,0)}(z_1,z_2)|_{z_1=1}=\sum_{i_1+i_2=u}\binom{u}{i_1}\sum_{r=1}^{\infty}\sum_{q=1}^{\infty}\frac{c_q(r)G_4^{(i_1)}(1,q)G_4(z_2,q)}{q^{1+z_2}r^{2s+1-z_2}}\left(\log\frac{r}{q}\right)^{i_2}.\tag{7.3}$$

Hence,

$$\begin{aligned}\mathscr{J}(s,t) &= \frac{1}{2\pi\mathrm{i}}\int_{\mathscr{B}_2}\zeta(z_2)^4\left(\frac{1}{2\pi\mathrm{i}}\int_{\mathscr{B}_1}\frac{1}{(z_1-1)^4}H(z_1)\,dz_1\right)dz_2\\ &= \frac{1}{2\pi\mathrm{i}}\int_{\mathscr{B}_2}\zeta(z_2)^4\sum_{i'+i=3}\frac{b_{i'}\mathscr{F}_{s,t}^{(i,0)}(1,z_2)}{i!}\,dz_2\\ &= \sum_{i'+i=3}\frac{b_{i'}}{i!}\frac{1}{2\pi\mathrm{i}}\int_{\mathscr{B}_2}\frac{1}{(z_2-1)^4}((z_2-1)\zeta(z_2))^4\mathscr{F}_{s,t}^{(i,0)}(1,z_2)\,dz_2\\ &= \sum_{i'+i=3}\frac{b_{i'}}{i!}\sum_{j'+j=3}\frac{b_{j'}}{j!}\mathscr{F}_{s,t}^{(i,j)}(1,1).\end{aligned}$$

From (7.2), it follows that

$$\begin{aligned}&\mathscr{F}_{s,t}^{(i,j)}(1,1)\\ &= \sum_{u+v+w=i}\binom{i}{u,v,w}\sum_{a+b+c=j}\binom{j}{a,b,c}\mathscr{H}_s^{(u,a)}(1,1)\mathscr{G}_{s,t}^{(v,b)}(1,1)(-1)^{w+c}\Gamma^{(w+c)}(2s),\end{aligned}$$

where

$$\mathscr{G}_{s,t}^{(v,b)}(1,1) = \frac{\partial^v}{\partial z_1^v}\frac{\partial^b}{\partial z_2^b}\left(\frac{\Gamma\left(z_2-s-\frac{1}{2}+\mathrm{i}t\right)}{\Gamma\left(s-z_1+\frac{3}{2}+\mathrm{i}t\right)}+\frac{\Gamma\left(z_1-s-\frac{1}{2}-\mathrm{i}t\right)}{\Gamma\left(s-z_2+\frac{3}{2}-\mathrm{i}t\right)}\right)\Bigg|_{z_1=z_2=1}. \tag{7.4}$$

By (7.3), we also know that for $\Re e(s) > 1/2 + 4\delta$,

$$\begin{aligned}&\mathscr{H}_s^{(u,a)}(1,1)\\ &\quad= \sum_{i_1+i_2=u}\binom{u}{i_1}\sum_{j_1+j_2=a}\binom{a}{j_1}\sum_{r=1}^{\infty}\sum_{q=1}^{\infty}\frac{c_q(r)G_4^{(i_1)}(1,q)G_4^{(j_1)}(1,q)}{q^2r^{2s}}\left(\log\frac{r}{q}\right)^{i_2+j_2}\\ &\quad= \sum_{i_1+i_2=u}\binom{u}{i_1}\sum_{j_1+j_2=a}\binom{a}{j_1}\mathscr{H}^{(i_1,j_1,i_2+j_2,0)}(0,0,0,s),\end{aligned} \tag{7.5}$$

where $\mathscr{H}$ is defined as in (6.26). Note that the above identity is also valid in the region $\Re e(s) > -1/4 + 2\delta$ since $\mathscr{H}_s^{(u,a)}(1,1)$ can be meromorphically extended to that region by Proposition 6.5 and the meromorphic continuation is unique. We then arrive at

$$\begin{aligned}\mathscr{J}(s,t) &= \sum_{i'+i=3}\frac{b_{i'}}{i!}\sum_{j'+j=3}\frac{b_{j'}}{j!}\mathscr{F}_{s,t}^{(i,j)}(1,1)\\ &= \sum_{i'+i=3}\frac{b_{i'}}{i!}\sum_{j'+j=3}\frac{b_{j'}}{j!}\sum_{u+v+w=i}\binom{i}{u,v,w}\sum_{a+b+c=j}\binom{j}{a,b,c}\\ &\qquad\qquad\times\mathscr{H}_s^{(u,a)}(1,1)\mathscr{G}_{s,t}^{(v,b)}(1,1)(-1)^{w+c}\Gamma^{(w+c)}(2s).\end{aligned}$$

Combining this with (7.1), it follows that

$$I_O = \int_{-\infty}^{\infty} \omega(t) \frac{2}{2\pi \mathrm{i}} \int_{(\varepsilon_3)} \frac{G(s)}{s} \left(\frac{U}{2\pi}\right)^{4s} \sum_{i'+i=3} \frac{b_{i'}}{i!} \sum_{j'+j=3} \frac{b_{j'}}{j!} \sum_{u+v+w=i} \binom{i}{u,v,w} \times \sum_{a+b+c=j} \binom{j}{a,b,c} \mathcal{H}_s^{(u,a)}(1,1) \mathcal{G}_{s,t}^{(v,b)}(1,1)(-1)^{w+c} \Gamma^{(w+c)}(2s)\, ds\, dt + O(T^{0.5}) + O\left(\left(\frac{T}{T_0}\right)^{1+C} T^{1-\varepsilon/2}\right)$$
$$= 2 \sum_{i'+i=3} \frac{b_{i'}}{i!} \sum_{j'+j=3} \frac{b_{j'}}{j!} \sum_{u+v+w=i} \binom{i}{u,v,w} \sum_{a+b+c=j} \binom{j}{a,b,c} \times \int_{-\infty}^{\infty} \omega(t) \frac{1}{2\pi \mathrm{i}} \int_{(\varepsilon_3)} \frac{G(s)}{s} \left(\frac{U}{2\pi}\right)^{4s} \mathcal{H}_s^{(u,a)}(1,1) \mathcal{G}_{s,t}^{(v,b)}(1,1)(-1)^{w+c} \Gamma^{(w+c)}(2s)\, ds\, dt + O(T^{0.5}) + O\left(\left(\frac{T}{T_0}\right)^{1+C} T^{1-\varepsilon/2}\right). \tag{7.6}$$

The last s-integral can be further evaluated by the following lemma.

Lemma 7.1. *Let* $0 < \delta < \frac{1}{10}$ *and* $0 < \delta < \varepsilon_3 < 0.15$. *Let* u, a, v, b, w, c *be non-negative integers such that* $a + b + c \le 3$ *and* $u + v + w \le 3$. *For* $\mathbf{x} = (u, a, v, b, w, c)$, *set*

$$i_{\mathbf{x}} := \frac{1}{2\pi \mathrm{i}} \int_{(\varepsilon_3)} \frac{G(s)}{s} \left(\frac{U}{2\pi}\right)^{4s} \mathcal{H}_s^{(u,a)}(1,1) \mathcal{G}_{s,t}^{(v,b)}(1,1)(-1)^{w+c} \Gamma^{(w+c)}(2s)\, ds.$$

Then

$$i_{\mathbf{x}} = i_{\mathbf{x},0} + i_{\mathbf{x},1} + O\left(\left(\frac{U^2}{t}\right)^{-1/2+4\delta}\right), \tag{7.7}$$

where

$$i_{\mathbf{x},0} := \sum_{i_1+i_2=u} \binom{u}{i_1} \sum_{j_1+j_2=a} \binom{a}{j_1} \sum_{x_0=0}^{i_2+j_2} \sum_{\substack{x_1+x_2+x_3=i_1 \\ x_1,x_2,x_3 \ge 0}} \sum_{\substack{y_1+y_2+y_3=j_1 \\ y_1,y_2,y_3 \ge 0}} \binom{i_2+j_2}{x_0} \times (-1)^{x_0+x_1+y_1} \binom{i_1}{x_1, x_2, x_3} \binom{j_1}{y_1, y_2, y_3} \times (-1)^{b+v} \sum_{\substack{r_1' \in \{0,\ldots,v\} \\ r_2' \in \{0,\ldots,b\} \\ v+b-r_1'-r_2' \equiv 0 \,(\mathrm{mod}\, 2)}} (-1)^{r_1'+r_2'} \binom{v}{r_1'} \binom{b}{r_2'} (\log t)^{r_1'+r_2'} \left(\mathrm{i}\frac{\pi}{2}\right)^{v+b-r_1'-r_2'} \times \frac{1}{2^{x_1+y_1+x_2+y_2+w+c+10}} \frac{1}{(x_1+y_1+x_2+y_2+w+c+10)!} \times \sum_{k=0}^{x_1+y_1+x_2+y_2+w+c+10} \binom{x_1+y_1+x_2+y_2+w+c+10}{k} \left(\log \frac{U^4}{t^2}\right)^k \times J_0^{(x_1+y_1+x_2+y_2+w+c+10-k)}(0), \tag{7.8}$$

and

$$i_{\mathbf{x},1} := \sum_{i_1+i_2=u} \binom{u}{i_1} \sum_{j_1+j_2=a} \binom{a}{j_1} \sum_{x_0=0}^{i_2+j_2} \sum_{\substack{x_1+x_2+x_3=i_1 \\ x_1,x_2,x_3\geq 0}} \sum_{\substack{y_1+y_2+y_3=j_1 \\ y_1,y_2,y_3\geq 0}} \binom{i_2+j_2}{x_0}$$
$$\times (-1)^{x_0+x_1+y_1} \binom{i_1}{x_1, x_2, x_3} \binom{j_1}{y_1, y_2, y_3}$$
$$\times (-1)^{b+v} \sum_{\substack{r_1'\in\{0,\dots,v\} \\ r_2'\in\{0,\dots,b\} \\ v+b-r_1'-r_2'\equiv 1 \pmod 2}} (-1)^{r_1'+r_2'} \binom{v}{r_1'} \binom{b}{r_2'} (\log t)^{r_1'+r_2'} \left(\mathrm{i}\frac{\pi}{2}\right)^{v+b-r_1'-r_2'}$$
$$\times \frac{\pi}{2^{x_1+y_1+x_2+y_2+w+c+10}} \frac{1}{(x_1+y_1+x_2+y_2+w+c+9)!}$$
$$\times \sum_{k=0}^{x_1+y_1+x_2+y_2+w+c+9} \binom{x_1+y_1+x_2+y_2+w+c+9}{k}$$
$$\times \left(\log\frac{U^4}{t^2}\right)^k J_1^{(x_1+y_1+x_2+y_2+w+c+9-k)}(0).$$

Here, $J_0(s)$ and $J_1(s)$ are defined by

$$\begin{aligned} J_0(s) &= J_0(s; x_0, x_1, y_1, x_2, y_2, x_3, y_3, i_2, j_2, w, c) \\ &:= G(s)\frac{1}{(2\pi)^{4s}}\zeta(1+2s)^2(2s)^2 g_{x_2}(s)(2s)^{x_2+3} g_{y_2}(s)(2s)^{y_2+3} \\ &\quad \times \zeta^{(x_1+y_1)}(1+2s)(2s)^{x_1+y_1+1} 2\cos(\pi s)(-1)^{w+c}\Gamma^{(w+c)}(2s)(2s)^{w+c+1} \\ &\quad \times \mathscr{I}^{(x_3,y_3,i_2+j_2-x_0,0)}(0,0,0,s)\zeta^{(x_0)}(2s), \end{aligned}$$

and

$$\begin{aligned} J_1(s) &= J_1(s; x_0, x_1, y_1, x_2, y_2, x_3, y_3, i_2, j_2, w, c) \\ &:= G(s)\frac{1}{(2\pi)^{4s}}\zeta(1+2s)^2(2s)^2 g_{x_2}(s)(2s)^{x_2+3} g_{y_2}(s)(2s)^{y_2+3} \\ &\quad \times \zeta^{(x_1+y_1)}(1+2s)(2s)^{x_1+y_1+1} \frac{2\mathrm{i}\sin(\pi s)}{\pi s}(-1)^{w+c}\Gamma^{(w+c)}(2s)(2s)^{w+c+1} \\ &\quad \times \mathscr{I}^{(x_3,y_3,i_2+j_2-x_0,0)}(0,0,0,s)\zeta^{(x_0)}(2s), \end{aligned}$$

where $g_{x_2}(s), g_{y_2}(s)$ are given by

$$\begin{aligned} g_0(s) &= \zeta(1+2s)^3, \\ g_1(s) &= 4\zeta'(1+2s)\zeta(1+2s)^2, \\ g_2(s) &= 20\zeta'(1+2s)^2\zeta(1+2s) - 4\zeta^{(2)}(1+2s)\zeta(1+2s)^2, \\ g_3(s) &= 120\zeta'(1+2s)^3 - 60\zeta'(1+2s)\zeta^{(2)}(1+2s)\zeta(1+2s) \\ &\quad + 4\zeta^{(3)}(1+2s)\zeta(1+2s)^2, \end{aligned} \tag{7.9}$$

and $\mathscr{I}(0,0,0,s)$ *is defined in* (6.28). *Clearly, both* $J_0(s)$ *and* $J_1(s)$ *are holomorphic for* $-1/4+2\delta<\Re\mathfrak{e}(s)<1/2$, *and* $J_0(s)$, $J_1(s)\ll 1/(1+|\Im\mathfrak{m}(s)|^{100})$ *uniformly in this region.*

In addition, we have

$$i_{\mathbf{x},1}\ll(\log T)^{15}. \tag{7.10}$$

To prove this lemma, we shall require the following two technical lemmata that give a further computation for $\mathcal{G}_{s,t}^{(v,b)}(1,1)$ and $\mathcal{H}_s^{(u,a)}(1,1)$. Their proofs are postponed to the next section.

Lemma 7.2. *Let* $\mathcal{G}_{s,t}^{(v,b)}(1,1)$ *be given as in* (7.4). *Assume* $\Re\mathfrak{e}(s)=\varepsilon_3$, $0<\delta<\frac{1}{10}$, *and* $0<\delta<\varepsilon_3<0.15$.

(i) *Let* $|\Im\mathfrak{m}(s)|\le t+1$. *We have*

$$\begin{aligned}\mathcal{G}_{s,t}^{(v,b)}(1,1)=(-1)^{b+v}\sum_{r_2'=0}^{b}\sum_{r_1'=0}^{v}(-1)^{r_1'+r_2'}\binom{v}{r_1'}\binom{b}{r_2'}(\log t)^{r_1'+r_2'}t^{-2s}\\ \times\left(\mathrm{i}\frac{\pi}{2}\right)^{v+b-r_1'-r_2'}\left((-1)^{v+b-r_1'-r_2'}\exp(-\mathrm{i}\pi s)+\exp(\mathrm{i}\pi s)\right)\\ +O\left(\delta^{-b-v}t^{-2\Re\mathfrak{e}(s)+2\delta}\exp(\pi|\Im\mathfrak{m}(s)|+\pi\delta)\left(\frac{1+2|s|^2}{t}\right)\right).\end{aligned}$$

(ii) *Let* $|\Im\mathfrak{m}(s)|>t+1$. *We have*

$$\mathcal{G}_{s,t}^{(v,b)}(1,1)\ll\delta^{-b-v}\frac{\Im\mathfrak{m}(s)^2}{t^2}e^{\pi|\Im\mathfrak{m}(s)|}.$$

Lemma 7.3. *Let* $\mathcal{H}_s^{(u,a)}(1,1)$ *be given as in* (7.5). *Assume* $\Re\mathfrak{e}(s)=\varepsilon_3$. *We have*

$$\begin{aligned}\mathcal{H}_s^{(u,a)}(1,1)=\zeta(1+2s)^2\sum_{i_1+i_2=u}\binom{u}{i_1}\sum_{j_1+j_2=a}\binom{a}{j_1}\\ \times\sum_{x_0=0}^{i_2+j_2}\sum_{\substack{x_1+x_2+x_3=i_1\\x_1,x_2,x_3\ge0}}\sum_{\substack{y_1+y_2+y_3=j_1\\y_1,y_2,y_3\ge0}}\binom{i_2+j_2}{x_0}(-1)^{x_0+x_1+y_1}\\ \times\zeta^{(x_0)}(2s)\binom{i_1}{x_1,x_2,x_3}\binom{j_1}{y_1,y_2,y_3}g_{x_2}(s)g_{y_2}(s)\zeta^{(x_1+y_1)}(1+2s)\\ \times\mathscr{I}^{(x_3,y_3,i_2+j_2-x_0,0)}(0,0,0,s),\end{aligned}$$

where $\mathscr{I}(0,0,0,s)$ *is defined in* (6.28), *and*

$$g_k(s):=\zeta(1+2s)^7\frac{\partial^k}{\partial z^k}\left(\frac{1}{\zeta(1+2s-z)^4}\right)\bigg|_{z=0}. \tag{7.11}$$

(*Note that the explicit expressions of* $g_k(s)$ *for* $k=0,\dots,4$ *are given in* (7.9).) *In addition, the* $g_k(s)$ *are holomorphic everywhere with the exception of poles at* $s=0$.

Proof of Lemma 7.1. From Lemmata 7.2 (ii) and 7.3, it follows that the $|\Im\mathfrak{m}(s)| > t+1$ part of $i_{\mathbf{x}}$ is

$$\begin{aligned}
&\frac{1}{2\pi \mathrm{i}} \int_{\substack{\Re\mathfrak{e}(s)=\varepsilon_3 \\ |\Im\mathfrak{m}(s)|>t+1}} \frac{G(s)}{s} \left(\frac{U}{2\pi}\right)^{4s} \mathscr{H}_s^{(u,a)}(1,1)\mathscr{G}_{s,t}^{(v,b)}(1,1)(-1)^{w+c}\Gamma^{(w+c)}(2s)\, ds \\
&\ll \sum_{i_1+i_2=u} \binom{u}{i_1} \sum_{j_1+j_2=a} \binom{a}{j_1} \\
&\quad \times \sum_{x_0=0}^{i_2+j_2} \sum_{\substack{x_1+x_2+x_3=i_1 \\ x_1,x_2,x_3\ge 0}} \sum_{\substack{y_1+y_2+y_3=j_1 \\ y_1,y_2,y_3\ge 0}} \binom{i_2+j_2}{x_0} \binom{i_1}{x_1,x_2,x_3} \binom{j_1}{y_1,y_2,y_3} \\
&\quad \times \frac{1}{2\pi \mathrm{i}} \int_{\substack{\Re\mathfrak{e}(s)=\varepsilon_3 \\ |\Im\mathfrak{m}(s)|>t+1}} \left| \frac{G(s)}{s} \left(\frac{U}{2\pi}\right)^{4s} \zeta(1+2s)^2 \zeta^{(x_0)}(2s) g_{x_2}(s) g_{y_2}(s) \zeta^{(x_1+y_1)}(1+2s) \right| \\
&\quad \times \left| \mathscr{I}^{(x_3,y_3,i_2+j_2-x_0,0)}(0,0,0,s) O\left(\frac{\Im\mathfrak{m}(s)^2}{t^2} e^{\pi|\Im\mathfrak{m}(s)|} \right) \Gamma^{(w+c)}(2s) \right| d|s| \\
&\ll T^{-2},
\end{aligned}$$

due to the rapid decay of $|G(s)|$. Therefore, it suffices to consider the $|\Im\mathfrak{m}(s)| \le t+1$ portion of $i_{\mathbf{x}}$. To do so, we first note that

$$(-1)^{v+b-r_1'-r_2'} \exp(-\mathrm{i}\pi s) + \exp(\mathrm{i}\pi s) = \begin{cases} 2\cos(\pi s) & \text{if } v+b-r_1'-r_2' \equiv 0 \pmod 2, \\ 2\mathrm{i}\sin(\pi s) & \text{if } v+b-r_1'-r_2' \equiv 1 \pmod 2. \end{cases}$$

Hence, by Lemmata 7.2 (i) and 7.3, together with the above bound, we obtain

$$i_{\mathbf{x}} = j_{\mathbf{x},0} + j_{\mathbf{x},1} + j_{\mathbf{x},\mathrm{error}} + O(T^{-2}),$$

where

$$\begin{aligned}
j_{\mathbf{x},0} := &\sum_{i_1+i_2=u} \binom{u}{i_1} \sum_{j_1+j_2=a} \binom{a}{j_1} \sum_{x_0=0}^{i_2+j_2} \sum_{\substack{x_1+x_2+x_3=i_1 \\ x_1,x_2,x_3\ge 0}} \sum_{\substack{y_1+y_2+y_3=j_1 \\ y_1,y_2,y_3\ge 0}} \binom{i_2+j_2}{x_0} \\
&\times (-1)^{x_0+x_1+y_1} \binom{i_1}{x_1,x_2,x_3} \binom{j_1}{y_1,y_2,y_3} (-1)^{b+v} \\
&\times \sum_{\substack{r_1'\in\{0,\dots,v\} \\ r_2'\in\{0,\dots,b\} \\ v+b-r_1'-r_2'\equiv 0 \,(\mathrm{mod}\, 2)}} (-1)^{r_1'+r_2'} \binom{v}{r_1'} \binom{b}{r_2'} (\log t)^{r_1'+r_2'} \left(\mathrm{i}\frac{\pi}{2}\right)^{v+b-r_1'-r_2'} \\
\times &\frac{1}{2\pi \mathrm{i}} \int_{\substack{\Re\mathfrak{e}(s)=\varepsilon_3 \\ |\Im\mathfrak{m}(s)|\le t+1}} \frac{1}{s} \left(\frac{U^2}{t}\right)^{2s} \frac{1}{(2s)^{2+x_2+3+y_2+3+x_1+y_1+1+w+c+1}} J_0(s)\, ds, \qquad (7.12)
\end{aligned}$$

$$j_{\mathbf{x},1} := \sum_{i_1+i_2=u} \binom{u}{i_1} \sum_{j_1+j_2=a} \binom{a}{j_1} \sum_{x_0=0}^{i_2+j_2} \sum_{\substack{x_1+x_2+x_3=i_1 \\ x_1,x_2,x_3\geq 0}} \sum_{\substack{y_1+y_2+y_3=j_1 \\ y_1,y_2,y_3\geq 0}} \binom{i_2+j_2}{x_0}$$
$$\times (-1)^{x_0+x_1+y_1} \binom{i_1}{x_1, x_2, x_3} \binom{j_1}{y_1, y_2, y_3} (-1)^{b+v}$$
$$\times \sum_{\substack{r_1'\in\{0,\dots,v\} \\ r_2'\in\{0,\dots,b\} \\ v+b-r_1'-r_2'\equiv 1 \pmod 2}} (-1)^{r_1'+r_2'} \binom{v}{r_1'} \binom{b}{r_2'} (\log t)^{r_1'+r_2'} \left(\mathrm{i}\frac{\pi}{2}\right)^{v+b-r_1'-r_2'}$$
$$\times \frac{1}{2\pi \mathrm{i}} \int_{\substack{\mathfrak{Re}(s)=\varepsilon_3 \\ |\mathfrak{Im}(s)|\leq t+1}} \frac{1}{s} \left(\frac{U^2}{t}\right)^{2s} \frac{\pi s}{(2s)^{2+x_2+3+y_2+3+x_1+y_1+1+w+c+1}} J_1(s)\, ds, \qquad (7.13)$$

and

$$j_{\mathbf{x},\mathrm{error}} := \sum_{i_1+i_2=u} \binom{u}{i_1} \sum_{j_1+j_2=a} \binom{a}{j_1} \sum_{x_0=0}^{i_2+j_2} \sum_{\substack{x_1+x_2+x_3=i_1 \\ x_1,x_2,x_3\geq 0}} \sum_{\substack{y_1+y_2+y_3=j_1 \\ y_1,y_2,y_3\geq 0}} \binom{i_2+j_2}{x_0}$$
$$\times (-1)^{x_0+x_1+y_1} \binom{i_1}{x_1, x_2, x_3} \binom{j_1}{y_1, y_2, y_3}$$
$$\times \frac{1}{2\pi \mathrm{i}} \int_{\substack{\mathfrak{Re}(s)=\varepsilon_3 \\ |\mathfrak{Im}(s)|\leq t+1}} \frac{G(s)}{s} \left(\frac{U}{2\pi}\right)^{4s} \zeta(1+2s)^2 \zeta^{(x_0)}(2s) g_{x_2}(s) g_{y_2}(s)$$
$$\times \zeta^{(x_1+y_1)}(1+2s) \mathscr{I}^{(x_3,y_3,i_2+j_2-x_0,0)}(0,0,0,s)$$
$$\times O\left(\delta^{-b-v} t^{-2\mathfrak{Re}(s)+2\delta} \exp(\pi|\mathfrak{Im}(s)| + \pi\delta)\left(\frac{1+2|s|^2}{t}\right)\right)(-1)^{w+c}\Gamma^{(w+c)}(2s)\, ds.$$

Clearly, $j_{\mathbf{x},\mathrm{error}} \ll U^{4\varepsilon_3} T^{-2\varepsilon_3+2\delta-1} \ll T^{-1/2}$ since $0 < \delta < \frac{1}{10}$ and $0 < \varepsilon_3 < 0.15$. Note that the integrals in $j_{\mathbf{x},0}$ and $j_{\mathbf{x},1}$ can be extended to the contour $\mathfrak{Re}(s) = \varepsilon_3$ at the cost of $O(T^{-2})$ since the $|\mathfrak{Im}(s)| > t+1$ parts of the integrals are $\ll T^{-2}$.

Observe that the integral in (7.12) (over the full line $\mathfrak{Re}(s) = \varepsilon_3$) is

$$\frac{1}{2\pi \mathrm{i}} \int_{(\varepsilon_3)} \frac{1}{2^{x_1+y_1+x_2+y_2+w+c+10}} \left(\frac{U^2}{t}\right)^{2s} \frac{1}{s^{x_1+y_1+x_2+y_2+w+c+11}} J_0(s)\, ds.$$

We then move this integral to $\mathfrak{Re}(s) = -1/4 + 2\delta$, where the integral on the new vertical line is $\ll (U^2/t)^{-1/2+4\delta}$ since

$$J_0(s) \ll \frac{1}{1+|\mathfrak{Im}(s)|^{100}},$$

and we see that the residue of the pole at $s = 0$ is

$$\frac{1}{2^{x_1+y_1+x_2+y_2+w+c+10}} \frac{1}{(x_1+y_1+x_2+y_2+w+c+10)!} \times \frac{d^{x_1+y_1+x_2+y_2+w+c+10}}{ds^{x_1+y_1+x_2+y_2+w+c+10}} \left(\left(\frac{U^2}{t} \right)^{2s} J_0(s) \right) \Big|_{s=0}$$
$$= \frac{1}{2^{x_1+y_1+x_2+y_2+w+c+10}} \frac{1}{(x_1+y_1+x_2+y_2+w+c+10)!} \times \sum_{k=0}^{x_1+y_1+x_2+y_2+w+c+10} \binom{x_1+y_1+x_2+y_2+w+c+10}{k} \left(\log \frac{U^4}{t^2} \right)^k \times J_0^{(x_1+y_1+x_2+y_2+w+c+10-k)}(0).$$

Similarly, the integral in (7.13) is

$$\frac{1}{2\pi i} \int_{(\varepsilon_3)} \frac{\pi}{2^{x_1+y_1+x_2+y_2+w+c+10}} \left(\frac{U^2}{t} \right)^{2s} \frac{1}{s^{x_1+y_1+x_2+y_2+w+c+10}} J_1(s)\, ds.$$

Moving the integral to $\Re e(s) = -1/4 + 2\delta$, we find that the new integral on $\Re e(s) = -1/4 + 2\delta$ is $\ll (U^2/t)^{-1/2+4\delta}$, and the residue of the pole at $s = 0$ is

$$\frac{\pi}{2^{x_1+y_1+x_2+y_2+w+c+10}} \frac{1}{(x_1+y_1+x_2+y_2+w+c+9)!} \times \sum_{k=0}^{x_1+y_1+x_2+y_2+w+c+9} \binom{x_1+y_1+x_2+y_2+w+c+9}{k} \times \left(\log \frac{U^4}{t^2} \right)^k J_1^{(x_1+y_1+x_2+y_2+w+c+9-k)}(0).$$

Gathering everything above together, we complete the proof of (7.7).

The functions $J_0(s)$ and $J_1(s)$ are holomophic for $-1/4 + 2\delta < \Re e(s) < 1/2$ because $\mathscr{I}(0, 0, 0, s)$ is holomorphic in this region due to Proposition 6.5. The upper bound for $J_0(s)$ and $J_1(s)$ is trivial since $|G(s)|$ decays exponentially.

For $i_{\mathbf{x},1}$, we know

$$i_{\mathbf{x},1} \ll (\log t)^{r_1'+r_2'} \left(\log \frac{U^4}{t^2} \right)^{x_1+y_1+x_2+y_2+w+c+9} \ll (\log T)^{r_1'+r_2'+x_1+y_1+x_2+y_2+w+c+9}.$$

Since

$$r_1' + r_2' + x_1 + y_1 + x_2 + y_2 + w + c + 9 \le v + b + x_1 + y_1 + x_2 + y_2 + w + c + 9 \le v + b + (i_1 + j_1) + w + c + 9,$$

which is $\le (a + b + c) + (u + v + w) + 9 \le 3 + 3 + 9 = 15$, we establish (7.10). ■

By (7.6) and Lemma 7.1, we have

$$I_O = 2\sum_{i'+i=3}\frac{b_{i'}}{i!}\sum_{j'+j=3}\frac{b_{j'}}{j!}\sum_{u+v+w=i}\binom{i}{u,v,w}\sum_{a+b+c=j}\binom{j}{a,b,c}\int_{-\infty}^{\infty}\omega(t)i_{\mathbf{x},0}\,dt + O(T(\log T)^{15}) + O\Big(T\Big(\frac{T}{T_0}\Big)^{1+C}\Big),$$

where $i_{\mathbf{x},0}$ is defined in (7.8). Note that we only need to consider the case $i=j=3$ since otherwise the above sum is $\ll T(\log T)^{i+j+10} \ll T(\log T)^{15}$. With the help of Maple, this gives

$$\begin{aligned} I_O = \int_{-\infty}^{\infty}\omega(t)\Big(&-\frac{(\log t)^4(\log(U^4/t^2))^{12}}{535088332800}+\frac{(\log t)^5(\log(U^4/t^2))^{11}}{122624409600} \\ &-\frac{(\log t)^6(\log(U^4/t^2))^{10}}{66886041600}-\frac{(\log t)^2(\log(U^4/t^2))^{14}}{119027426918400} \\ &+\frac{(\log t)^3(\log(U^4/t^2))^{13}}{6376469299200}-\frac{(\log(U^4/t^2))^{16}}{171399494762496000}\Big)\mathscr{I}(0,0,0,0)\,dt \\ &+ O(T(\log T)^{15}) + O\Big(T\Big(\frac{T}{T_0}\Big)^{1+C}\Big). \end{aligned} \tag{7.14}$$

Let $n_1, n_2 \geq 0$ be integers satisfying $n_1+n_2=16$. As we work over $c_1T \leq t \leq c_2T$, we know $\log t = \log T + O(1)$ and hence

$$\begin{aligned} &\int_{-\infty}^{\infty}\omega(t)(\log t)^{n_1}(\log(U^4/t^2))^{n_2}\,dt \\ &= (\log T + O(1))^{n_1}\big(4(1-\varepsilon)\log T - 2\log T + O(1)\big)^{n_2}\int_{-\infty}^{\infty}\omega(t)\,dt \\ &= ((\log T)^{n_1} + O((\log T)^{n_1-1}))\big((2-4\varepsilon)^{n_2}(\log T)^{n_2} + O((\log T)^{n_2-1})\big)\int_{-\infty}^{\infty}\omega(t)\,dt \\ &= \big((2-4\varepsilon)^{n_2}(\log T)^{n_1+n_2} + O((\log T)^{n_1+n_2-1})\big)\int_{-\infty}^{\infty}\omega(t)\,dt \\ &= \big(2^{n_2}(\log T)^{16} + O(\varepsilon(\log T)^{16}) + O((\log T)^{15})\big)\int_{-\infty}^{\infty}\omega(t)\,dt. \end{aligned} \tag{7.15}$$

Note that in the second equality, the terms $O((\log T)^{n_1-1})$ and $O((\log T)^{n_2-1})$ are valid even if $n_1=0$ or $n_2=0$.

Finally, plugging (7.15) into (7.14), we derive

$$\begin{aligned} I_O = \Big(&-\frac{2^{12}}{535088332800}+\frac{2^{11}}{122624409600}-\frac{2^{10}}{66886041600}-\frac{2^{14}}{119027426918400} \\ &+\frac{2^{13}}{6376469299200}-\frac{2^{16}}{171399494762496000}\Big)\mathscr{I}(0,0,0,0)\int_{-\infty}^{\infty}\omega(t)\,dt \\ &+ O(\varepsilon T(\log T)^{16}) + O\Big(T\Big(\frac{T}{T_0}\Big)^{1+C}\Big) \end{aligned}$$

$$= -\frac{13381\mathscr{I}(0,0,0,0)}{2615348736000}\int_{-\infty}^{\infty}\omega(t)(\log T)^{16}\,dt$$
$$+ O(\varepsilon T(\log T)^{16}) + O\left(T\left(\frac{T}{T_0}\right)^{1+C}\right).$$

We see from (8.9) that $\mathscr{I}(0,0,0,0) = a_4$. This completes the proof of Proposition 3.3.

8. Proofs of Proposition 6.5 and Lemmata 7.2 and 7.3

In this section, we prove Proposition 6.5. Then we obtain an asymptotic formula for $\mathscr{G}_{s,t}^{(v,b)}(1,1)$ in (7.4). We shall also simplify $\mathscr{H}_s^{(u,a)}(1,1)$ which appeared in (7.5).

8.1. *Proof of Proposition 6.5*

In this proof, we let $\sigma = \Re e(s)$. From the bound $|c_q(r)| \le (q,r)$ and (2.6) of Lemma 2.3, it follows that $\mathscr{H}(u_1,u_2,u_3,s)$, defined in (6.26), is absolutely convergent in $\Re e(s) > 1/2 + 2\delta'$. Furthermore, since $c_q(r) = \sum_{d|(q,r)} d\mu(q/d)$, we have

$$\mathscr{H}(u_1,u_2,u_3,s) = \sum_{r=1}^{\infty}\sum_{q=1}^{\infty}\frac{c_q(r)G_4(1+u_1,q)G_4(1+u_2,q)}{q^{2+u_3}r^{2s-u_3}}$$
$$= \sum_{q=1}^{\infty}\alpha_q\sum_{r=1}^{\infty}\frac{1}{r^c}\sum_{d|q,d|r} d\mu(q/d),$$

where $\alpha_q = \frac{G_4(1+u_1,q)G_4(1+u_2,q)}{q^{2+u_3}}$ and $c = 2s - u_3$. Thus,

$$\mathscr{H}(u_1,u_2,u_3,s) = \sum_{q=1}^{\infty}\alpha_q\sum_{d|q} d\mu\left(\frac{q}{d}\right)\sum_{r\ge 1,\,d|r}\frac{1}{r^c} = \sum_{q=1}^{\infty}\alpha_q\sum_{d|q}\frac{d\mu(q/d)}{d^c}\zeta(c)$$
$$= \zeta(c)\mathscr{H}^*(u_1,u_2,u_3,s),$$

where

$$\mathscr{H}^*(u_1,u_2,u_3,s) = \sum_{q=1}^{\infty}\alpha_q q^{1-c}\sum_{d|q}\frac{\mu(d)}{d^{1-c}}.$$

For any prime p and $j \ge 1$, we have $\sum_{d|p^j}\frac{\mu(d)}{d^{1-c}} = 1 - \frac{1}{p^{1-c}}$. By multiplicativity,

$$\mathscr{H}^*(u_1,u_2,u_3,s)$$
$$= \prod_p\left(1 + \sum_{j=1}^{\infty}\frac{G_4(1+u_1,p^j)G_4(1+u_2,p^j)}{(p^j)^{2+u_3}}(p^j)^{1+u_3-2s}(1-p^{2s-1-u_3})\right)$$
$$= \prod_p\left(1 + \sum_{j=1}^{\infty}G_4(1+u_1,p^j)G_4(1+u_2,p^j)\cdot\frac{1-p^{2s-1-u_3}}{(p^j)^{1+2s}}\right). \tag{8.1}$$

We aim to simplify the above expression with the brackets. At this point, it will be convenient to introduce the following notation. Let

$$U = p^{-1}, \quad W = p^{-1-2s}, \quad X_i = p^{-u_i}, \tag{8.2}$$

for $i = 1, 2, 3$, where $|u_i| \le \delta'$ as required in (6.27). Observe that

$$|W| \le p^{-1-2\sigma}, \quad p^{-\delta'} \le |X_i| \le p^{\delta'}$$

for $i = 1, 2, 3$. Also, we shall set

$$T_j = G_4(1 + u_1, p^j)G_4(1 + u_2, p^j) \quad \text{and} \quad f_j = \frac{1 - p^{2s-1-u_3}}{(p^j)^{1+2s}} \tag{8.3}$$

for $j \in \mathbb{Z}_{\ge 0}$. Note that

$$\mathscr{H}^*(u_1, u_2, u_3, s) = \prod_p \Big(1 + \sum_{j=1}^{\infty} T_j f_j\Big).$$

We aim to simplify this further. It follows from (8.2) that for $j \in \mathbb{N}$,

$$f_j = W^j - X_3 U^2 W^{j-1}. \tag{8.4}$$

Also, by (2.9) and (2.10), for $i = 1, 2$,

$$\begin{aligned} G_4(1 + u_i, p) &= \frac{1}{1-U} \sum_{j=0}^{3} \alpha_j(X_i) U^j, \\ G_4(1 + u_i, p^2) &= \frac{1}{1-U} \sum_{j=0}^{3} \beta_j(X_i) U^j, \end{aligned} \tag{8.5}$$

where

$$\alpha_0(X) = 4 - X^{-1}, \quad \alpha_1(X) = -6X, \quad \alpha_2(X) = 4X^2, \quad \alpha_3(X) = -X^3,$$
$$\beta_0(X) = 10 - 4X^{-1}, \quad \beta_1(X) = -20X + 6,$$
$$\beta_2(X) = 15X^2 - 4X, \quad \beta_3(X) = -4X^3 + X^2.$$

Note that we have the bounds

$$\alpha_0(X_i), \beta_0(X_i) \ll p^{\delta'} \quad \text{and} \quad \alpha_j(X_i), \beta_j(X_i) \ll p^{j\delta'} \tag{8.6}$$

for $j = 1, 2, 3$. By (8.3) and (8.5), we know

$$\begin{aligned} T_1 &= (1-U)^{-2} \Big(\sum_{j=0}^{3} \alpha_j(X_1) U^j\Big)\Big(\sum_{j'=0}^{3} \alpha_{j'}(X_2) U^{j'}\Big) \\ &= (1-U)^{-2} \sum_{k=0}^{6} A_k U^k = \sum_{k=0}^{\infty} \tilde{A}_k U^k, \end{aligned}$$

where
$$A_k = \sum_{\substack{j+j'=k \\ 0\le j,j'\le 3}} \alpha_j(X_1)\alpha_{j'}(X_2) \quad \text{and} \quad \tilde{A}_k = \sum_{\substack{i+j=k \\ 0\le j\le 6}} (i+1)A_j. \tag{8.7}$$
It follows from (8.6) and (8.7) that
$$\tilde{A}_0 \ll p^{2\delta'}, \quad \tilde{A}_j \ll \begin{cases} p^{(j+1)\delta'} & \text{for } j = 1,\dots,6, \\ (j+1)p^{7\delta'} & \text{for } j \ge 7. \end{cases}$$
We now have
$$T_1 f_1 = \Big(\sum_{k=0}^{\infty} \tilde{A}_k U^k\Big)(W - X_3U^2).$$
Expanding this out, we find that
$$T_1 f_1 = \tilde{A}_0 W + O(p^{3\delta'-2}) + O\Big(\Big(\sum_{k=1}^{\infty} |\tilde{A}_k| U^k\Big)(p^{-1-2\sigma} + p^{-2+\delta'})\Big).$$
Observe that
$$\sum_{k=1}^{\infty} |\tilde{A}_k| U^k \ll \sum_{k=1}^{6} p^{(k+1)\delta'-k} + p^{14\delta'-7} \ll p^{2\delta'-1},$$
as long as $\delta' < 1/2$, and thus
$$\begin{aligned} T_1 f_1 &= \tilde{A}_0 W + O(p^{3\delta'-2} + p^{2\delta'-2-2\sigma} + p^{3\delta'-3}) \\ &= \tilde{A}_0 W + O(p^{3\delta'-2} + p^{2\delta'-2-2\sigma}). \end{aligned} \tag{8.8}$$
Since
$$\tilde{A}_0 = A_0 = \alpha_0(X_1)\alpha_0(X_2) = (4 - X_1^{-1})(4 - X_2^{-1}) = 16 - 4X_1^{-1} - 4X_2^{-1} + (X_1X_2)^{-1},$$
and the error term in (8.8) is $O(p^{3\delta'+\vartheta(\sigma)})$, it follows that
$$1 + T_1 f_1 = 1 + \frac{16}{p^{1+2s}} - \frac{4}{p^{1+2s-u_1}} - \frac{4}{p^{1+2s-u_2}} + \frac{1}{p^{1+2s-u_1-u_2}} + O(p^{3\delta'+\vartheta(\sigma)}).$$
Therefore, for $\sigma > 1/2 + 2\delta'$, we can factor out some zeta factors from (8.1), which leads to
$$\mathscr{H}^*(u_1,u_2,u_3,s) = \frac{\zeta(1+2s)^{16}\zeta(1+2s-u_1-u_2)}{\zeta(1+2s-u_1)^4\zeta(1+2s-u_2)^4}\mathscr{I}(u_1,u_2,u_3,s),$$
where $\mathscr{I}(u_1,u_2,u_3,s) = \prod_p \mathscr{I}_p(u_1,u_2,u_3,s)$ is as in (6.28), and $\mathscr{I}_p(u_1,u_2,u_3,s)$ is defined as in (6.29). We note that
$$\prod_p \mathscr{I}_p(0,0,0,0) = \prod_p \Big(1 - \frac{1}{p}\Big)^9 \times \Big(1 + \sum_{j=1}^{\infty} G_4(1,p^j)^2 \cdot \frac{1-p^{-1}}{(p^j)},\Big) = a_4, \tag{8.9}$$

which is shown in [7, p. 595, starting at (43)]. We now aim to provide an analytic continuation of $\mathscr{I}(u_1, u_2, u_3, s)$. Observe that

$$\mathscr{I}_p(u_1, u_2, u_3, s) = \Big(1 + \sum_{j=1}^{\infty} T_j f_j\Big)\Pi, \tag{8.10}$$

where

$$\Pi = (1-W)^{16}(1 - X_1^{-1}X_2^{-1}W)(1 - X_1^{-1}W)^{-4}(1 - X_2^{-1}W)^{-4}. \tag{8.11}$$

To simplify the local factor (8.10), we first write $\Pi = \sum_{j=0}^{\infty} a_j W^j$ with

$$\begin{aligned} a_0 &= 1, \\ a_1 &= 4X_1^{-1} + 4X_2^{-1} - (X_1X_2)^{-1} - 16, \\ a_2 &= 10/X_2^2 + 4(4/X_1 - 1/(X_1X_2) - 16)/X_2 + 10/X_1^2 \\ &\quad + 4(-1/(X_1X_2) - 16)/X_1 + 16/(X_1X_2) + 120. \end{aligned}$$

Observe that $a_1 = -\tilde{A}_0$. It follows from (8.11) that $|a_j| \ll j^7 p^{(j+1)\delta'}$ and thus

$$\sum_{j=3}^{\infty} a_j W^j \ll \sum_{j=3}^{\infty} j^7 p^{(j+1)\delta'}\Big(\frac{1}{p^{1+2\sigma}}\Big)^j \ll p^{\delta'}\Big(\frac{p^{\delta'}}{p^{1+2\sigma}}\Big)^3 = \frac{p^{4\delta'}}{p^{3+6\sigma}} \ll p^{4\delta'+\vartheta(\sigma)}.$$

Therefore, we have

$$\Pi = (1 + a_1W + a_2W^2) + O(p^{4\delta'+\vartheta(\sigma)}).$$

To bound the terms with $j \geq 3$ in (8.10), we make use of $|T_j| \leq (32)^2\tau_4(p^j)^2(p^j)^{2\delta'}$ and (8.4) to obtain

$$\begin{aligned} \sum_{j=3}^{\infty} T_j f_j &\ll \sum_{j=3}^{\infty} \tau_4(p^j)^2 p^{j2\delta'}\Big(\Big(\frac{1}{p^{1+2\sigma}}\Big)^j + \frac{p^{\delta'}}{p^2}\Big(\frac{1}{p^{1+2\sigma}}\Big)^{j-1}\Big) \\ &\ll p^{6\delta'}\Big(\Big(\frac{1}{p^{1+2\sigma}}\Big)^3 + \frac{p^{\delta'}}{p^2}\Big(\frac{1}{p^{1+2\sigma}}\Big)^2\Big) \leq p^{7\delta'}\Big(\frac{1}{p^{3+6\sigma}} + \frac{1}{p^{4+4\sigma}}\Big) \\ &\ll p^{7\delta'}p^{\vartheta(\sigma)}. \end{aligned}$$

We now analyse $T_2 f_2$. By (8.3) and (8.5), we get

$$\begin{aligned} T_2 &= (1-U)^{-2}\Big(\sum_{j=0}^{3}\beta_j(X_1)U^j\Big)\Big(\sum_{j'=0}^{3}\beta_{j'}(X_2)U^{j'}\Big) \\ &= (1-U)^{-2}\sum_{k=0}^{6} B_k U^k = \sum_{k=0}^{\infty}\tilde{B}_k U^k, \end{aligned}$$

where

$$B_k = \sum_{\substack{j+j'=k \\ 0\leq j,j'\leq 3}} \beta_j(X_1)\beta_{j'}(X_2) \quad \text{and} \quad \tilde{B}_k = \sum_{\substack{i+j=k \\ 0\leq j\leq 6}} (i+1)B_j.$$

With these observations in hand, we find

$$B_0 \ll p^{2\delta'}, \quad B_j \ll p^{(j+1)\delta'} \quad \text{for } j = 1, \dots, 6,$$

$$\tilde{B}_0 \ll p^{2\delta'}, \quad \tilde{B}_j \ll \begin{cases} p^{(j+1)\delta'} & \text{for } j = 1, \dots, 6, \\ p^{7\delta'} & \text{for } j \geq 7. \end{cases}$$

We then arrive at

$$T_2 f_2 = \Big(\sum_{k=0}^{\infty} \tilde{B}_k U^k\Big)(W^2 - X_3 U^2 W) = \tilde{B}_0 W^2 + O(p^{3\delta'-3-2\sigma})$$
$$+ O\Big(\Big(\sum_{k=1}^{\infty} |\tilde{B}_k| U^k\Big)(p^{-2-4\sigma} + p^{\delta'-3-2\sigma})\Big),$$

which gives

$$T_2 f_2 = \tilde{B}_0 W^2 + O(p^{3\delta'-3-2\sigma} + p^{2\delta'-3-4\sigma} + p^{3\delta'-4-2\sigma})$$
$$= \tilde{B}_0 W^2 + O(p^{3\delta'+\vartheta(\sigma)}).$$

Putting everything together, we find from (8.10) that

$$\mathscr{I}_p(u_1, u_2, u_3, s)$$
$$= \big(1 + \tilde{A}_0 W + \tilde{B}_0 W^2 + O(p^{7\delta'} p^{\vartheta(\sigma)})\big)\big(1 + a_1 W + a_2 W^2 + O(p^{4\delta'+\vartheta(\sigma)})\big)$$
$$= 1 + (\tilde{B}_0 + \tilde{A}_0 a_1 + a_2) W^2 + O(p^{7\delta'} p^{\vartheta(\sigma)}).$$

Using Maple, one may check

$$\tilde{B}_0 + \tilde{A}_0 a_1 + a_2 = Y_p,$$

where Y_p is defined in (6.30). From this, we see $\mathscr{I}(u_1, u_2, u_3, s)$ is holomorphic in $\Re e(s) > -1/4 + \delta'$ and $\mathscr{H}(u_1, u_2, u_3, s)$ has a meromorphic continuation to $\Re e(s) > -1/4 + \delta'$ with the exception of the poles listed in (6.31).

8.2. *Proofs of Lemmata 7.2 and 7.3*

We assume $\Re e(s) = \varepsilon_3$ and recall from (7.4) that

$$\mathscr{G}_{s,t}^{(v,b)}(1,1) = \frac{\partial^v}{\partial z_1^v} \frac{\partial^b}{\partial z_2^b} \left(\frac{\Gamma\big(z_2 - s - \frac{1}{2} + it\big)}{\Gamma\big(s - z_1 + \frac{3}{2} + it\big)} + \frac{\Gamma\big(z_1 - s - \frac{1}{2} - it\big)}{\Gamma\big(s - z_2 + \frac{3}{2} - it\big)} \right)\Bigg|_{z_1 = z_2 = 1}.$$

Making the change of variable $z_2 = 1 - \beta$ and $z_1 = 1 - \alpha$ for the first term and $z_1 = 1 - \beta'$ and $z_2 = 1 - \alpha'$ for the second term, we obtain

$$\mathscr{G}_{s,t}^{(v,b)}(1,1) = (-1)^{b+v} \frac{\partial^b}{\partial \beta^b} \frac{\partial^v}{\partial \alpha^v} \frac{\Gamma\big(\frac{1}{2} - \beta - s + it\big)}{\Gamma\big(\frac{1}{2} + \alpha + s + it\big)}\Bigg|_{\alpha = \beta = 0}$$
$$+ (-1)^{b+v} \frac{\partial^b}{\partial (\alpha')^b} \frac{\partial^v}{\partial (\beta')^v} \frac{\Gamma\big(\frac{1}{2} - \beta' - s - it\big)}{\Gamma\big(\frac{1}{2} + \alpha' + s - it\big)}\Bigg|_{\alpha' = \beta' = 0}. \tag{8.12}$$

Now, by using Lemma 6.4, we can derive Lemma 7.2, which in particular expresses $\mathscr{G}_{s,t}^{(v,b)}(1,1)$ as a combinatorial sum.

Proof of Lemma 7.2. Recall (8.12). It follows from Lemma 6.4 (i) that for $|\mathfrak{Im}(s)| \le t+1$,

$$\frac{\Gamma\big(\frac{1}{2}-\beta-s+\mathrm{i}t\big)}{\Gamma\big(\frac{1}{2}+\alpha+s+\mathrm{i}t\big)} = t^{-(2s+\alpha+\beta)}\exp\Big(-\mathrm{i}\frac{\pi}{2}(2s+\alpha+\beta)\Big)\Big(1+O\Big(\frac{1+2|s|^2}{t}\Big)\Big). \tag{8.13}$$

Observe that the v-th derivative, with respect to α, of the main term above is

$$\begin{aligned}&\frac{\partial^v}{\partial\alpha^v}\Big(t^{-(2s+\alpha+\beta)}\exp\Big(-\mathrm{i}\frac{\pi}{2}(2s+\alpha+\beta)\Big)\Big)\\ &\qquad = \sum_{r_1'=0}^{v}(-1)^{r_1'}\binom{v}{r_1'}(\log t)^{r_1'}t^{-(2s+\alpha+\beta)}\exp\Big(-\mathrm{i}\frac{\pi}{2}(2s+\alpha+\beta)\Big)\Big(-\mathrm{i}\frac{\pi}{2}\Big)^{v-r_1'}.\end{aligned}$$

By a direct computation, we see that the b-th derivative, with respect to β, of the above expression is

$$\begin{aligned}&\frac{\partial^b}{\partial\beta^b}\frac{\partial^v}{\partial\alpha^v}\Big(t^{-(2s+\alpha+\beta)}\exp\Big(-\mathrm{i}\frac{\pi}{2}(2s+\alpha+\beta)\Big)\Big)\\ &= \sum_{r_2'=0}^{b}\sum_{r_1'=0}^{v}(-1)^{r_1'+r_2'}\binom{v}{r_1'}\binom{b}{r_2'}(\log t)^{r_1'+r_2'}t^{-(2s+\alpha+\beta)}\Big(-\mathrm{i}\frac{\pi}{2}\Big)^{v+b-r_1'-r_2'}\\ &\quad\times\exp\Big(-\mathrm{i}\frac{\pi}{2}(2s+\alpha+\beta)\Big).\end{aligned}\tag{8.14}$$

Now, we handle the error term in (8.13). Note that

$$t^{-(2s+\alpha+\beta)}\exp\Big(-\mathrm{i}\frac{\pi}{2}(2s+\alpha+\beta)\Big)O\Big(\frac{1+2|s|^2}{t}\Big)$$

is holomorphic in the region $|\alpha|, |\beta| \le \delta$. By the Cauchy integral formula, we have

$$\begin{aligned}&\frac{\partial^b}{\partial\beta^b}\frac{\partial^v}{\partial\alpha^v}\Big(t^{-(2s+\alpha+\beta)}\exp\Big(-\mathrm{i}\frac{\pi}{2}(2s+\alpha+\beta)\Big)O\Big(\frac{1+2|s|^2}{t}\Big)\Big)\Big|_{\alpha=\beta=0}\\ &= \frac{b!v!}{(2\pi\mathrm{i})^2}\int_{C(0,\delta)}\int_{C(0,\delta)}\Big(t^{-(2s+\alpha+\beta)}\exp\Big(-\mathrm{i}\frac{\pi}{2}(2s+\alpha+\beta)\Big)O\Big(\frac{1+2|s|^2}{t}\Big)\Big)\\ &\qquad\qquad\times\frac{1}{\beta^{b+1}\alpha^{v+1}}\,d\alpha\,d\beta\\ &\ll \delta^{-b-v}t^{-2\mathfrak{Re}(s)+2\delta}\exp(\pi\mathfrak{Im}(s)+\pi\delta)\Big(\frac{1+2|s|^2}{t}\Big),\end{aligned}$$

where $C(0,\delta)$ is the circle centred at 0 with radius δ. This estimate, together with (8.14), gives

$$\frac{\partial^b}{\partial\beta^b}\frac{\partial^v}{\partial\alpha^v}\frac{\Gamma\big(\frac{1}{2}-\beta-s+\mathrm{i}t\big)}{\Gamma\big(\frac{1}{2}+\alpha+s+\mathrm{i}t\big)}\bigg|_{\alpha=\beta=0}$$
$$=\sum_{r_2'=0}^{b}\sum_{r_1'=0}^{v}(-1)^{r_1'+r_2'}\binom{v}{r_1'}\binom{b}{r_2'}(\log t)^{r_1'+r_2'}t^{-2s}\Big(-\mathrm{i}\frac{\pi}{2}\Big)^{v+b-r_1'-r_2'}\exp(-\mathrm{i}\pi s)$$
$$+O\bigg(\delta^{-b-v}t^{-2\Re\mathrm{e}(s)+2\delta}\exp(\pi\Im\mathrm{m}(s)+\pi\delta)\Big(\frac{1+2|s|^2}{t}\Big)\bigg). \tag{8.15}$$

Similarly, by Lemma 6.4 (i), we have

$$\frac{\Gamma\big(\frac{1}{2}-\beta'-s-\mathrm{i}t\big)}{\Gamma\big(\frac{1}{2}+\alpha'+s-\mathrm{i}t\big)}=t^{-(2s+\alpha'+\beta')}\exp\Big(\mathrm{i}\frac{\pi}{2}(2s+\alpha'+\beta')\Big)\bigg(1+O\Big(\frac{1+2|s|^2}{t}\Big)\bigg). \tag{8.16}$$

It can be derived that

$$\frac{\partial^b}{\partial\alpha'^b}\frac{\partial^v}{\partial\beta'^v}\bigg(t^{-(2s+\alpha'+\beta')}\exp\Big(\mathrm{i}\frac{\pi}{2}(2s+\alpha'+\beta')\Big)\bigg)$$
$$=\sum_{r_2'=0}^{b}\sum_{r_1'=0}^{v}(-1)^{r_1'+r_2'}\binom{v}{r_1'}\binom{b}{r_2'}(\log t)^{r_1'+r_2'}t^{-(2s+\alpha'+\beta')}\Big(\mathrm{i}\frac{\pi}{2}\Big)^{v+b-r_1'-r_2'}$$
$$\times\exp\Big(\mathrm{i}\frac{\pi}{2}(2s+\alpha'+\beta')\Big).$$

Again, applying the Cauchy integral formula gives

$$\frac{\partial^b}{\partial\alpha'^b}\frac{\partial^v}{\partial\beta'^v}\bigg(t^{-(2s+\alpha'+\beta')}\exp\Big(\mathrm{i}\frac{\pi}{2}(2s+\alpha'+\beta')\Big)O\Big(\frac{1+2|s|^2}{t}\Big)\bigg)\bigg|_{\alpha'=\beta'=0}$$
$$\ll\delta^{-b-v}t^{-2\Re\mathrm{e}(s)+2\delta}\exp(-\pi\Im\mathrm{m}(s)+\pi\delta)\Big(\frac{1+2|s|^2}{t}\Big),$$

and thus

$$\frac{\partial^b}{\partial\alpha'^b}\frac{\partial^v}{\partial\beta'^v}\bigg(\frac{\Gamma\big(\frac{1}{2}-\beta'-s-\mathrm{i}t\big)}{\Gamma\big(\frac{1}{2}+\alpha'+s-\mathrm{i}t\big)}\bigg)\bigg|_{\alpha'=\beta'=0}$$
$$=\sum_{r_2'=0}^{b}\sum_{r_1'=0}^{v}(-1)^{r_1'+r_2'}\binom{v}{r_1'}\binom{b}{r_2'}(\log t)^{r_1'+r_2'}t^{-2s}\Big(\mathrm{i}\frac{\pi}{2}\Big)^{v+b-r_1'-r_2'}\exp(\mathrm{i}\pi s)$$
$$+O\bigg(\delta^{-b-v}t^{-2\Re\mathrm{e}(s)+2\delta}\exp(-\pi\Im\mathrm{m}(s)+\pi\delta)\Big(\frac{1+2|s|^2}{t}\Big)\bigg). \tag{8.17}$$

Finally, combining (8.15) and (8.17), we complete the proof of the first part of the lemma.

Now we prove the second part. By the Cauchy integral formula, it follows that for $|\mathfrak{Im}(s)| > t+1$,

$$\frac{\partial^b}{\partial\beta^b}\frac{\partial^v}{\partial\alpha^v}\left(\frac{\Gamma\big(\frac{1}{2}-\beta-s+\mathrm{i}t\big)}{\Gamma\big(\frac{1}{2}+\alpha+s+\mathrm{i}t\big)}\right)\Bigg|_{\alpha=\beta=0} = \frac{b!v!}{(2\pi\mathrm{i})^2}\int_{C(0,\delta)}\int_{C(0,\delta)}\frac{\Gamma\big(\frac{1}{2}-\beta-s+\mathrm{i}t\big)}{\Gamma\big(\frac{1}{2}+\alpha+s+\mathrm{i}t\big)}\frac{1}{\beta^{b+1}\alpha^{v+1}}\,d\alpha\,d\beta.$$

Note that by Lemma 6.4 (ii),

$$\left|\frac{\Gamma\big(\frac{1}{2}-\beta-s+\mathrm{i}t\big)}{\Gamma\big(\frac{1}{2}+\alpha+s+\mathrm{i}t\big)}\right| \ll \frac{\mathfrak{Im}(s)^2}{t^2}e^{\pi|\mathfrak{Im}(s)|}.$$

Therefore,

$$\frac{\partial^b}{\partial\beta^b}\frac{\partial^v}{\partial\alpha^v}\left(\frac{\Gamma\big(\frac{1}{2}-\beta-s+\mathrm{i}t\big)}{\Gamma\big(\frac{1}{2}+\alpha+s+\mathrm{i}t\big)}\right)\Bigg|_{\alpha=\beta=0} \ll \delta^{-b-v}\frac{\mathfrak{Im}(s)^2}{t^2}e^{\pi|\mathfrak{Im}(s)|}.$$

In a similar way, we can derive

$$\frac{\partial^b}{\partial\alpha'^b}\frac{\partial^v}{\partial\beta'^v}\left(\frac{\Gamma\big(\frac{1}{2}-\beta'-s-\mathrm{i}t\big)}{\Gamma\big(\frac{1}{2}+\alpha'+s-\mathrm{i}t\big)}\right)\Bigg|_{\alpha'=\beta'=0} \ll \delta^{-b-v}\frac{\mathfrak{Im}(s)^2}{t^2}e^{\pi|\mathfrak{Im}(s)|}.$$

This completes the proof. ∎

To close this section, we prove Lemma 7.3.

Proof of Lemma 7.3. Recall the relation between $\mathcal{H}_s^{(u,a)}(1,1)$ and $\mathscr{H}(u_1,u_2,u_3,s)$ from (7.5) and the remark below it. We only need to handle $\mathscr{H}(u_1,u_2,u_3,s)$. By Proposition 6.5, for $\mathfrak{Re}(s)=\varepsilon_3$, it follows that

$$\frac{\partial^{i_2+j_2}}{\partial {u_3}^{i_2+j_2}}\mathscr{H}(u_1,u_2,u_3,s) = \frac{\zeta(1+2s)^{16}\zeta(1+2s-u_1-u_2)}{\zeta(1+2s-u_1)^4\zeta(1+2s-u_2)^4}\sum_{x_0=0}^{i_2+j_2}\binom{i_2+j_2}{x_0}(-1)^{x_0} \times \zeta^{(x_0)}(2s-u_3)\mathscr{I}^{(0,0,i_2+j_2-x_0,0)}(u_1,u_2,u_3,s).$$

By a direct calculation, we see that $\frac{\partial^{i_1}}{\partial {u_1}^{i_1}}\frac{\partial^{i_2+j_2}}{\partial {u_3}^{i_2+j_2}}\mathscr{H}(u_1,u_2,u_3,s)$ equals

$$\frac{\zeta(1+2s)^{16}}{\zeta(1+2s-u_2)^4}\sum_{x_0=0}^{i_2+j_2}\binom{i_2+j_2}{x_0}(-1)^{x_0}\zeta^{(x_0)}(2s-u_3) \times \sum_{\substack{x_1+x_2+x_3=i_1\\ x_1,x_2,x_3\geq 0}}\binom{i_1}{x_1,x_2,x_3}(-1)^{x_1}\zeta^{(x_1)}(1+2s-u_1-u_2) \times \frac{\partial^{x_2}}{\partial {u_1}^{x_2}}\left(\frac{1}{\zeta(1+2s-u_1)^4}\right)\mathscr{I}^{(x_3,0,i_2+j_2-x_0,0)}(u_1,u_2,u_3,s),$$

and thus $\frac{\partial^{j_1}}{\partial {u_2}^{j_1}} \frac{\partial^{i_1}}{\partial {u_1}^{i_1}} \frac{\partial^{i_2+j_2}}{\partial {u_3}^{i_2+j_2}} \mathscr{H}(u_1,u_2,u_3,s)$ is equal to

$$\zeta(1+2s)^{16} \sum_{x_0=0}^{i_2+j_2} \binom{i_2+j_2}{x_0} (-1)^{x_0} \zeta^{(x_0)}(2s-u_3)$$
$$\times \sum_{\substack{x_1+x_2+x_3=i_1 \\ x_1,x_2,x_3\geq 0}} \binom{i_1}{x_1,x_2,x_3} (-1)^{x_1} \frac{\partial^{x_2}}{\partial {u_1}^{x_2}} \left(\frac{1}{\zeta(1+2s-u_1)^4} \right)$$
$$\times \sum_{\substack{y_1+y_2+y_3=j_1 \\ y_1,y_2,y_3\geq 0}} \binom{j_1}{y_1,y_2,y_3} (-1)^{y_1} \zeta^{(x_1+y_1)}(1+2s-u_1-u_2)$$
$$\times \frac{\partial^{y_2}}{\partial {u_2}^{y_2}} \left(\frac{1}{\zeta(1+2s-u_2)^4} \right) \mathscr{I}^{(x_3,y_3,i_2+j_2-x_0,0)}(u_1,u_2,u_3,s).$$

Taking $u_1=u_2=u_3=0$, we obtain

$$\frac{\partial^{j_1}}{\partial {u_2}^{j_1}} \frac{\partial^{i_1}}{\partial {u_1}^{i_1}} \frac{\partial^{i_2+j_2}}{\partial {u_3}^{i_2+j_2}} \mathscr{H}(u_1,u_2,u_3,s) \Big|_{u_1=u_2=u_3=0}$$
$$= \zeta(1+2s)^2 \sum_{x_0=0}^{i_2+j_2} \sum_{\substack{x_1+x_2+x_3=i_1 \\ x_1,x_2,x_3\geq 0}} \sum_{\substack{y_1+y_2+y_3=j_1 \\ y_1,y_2,y_3\geq 0}} \binom{i_2+j_2}{x_0} (-1)^{x_0+x_1+y_1} \zeta^{(x_0)}(2s)$$
$$\times \binom{i_1}{x_1,x_2,x_3} \binom{j_1}{y_1,y_2,y_3} g_{x_2}(s) g_{y_2}(s) \zeta^{(x_1+y_1)}(1+2s)$$
$$\times \mathscr{I}^{(x_3,y_3,i_2+j_2-x_0,0)}(0,0,0,s),$$

where $g_k(s)$ is defined as in (7.11). Together with (7.5), this completes the proof. ∎

9. Appendix 1: Additive divisor conjecture for τ_k and τ_ℓ

This appendix is based on the ideas of [9] and follows closely their notation and presentation. Let $f(x,y)$ be a smooth function compactly supported on $[X,2X]\times[Y,2Y]$, and let ϕ be an even smooth compactly supported function ϕ which satisfies $\phi(0)=1$. Note that

$$D_{f;k,\ell}(r) = \sum_{m-n=r} \tau_k(m)\tau_\ell(n) f(m,n)\phi(m-n-r).$$

We set $F(x,y)=f(x,y)\phi(x-y-r)$, and we define $\delta(u)=1$ if $u=0$ and $\delta(u)=0$ otherwise. We have

$$D_{f;k,\ell}(r) = \sum_{m,n\geq 1} \tau_k(m)\tau_\ell(n) F(m,n)\delta(m-n-r).$$

Since $\delta(n)=\sum_{q=1}^{\infty}\sum^{*}_{d \pmod q} e(\frac{dn}{q})\Delta_q(n)$, it follows that

$$D_{f;k,\ell}(r) = \sum_{m,n\geq 1} \tau_k(m)\tau_\ell(n) F(m,n) \sum_{q=1}^{\infty} \sum^{*}_{d \pmod q} e\big(\tfrac{d}{q}(m-n-r)\big)\Delta_q(m-n-r).$$

Set $E(x, y) = F(x, y)\Delta_q(x - y - r)$ and thus

$$D_{f;k,\ell}(r) = \sum_{q=1}^{\infty} \sideset{}{^*}\sum_{d \pmod q} e\big(\tfrac{-dr}{q}\big) \sum_{m\ge 1} \tau_k(m) e\big(\tfrac{md}{q}\big) \sum_{n\ge 1} \tau_\ell(n) e\big(\tfrac{-nd}{q}\big) E(m, n). \tag{9.1}$$

Recall the Mellin transform of the smooth function $E(x, y)$ is

$$\tilde{E}(z_1, z_2) := \int_0^\infty \int_0^\infty E(x, y) x^{z_1-1} y^{z_2-1}\, dx\, dy.$$

By the inverse Mellin transform, we have

$$E(x, y) = \frac{1}{(2\pi \mathrm{i})^2} \int_{(c_1)} \int_{(c_2)} \tilde{E}(z_1, z_2) x^{-z_1} y^{-z_2}\, dz_2\, dz_1,$$

where $c_1, c_2 > 0$. Inserting this into (9.1) gives

$$\begin{aligned} D_{f;k,\ell}(r) = \sum_{q=1}^{\infty} \sideset{}{^*}\sum_{d \pmod q} e\big(\tfrac{-dr}{q}\big) \frac{1}{(2\pi \mathrm{i})^2} \int_{(c_1)} \int_{(c_2)} \tilde{E}(z_1, z_2) \sum_{m\ge 1} \frac{\tau_k(m) e\big(\frac{md}{q}\big)}{m^{z_1}} \\ \times \sum_{n\ge 1} \frac{\tau_\ell(n) e\big(\frac{-nd}{q}\big)}{n^{z_2}}\, dz_2\, dz_1. \end{aligned}$$

Define $\mathcal{D}_k(s, \frac{d}{q}) := \sum_{n=1}^{\infty} \tau_k(n) e(\frac{nd}{q}) n^{-s}$. Then we have

$$\begin{aligned} &D_{f;k,\ell}(r) \\ &= \sum_{q=1}^{\infty} \sideset{}{^*}\sum_{d \pmod q} e\big(\tfrac{-dr}{q}\big) \frac{1}{(2\pi \mathrm{i})^2} \int_{(c_1)} \int_{(c_2)} \tilde{E}(z_1, z_2) \mathcal{D}_k\big(z_1, \tfrac{d}{q}\big) \mathcal{D}_\ell\big(z_2, \tfrac{-d}{q}\big)\, dz_2\, dz_1. \end{aligned}$$

Note that (see [7]) since $\mathcal{D}_k(s, \frac{a}{q}) \sim q^{-s} \zeta(s)^k G_k(s, q)$, we expect that

$$\begin{aligned} D_{f;k,\ell}(r) \sim \sum_{q=1}^{\infty} \sideset{}{^*}\sum_{d \pmod q} e\big(\tfrac{-dr}{q}\big) \frac{1}{(2\pi \mathrm{i})^2} \int_{(c_1)} \int_{(c_2)} \tilde{E}(z_1, z_2) q^{-z_1} \zeta(z_1)^k G_k(z_1, q) \\ \times q^{-z_2} \zeta(z_2)^\ell G_\ell(z_2, q)\, dz_2\, dz_1. \end{aligned}$$

We next simplify $\tilde{E}(z_1, z_2)$. Note that

$$\begin{aligned} \int_{\mathbb{R}} E(x, y) y^{z_2-1}\, dy &= \int_{\mathbb{R}} F(x, y) \Delta_q(x - y - r) y^{z_2-1} dy \\ &= \int_{\mathbb{R}} F(x, x - u - r) \Delta_q(u) (x - u - r)^{z_2-1}\, du \end{aligned}$$

and thus

$$\int_{\mathbb{R}} E(x, y) y^{z_2-1}\, dy \sim F(x, x - r)(x - r)^{z_2-1} = f(x, x - r)(x - r)^{z_2-1}$$

since the behaviour of $\Delta_q(u)$ is similar to that of the Dirac delta function. Hence,

$$\tilde{E}(z_1, z_2) \sim \int_0^\infty f(x, x-r)x^{z_1-1}(x-r)^{z_2-1}dx.$$

From the definition of the Ramanujan sum, it follows that $D_{f;k,\ell}(r)$ is equal to

$$\frac{1}{(2\pi i)^2}\int_{(c_1)}\int_{(c_2)}\zeta(z_1)^k\zeta(z_2)^\ell\sum_{q=1}^\infty\frac{c_q(r)G_k(z_1,q)G_\ell(z_2,q)}{q^{z_1+z_2}} \times\int_0^\infty f(x,x-r)x^{z_1-1}(x-r)^{z_2-1}\,dx\,dz_2\,dz_1.$$

Since $\zeta(z_1)^k$ and $\zeta(z_2)^\ell$ have poles at $z_1 = 1$ and $z_2 = 1$, respectively, we expect that $D_{f;k,\ell}(r)$ is asymptotic to

$$\frac{1}{(2\pi i)^2}\int_{\mathcal{B}_1}\int_{\mathcal{B}_2}\zeta(z_1)^k\zeta(z_2)^\ell\sum_{q=1}^\infty\frac{c_q(r)G_k(z_1,q)G_\ell(z_2,q)}{q^{z_1+z_2}} \times\int_0^\infty f(x,x-r)x^{z_1-1}(x-r)^{z_2-1}\,dx\,dz_2\,dz_1,$$

where $\mathcal{B}_j = \{z_j \in \mathbb{C} \mid |z_j - 1| < r_j\}$ for $j = 1, 2$. We believe that this last expression is the main term in the additive divisor conjecture for τ_k and τ_ℓ.

Conjecture 2 (k-ℓ additive divisor conjecture). *There exists $C > 0$ for which the following holds. Let ε_0 and ε' be arbitrarily small positive constants. Let $P > 1$, and let $X, Y > 1/2$ satisfy $Y \asymp X$. Let f be a smooth function satisfying* (1.9) *and* (1.10)*. Then, in those cases where X is sufficiently large* (*in absolute terms*)*, one has*

$$D_{f;k,\ell}(r) = \frac{1}{(2\pi i)^2}\int_{\mathcal{B}_1}\int_{\mathcal{B}_2}\zeta(z_1)^k\zeta(z_2)^\ell\sum_{q=1}^\infty\frac{c_q(r)G_k(z_1,q)G_\ell(z_2,q)}{q^{z_1+z_2}} \times\int_0^\infty f(x,x-r)x^{z_1-1}(x-r)^{z_2-1}\,dx\,dz_2\,dz_1 + O(P^C X^{1/2+\varepsilon_0}),$$

uniformly for $1 \le |r| \ll X^{1-\varepsilon'}$, where for $i = 1, 2$, $\mathcal{B}_i = \{z_i \in \mathbb{C} \mid |z_i - 1| = r_i\} \subset \mathbb{C}$ are circles, centred at 1, of radii $r_i \in (\frac{1}{100}, \frac{1}{10})$, and $c_q(r) = \sum^*_{d \,(\mathrm{mod}\, q)} e(\frac{-dr}{q})$ is the Ramanujan sum.

10. Appendix 2: Proof of Lemma 6.2

For $\varepsilon_1 \in (0, \frac{1}{2}]$, $M \ll U^{2+\varepsilon_2}$, $N \asymp M$, $0 \ne r \ll \frac{M}{T_0}T^{\varepsilon_1}$, and $(x, y) \in [M, 2M] \times [N, 2N]$, we claim

$$x^m y^n f_r^{(m,n)}(x, y) \ll T^{4\varepsilon_1}P^n, \quad \text{where} \quad P = T^{1+\varepsilon_1}T_0^{-1}. \tag{10.1}$$

By (6.7), the definition of f_r, we can write $f_r(x, y) = W(\frac{x}{M})W(\frac{y}{N})\phi(x, y)$, where

$$\phi(x, y) = \frac{1}{2\pi i}\int_{(\varepsilon_1)} \frac{G(s)}{s}\left(\frac{1}{\pi^4 xy}\right)^s \frac{1}{T}\int_{-\infty}^{\infty}\left(1+\frac{r}{y}\right)^{-it} g(s,t)\left(\frac{U}{t}\right)^{4s} w(t)\,dt\,ds$$

for $x, y > 0$ (and $\phi(x, y) = 0$ otherwise). It suffices to prove that for $x \asymp M$ and $y \asymp N$,

$$x^m y^n \phi^{(m,n)}(x, y) \ll T^{4\varepsilon_1} P^n. \tag{10.2}$$

This is because by the generalized product rule and (10.2), we have

$$\begin{aligned}
&|f_r^{(m,n)}(x, y)| \\
&= \left|\sum_{i_1+i_2=m}\binom{m}{i_1}W^{(i_1)}\left(\frac{x}{M}\right)M^{-i_1}\sum_{j_1+j_2=n}\binom{n}{j_1}W^{(j_1)}\left(\frac{y}{N}\right)N^{-j_1}\phi^{(i_2,j_2)}(x, y)\right| \\
&\le \sum_{i_1+i_2=m} 2^m O_{i_1}(1)\left(\frac{x}{2}\right)^{-i_1}\sum_{j_1+j_2=n} 2^n O_{j_1}(1)\left(\frac{y}{2}\right)^{-j_1}\cdot x^{-i_2}y^{-j_2}\cdot O_{i_2,j_2}(T^{4\varepsilon_1}P^{j_2}) \\
&= \Big(\sum_{i_1=0}^{m}\sum_{j_1=0}^{n} O_{i_1,j_1,m,n}(1)\cdot P^{-j_1}\Big)T^{4\varepsilon_1}P^n x^{-m}y^{-n},
\end{aligned}$$

where we have used the fact that $W(u) = 0$ for $u \ge 2$. By (iii), for all sufficiently large T, we know $P \ge 1$ and thus we obtain (10.1).

Now, we shall prove (10.2). We first write

$$\begin{aligned}
\phi^{(m,n)}(x, y) = \frac{1}{2\pi i}\int_{(\varepsilon_1)}\frac{G(s)}{s}\left(\frac{1}{\pi^4}\right)^s\frac{1}{T}\int_{-\infty}^{\infty}\frac{\partial^m}{\partial x^m}\frac{\partial^n}{\partial y^n}\left(x^{-s}y^{-s}\left(1+\frac{r}{y}\right)^{-it}\right)g(s,t) \\
\times\left(\frac{U}{t}\right)^{4s}\omega(t)\,dt\,ds.
\end{aligned} \tag{10.3}$$

As shown in [26, pp. 56–57], when $\Re e(s) = \varepsilon_1$, $x \asymp M$, $y \asymp N$, $t \asymp T$, $1 \le |r| \ll \frac{M}{T_0}T^{\varepsilon_1} = o(M)$, and $P = \frac{T}{T_0}T^{\varepsilon_1} \ge 1$ (by (iii)), one has

$$\frac{\partial^m}{\partial x^m}\frac{\partial^n}{\partial y^n}\left(x^{-s}y^{-s}\left(1+\frac{r}{y}\right)^{-it}\right) \ll M^{-2\varepsilon_1}(1+|s|)^{m+n}P^n x^{-m}y^{-n}. \tag{10.4}$$

Note that $|G(s)/s| \le |G(s)|/\varepsilon_1$ for $\Re e(s) = \varepsilon_1$, and

$$|g(s,t)| \ll \left(\frac{t}{2}\right)^{4\varepsilon_1}(1 + O(|s|^2+1)) \ll (1+|s|)^2 t^{4\varepsilon_1}$$

for $\Re e(s) = \varepsilon_1$ and $t \asymp T > 1$ (by Lemma 2.2 (i)). Using (10.3), (10.4), and these two bounds, combined with (ii) and (iii), we derive

$$\begin{aligned}
&x^m y^n \phi^{(m,n)}(x, y) \\
&\ll P^n M^{-2\varepsilon_1}\int_{(\varepsilon_1)}|G(s)|\left(\frac{1}{T}\int_{-\infty}^{\infty}(1+|s|)^{m+n+2}T^{4\varepsilon_1}|\omega(t)|\,dt\right)|ds| \ll \left(\frac{T^4}{M^2}\right)^{\varepsilon_1}P^n
\end{aligned}$$

for $x \asymp M$, $y \asymp N$ and $m, n \in \mathbb{N} \cup \{0\}$. Finally, as $M \gg \frac{T_0}{T^{\varepsilon_1}}|r| \geq \frac{T_0}{T^{\varepsilon_1}} \geq T^{1/2-\varepsilon_1}$ (by (iii)) and thus $T^4/M^2 \ll T^{3+2\varepsilon_1} \leq T^4$ whenever $\varepsilon_1 \in (0, 1/2]$, we complete the proof of Lemma 6.2. ∎

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