Journal of Nonlinear and Convex Analysis
Volume 20, Number Proof, 2019, 1–

NORM OF POSITIVE SUM PRESERVERS OF SMOOTH BANACH LATTICES AND $L^p(\mu)$ SPACES

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In the memory of our beloved friend, Siegfried Schaible

Abstract. In this paper, we study surjective maps $\varphi : E_+ \to F_+$ between positive cones of two ordered Banach spaces E and F, which preserve norm of sums, i.e.,

$$\|\varphi(x) + \varphi(y)\| = \|x + y\|, \quad \forall x, y \in E_+.$$

In the case when E, F are strictly convex smooth Banach lattices, as well as $L^p(\mu)$ spaces $(1 < p \leq +\infty)$, we show that φ can be extended to a real linear map/isometry from E onto F. A counter example for the case when $p = 1$ is presented.

1. Introduction

Let $\varphi : E \to F$ be a surjective map between two Banach spaces. The classical Mazur-Ulam theorem [6] states that φ is an affine map when it preserves norm of differences, i.e.,

$$\|\varphi(x) - \varphi(y)\| = \|x - y\|, \quad \forall x, y \in E.$$

One might ask what happens if φ preserves norm of sums instead, i.e.,

$$\|\varphi(x) + \varphi(y)\| = \|x + y\|, \quad \forall x, y \in E.$$

But this is indeed a trivial question. In fact, putting $y = 0, \pm x$ into the above condition, we have $\varphi(0) = 0$, as well as $\|\varphi(x)\| = \|x\|$ and $\varphi(-x) = -\varphi(x)$ for all x in E. Consequently,

$$\|\varphi(x) - \varphi(y)\| = \|\varphi(x) + \varphi(-y)\| = \|x + (-y)\| = \|x - y\|, \quad \forall x, y \in E.$$

The Mazur-Ulam theorem ensures that φ is a real linear isometry.

To formulate a meaningful problem of norm of positive sum preservers, motivated by several well-known Banach-Stone and Lamperti type theorems (see, e.g., [2, 3]) and recent development of various preserver problems (see, e.g., [7]), we propose the following

Problem 1.1. Let E, F be ordered Banach spaces with positive cones E_+, F_+. Let $\varphi : E_+ \to F_+$ be a surjective map preserving norm of sums, i.e.,

$$\|\varphi(x) + \varphi(y)\| = \|x + y\|, \quad \forall x, y \in E_+.$$

Can we extend φ to a positive linear map from E onto F?

2000 *Mathematics Subject Classification.* 46C05, 46C50, 47B49, 47D25.
Key words and phrases. Norm preservers, Banach lattices, Lamperti type theorems.
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This research is supported partially by Taiwan MOST grants 105-2115-M-027-002-MY2 (for Tsai) and 106-2115-M-110-006-MY2 (for Wong), respectively.

There are already some efforts appeared in the literature. For example, Nagy [9, 10], and Kuo, Tsai, Wong and Zhang [5] have studied similar problems for Schatten p-class operators. In this paper, we discuss norm of positive sum preservers of smooth Banach lattices in Section 2. In particular, every such map of strictly convex smooth Banach lattices extend to a linear map (Corollary 2.5). We answer Problem 1.1 affirmatively for the case when E, F are $L^p(\mu)$ spaces ($1 < p \leq +\infty$) in Section 3. A detail analysis on such preservers of the finite dimensional positive cones ℓ^p_{n+} are given in Section 4. In particular, a counter example to Problem 1.1 for the case when $p = 1$ is presented. In a forthcoming paper [13], we will answer Problem 1.1 for the case when E, F are noncommutative $L^p(M), L^p(N)$ spaces associated to von Neumann algebras M, N.

2. The case of smooth Banach lattices

In this paper, without loss of generality, we consider only real vector spaces.

Let E be a Banach space. We say that E is *strictly convex* if

$$\|x+y\| = \|x\| + \|y\| \quad \Longrightarrow \quad x = \delta y \text{ for some } \delta > 0, \quad \forall x \neq 0, y \neq 0.$$

We say that E is *smooth* if its norm is Gâteaux differentiable, namely, the limit

$$G(x)y := \lim_{t \to 0^+} \frac{\|x+ty\| - \|x\|}{t}$$

exists in $\mathbb{R}$ whenever $x, y \in E$ with $x \neq 0$.

Let $E = (E, \leq, \|\cdot\|)$ be a Banach lattice with a partial order $\leq$. Let $E_+ = \{x \in E : x \geq 0\}$ be the positive cone of E. We say that E has a *strictly monotone* norm if

$$x \leq y \text{ and } x \neq y \quad \Longrightarrow \quad \|x\| < \|y\|, \qquad \forall x, y \in E_+.$$

Two vectors x and y in a Banach lattice are said to be *disjoint*, denoted by

$$x \perp y, \quad \text{if} \quad |x| \wedge |y| = 0.$$

For the general theory of Banach lattices and positive operators, see, e.g., Aliprantis [1], Nieberg [11] and Hudzik [4]. In particular, we collect some well known facts in the following lemma.

Lemma 2.1. *Suppose that E is a smooth Banach lattice. We define*

$$G(x)y = \lim_{t \to 0^+} \frac{\|x+ty\| - \|x\|}{t}, \quad \forall x, y \in E \text{ with } x \neq 0.$$

(1) *$G(x) : E \to \mathbb{R}$ is a linear operator.*
(2) *$G(x)x = \|x\|$, $\|G(x)\| = 1$ and $G(\lambda x) = G(x)$, for every $x \in E_+$ and $\lambda > 0$.*
(3) *$G(x)y \geq 0$ for every $x, y \in E_+$.*
(4) *If $y_1 \geq y_2$ then $G(x)y_1 \geq G(x)y_2$, for every $x, y_1, y_2 \in E_+$.*

The following result can be found in [12, Theorem 1]. We include the proof here for completeness.

Lemma 2.2. *Suppose that E is a smooth Banach lattice. We have*

$$x \perp y \quad \Longrightarrow \quad G(x)y = 0, \quad \forall x, y \in E_+.$$

The converse holds when E has a strictly monotone norm.

Proof. Define $f(t) = \|x + ty\|$ for any fixed $x, y \in E_+$. Since $f(\frac{t_1+t_2}{2}) \leq (f(t_1) + f(t_2))/2$, we see that f is a convex function of $\mathbb{R}$. Suppose $x \perp y = 0$ for $x, y \in E_+$. We have

$$|x + ty| = x + ty \geq x = |x|, \text{ for } t \geq 0,$$
$$|x + ty| = (x + ty)^+ + (x + ty)^- = x - ty \geq x = |x|, \text{ for } t \leq 0.$$

Hence $\|x + ty\| \geq \|x\|$ for all $t \in \mathbb{R}$, and $f(t)$ attains its minimum at $t = 0$. Since E is smooth, we have

$$G(x)y = \lim_{t\to 0^+} \frac{\|x + ty\| - \|x\|}{t} = f'_+(0) = 0.$$

Assume now E is strictly monotone. Suppose $G(x)y = 0$ but $x \wedge y = z \neq 0$. It follows from Lemma 2.1(4) that $0 \leq G(x)z \leq G(x)y = 0$. It forces $G(x)z = 0$. Hence,

$$\|x\| = G(x)x = G(x)(x - z) \leq \|G(x)\|\|x - z\| \leq \|x - z\|.$$

Since $0 \leq x - z \leq x$, and $x - z \neq x$, the strict monotonicity of the norm implies that $\|x - z\| < \|x\|$. This contradiction tells us that $z = x \wedge y = 0$. □

Lemma 2.3. *Suppose that E, F are Banach lattices and F is strictly convex. Let $\varphi : E_+ \mapsto F_+$ be a map preserving norm of sums, i.e.,*

$$\|\varphi(x) + \varphi(y)\| = \|x + y\|, \quad \forall x, y \in E_+. \tag{2.1}$$

Then, φ is nonnegatively homogeneous, i.e.,

$$\varphi(\lambda x) = \lambda\varphi(x), \quad \forall x \in E_+, \forall\lambda \geq 0,$$

and preserves norm of convex combinations, i.e.,

$$\|(1 - t)x + ty\| = \|(1 - t)\varphi(x) + t\varphi(y)\|, \quad \forall t \in [0, 1].$$

Proof. Taking $x = y$ in equation (2.1), one has $\|x\| = \|\varphi(x)\|$. For x in E_+ and $\lambda > 0$, we have

$$\|\varphi(x) + \varphi(\lambda x)\| = \|x + \lambda x\| = \|x\| + \|\lambda x\| = \|\varphi(x)\| + \|\varphi(\lambda x)\|.$$

From the strict convexity of F, we have $\varphi(\lambda x) = \delta\varphi(x)$ for some $\delta > 0$. Then

$$\lambda\|x\| = \|\varphi(\lambda x)\| = \|\delta\varphi(x)\| = \delta\|x\|$$

ensures that $\lambda = \delta$. Hence, $\varphi(\lambda x) = \lambda\varphi(x)$, and thus

$$\|(1 - t)\varphi(x) + t\varphi(y)\| = \|\varphi((1 - t)x) + \varphi(ty)\| = \|(1 - t)x + ty\|, \quad \forall t \in [0, 1].$$

□

Below are two answers to Problem 1.1 for the case of Banach lattices.

Theorem 2.4. *Let E, F be smooth Banach lattices. Suppose that $\varphi : E_+ \mapsto F_+$ is a surjective map preserving norm of convex combinations, that is,*

$$\|(1 - t)\varphi(x) + t\varphi(y)\| = \|(1 - t)x + ty\|, \quad \forall x, y \in E_+, \forall t \in [0, 1]. \tag{2.2}$$

(1) *For all x, y in E_+, we have*

$$x \perp y \quad \Longrightarrow \quad G(\varphi(x))(\varphi(y)) = 0.$$

(2) *φ is nonnegative homogenous and additive; that is,*

(i) *$\varphi(\lambda y) = \lambda\varphi(y)$ for all $\lambda \geq 0$ and $y \in E_+$;*
(ii) *$\varphi(y_1 + y_2) = \varphi(y_1) + \varphi(y_2)$, for all $y_1, y_2 \in E_+$.*

Consequently, φ has a unique positive surjective linear extension $\hat{\varphi} : E \mapsto F$ given by the formula $\hat{\varphi}(x) = \varphi(x^+) - \varphi(x^-)$.

Proof. (1) Differentiating both side of (2.2) at $t = 0^+$, we get

$$G(\varphi(x))(\varphi(y) - \varphi(x)) = G(x)(y - x).$$

Putting $t = 0$ in (2.2), we have $\|\varphi(x)\| = \|x\|$. By Lemma 2.1(2), we have $G(x)x = G(\varphi(x))\varphi(x)$. Hence,

$$G(x)y = G(\varphi(x))\varphi(y), \quad \text{for all } x, y \in E_+. \tag{2.3}$$

It follows from Lemma 2.2 that

$$x \perp y \quad \Longrightarrow \quad G(x)y = 0 \quad \Longrightarrow \quad G(\varphi(x))\varphi(y) = 0.$$

(2) For any $x, y \in E_+$ and $\lambda > 0$, it follows from (2.3) that

$$\begin{aligned} \lambda G(x)y &= \lambda G(\varphi(x))(\varphi(y)) = G(\varphi(x))(\lambda\varphi(y)), \\ G(x)(\lambda y) &= G(\varphi(x))(\varphi(\lambda y)). \end{aligned}$$

Subtracting the two equations, we have

$$G(\varphi(x))(\lambda\varphi(y) - \varphi(\lambda y)) = 0. \tag{2.4}$$

Since φ is surjective, we can choose x from E_+ such that $\varphi(x) = (\lambda\varphi(y) - \varphi(\lambda y))^+$. It follows from part (1) and (2.4) that

$$\begin{aligned} &G(\varphi(x))(\lambda\varphi(y) - \varphi(\lambda y)) \\ &= G((\lambda\varphi(y) - \varphi(\lambda y))^+)((\lambda\varphi(y) - \varphi(\lambda y))^+ - (\lambda\varphi(y) - \varphi(\lambda y))^-) \\ &= G((\lambda\varphi(y) - \varphi(\lambda y))^+)(\lambda\varphi(y) - \varphi(\lambda y))^+ \\ &= \|(\lambda\varphi(y) - \varphi(\lambda y))^+\| = 0. \end{aligned}$$

This forces that $(\lambda\varphi(y) - \varphi(\lambda y))^+ = 0$. Similar argument shows that $(\lambda\varphi(y) - \varphi(\lambda y))^- = 0$. Hence, $\varphi(\lambda y) = \lambda\varphi(y)$ for all $\lambda > 0$ and $y \in E_+$.

With a similar argument we can show that $\varphi(y_1 + y_2) = \varphi(y_1) + \varphi(y_2)$ for all y_1, y_2 in E_+. □

Corollary 2.5. *Suppose that E, F be smooth Banach lattices and F is strictly convex. Let $\varphi : E_+ \to F_+$ be a surjective map preserving norm of sums, i.e.,*

$$\|\varphi(x) + \varphi(y)\| = \|x + y\|, \quad \forall x, y \in E_+.$$

Then φ can be extended to a positive linear map from E onto F.

Proof. The assertion follows from Lemma 2.3 and Theorem 2.4. □

3. Norm of sums preservers of $L^p_+(\Omega, \Sigma, \mu)$

In this section, we consider norm of positive sum preservers between positive cones $L^p_+(\Omega, \Sigma, \mu)$ of $L^p(\Omega, \Sigma, \mu)$ spaces. Note that $L^p(\Omega, \Sigma, \mu)$ spaces are strictly convex Banach lattices with smooth and strictly monotone norms, when $1 < p < \infty$.

Noting that the disjointness $x \perp y$ here is equivalent to that $xy = 0$ (almost everywhere) on Ω, we have the following well-known fact.

Lemma 3.1. *Let $1 < p < \infty$.*

$$x \perp y \quad \Longrightarrow \quad \|x + y\|^p = \|x\|^p + \|y\|^p, \qquad \forall x, y \in L^p(\Omega, \Sigma, \mu).$$

The converse holds whenever $x, y \in L^p_+(\Omega, \Sigma, \mu)$.

Definition 3.2. Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be two measure spaces. A bijective set-to-set map Ψ from Σ_1 onto Σ_2, defined modulo null sets, is called a *regular set isomorphism* if

(i) $\Psi(\Omega_1 \setminus A) = \Psi(\Omega_1) \setminus \Psi(A)$ for all $A \in \Sigma_1$;
(ii) $\Psi(\bigcup_{n=1}^\infty A_n) = \bigcup_{n=1}^\infty \Psi(A_n)$, for disjoint $A_n \in \Sigma_1$;
(iii) $\mu_2(\Psi(A)) = 0$ if and only if $\mu_1(A) = 0$.

Every regular set isomorphism Ψ induces a unique bijective linear transformation ψ sending Σ_1-measurable functions to Σ_2-measurable functions satisfying that

$$\psi(\mathbf{1}_A) = \mathbf{1}_{\Psi(A)} \quad \text{for all } A \in \Sigma_1.$$

Here $\mathbf{1}_A$ denotes the indicator function of the measurable set A.

Theorem 3.3. *Let $\varphi : L^p_+(\Omega_1, \Sigma_1, \mu_1) \to L^p_+(\Omega_2, \Sigma_2, \mu_2)$ be a bijective map, where $1 < p \leq \infty$. Suppose φ preserves norm of sums of positive functions, that is,*

$$\|x + y\| = \|\varphi(x) + \varphi(y)\|, \quad x, y \in L^p_+(\Omega_1, \Sigma_1, \mu_1). \tag{3.1}$$

Then φ extends to a surjective linear isometry from $L^p(\Omega_1, \Sigma_1, \mu_1)$ onto $L^p(\Omega_2, \Sigma_2, \mu_2)$. More precisely, there exists a regular set isomorphism Ψ from Σ_1 onto Σ_2 inducing a surjective positive linear map $\psi : L^p(\Omega_1, \Sigma_1, \mu_1) \to L^p(\Omega_2, \Sigma_2, \mu_2)$, and a locally measurable function h on Ω_2 such that

$$\varphi(x) = h \cdot \psi(x), \quad \forall x \in L^p_+(\Omega_1, \Sigma_1, \mu_1). \tag{3.2}$$

When $1 < p < +\infty$, we have

$$\int_{\Psi(A)} |h(t)|^p d\mu_2 = \mu_1(A), \quad \text{for each } \sigma\text{-finite } A \in \Sigma_1. \tag{3.3}$$

In other words, $|h|^p = \frac{d(\mu_1 \circ \Psi^{-1})}{d\mu_2}$ is the Radon-Nikodym derivative of $\mu_1 \circ \Psi^{-1}$ with respect to μ_2. When $p = +\infty$, we have

$$h(y) = 1, \quad \text{locally almost everywhere on } \Omega_2. \tag{3.4}$$

The case $p = +\infty$ of Theorem 3.3 can be derived from a current result of Molnár [8, Theorem 2.7] which states that every surjective norm of sum preserver $\varphi : M_+ \to N_+$ between positive cones of von Neumann algebras M and N extends to a Jordan isomorphism $J : M \to N$. In the abelian case, $J = h\psi$ satisfies the conditions (3.2) and (3.4).

Proof of Theorem 3.3 when $1 < p < +\infty$. Since $L^p(\mu)$ spaces are strictly convex Banach lattice with smooth and strictly monotone norms, by Lemma 2.2 and Theorem 2.4 we have

(I) φ preserves disjointness, i.e., $\varphi(x) \perp \varphi(y)$ if and only if $x \perp y$ for all $x, y \in L_+^p(\Omega_1, \Sigma_1, \mu_1)$;

(II) $\varphi(x+y) = \varphi(x) + \varphi(y)$ and $\varphi(\lambda x) = \lambda \varphi(x)$ for all $x, y \in L_+^p(\Omega_1, \Sigma_1, \mu_1)$ and $\lambda \geq 0$.

We first assume that $\mu_1(\Omega_1) < +\infty$. Set

$$\Psi(A) = \operatorname{supp} \varphi(\mathbf{1}_A), \quad \forall A \in \Sigma_1,$$

where the support of any measurable function is the measurable set $\operatorname{supp}(x) = \{t \in \Omega : x(t) > 0\}$. We claim that Ψ defines a regular set isomorphism from Σ_1 to Σ_2.

(1) If $A, B \in \Sigma_1$ are disjoint, then $\mathbf{1}_A \cdot \mathbf{1}_B = 0$, and by (I), $\varphi(\mathbf{1}_A) \cdot \varphi(\mathbf{1}_B) = 0$. In other words, $\varphi(\mathbf{1}_A)$ and $\varphi(\mathbf{1}_B)$ have disjoint supports (modulo sets of measure zero). It follows that $\Psi(A \cup B) = \Psi(A) \cup \Psi(B)$ (modulo sets of measure zero). It follows that $\Psi(\Omega \setminus A) = \Psi(\Omega) \setminus \Psi(A)$.

(2) Let $A = \bigcup_{n=1}^{\infty} A_n$ be a countable disjoint union of sets in Σ_1. Setting $B = A \setminus \bigcup_{i=1}^{n} A_i$, we have

$$\varphi(\mathbf{1}_{A\setminus B}) + \varphi(\mathbf{1}_B) = \varphi(\mathbf{1}_A) \quad \text{and} \quad \varphi(\mathbf{1}_{A\setminus B}) = \varphi(\sum_{i=1}^{n} \mathbf{1}_{A_i}) = \sum_{i=1}^{n} \varphi(\mathbf{1}_{A_i}).$$

Therefore

$$\begin{aligned} \|\varphi(\mathbf{1}_A) - \sum_{i=1}^{n} \varphi(\mathbf{1}_{A_i})\| &= \|\varphi(\mathbf{1}_A) - \varphi(\mathbf{1}_{A\setminus B})\| \\ &= \|\varphi(\mathbf{1}_B)\| = \|\mathbf{1}_B\| \to 0, \quad \textit{as } n \to +\infty. \end{aligned}$$

Hence, $\varphi(\mathbf{1}_A) = \sum_{n=1}^{\infty} \varphi(\mathbf{1}_{A_n})$, and then $\Psi(A) = \bigcup_{n=1}^{\infty} \Psi(A_n)$.

(3) Final, we observe that if $\mu_2(\Psi(A)) = 0$, then $\varphi(\mathbf{1}_A) = 0$ μ_2-a.e., and so $\mu_1(A) = 0$ because $\|\mathbf{1}_A\| = \|\varphi(\mathbf{1}_A)\|$. Dealing with φ^{-1} we see that $\mu_1(A) = 0$ implies $\mu_2(\Psi(A)) = 0$.

We conclude from (1)-(3) that Ψ is a regular set isomorphism from Σ_1 onto Σ_2.

Observe

$$\|\varphi(\mathbf{1}_{\Omega_1})\| = \|\mathbf{1}_{\Omega_1}\| = \mu_1(\Omega_1) < +\infty.$$

Therefore the support Ω_2 of the p-integrable function $\varphi(\mathbf{1}_{\Omega_1})$ is σ-finite. Let $h(t) = \varphi(\mathbf{1}_{\Omega_1})$. For any $A \in \Sigma_1$, we have

$$h = \varphi(\mathbf{1}_A) + \varphi(\mathbf{1}_{\Omega_1 \setminus A}).$$

Since the functions on the right have disjoint supports, $\varphi(\mathbf{1}_A)(t)$ agrees with $h(t)$ whenever $\varphi(\mathbf{1}_A)(t)$ is not zero (μ_2-a.e.). Therefore,

$$\varphi(\mathbf{1}_A)(t) = h(t)\mathbf{1}_{\Psi(A)}(t) = h(t)\psi(\mathbf{1}_A)(t) \ (\mu_2\text{-a.e.}).$$

By the positive linearity (II) of φ, the equality (3.2) holds for every nonnegative simple function x on Ω_1. Since φ is an isometry for all $x \in L_+^p(\Omega_1, \Sigma_1, \mu_1)$, we have for each $A \in \Sigma_1$ that

$$\int_{\Psi(A)} |h(t)|^p d\mu_2 = \|\varphi(\mathbf{1}_A)\|^p = \|\mathbf{1}_A\|^p = \mu_1(A).$$

Since $|h|^p = \frac{d(\mu_1 \circ \Psi^{-1})}{d\mu_2}$, the map $\mathbf{1}_A \mapsto h\mathbf{1}_{\Psi(A)}$ extends to a surjective positive linear isometry from $L^p(\Omega_1, \Sigma_1, \mu_1)$ onto $L^p(\Omega_2, \Sigma_2, \mu_2)$ sending x to $h\psi(x)$. Composing the inverse of this map with φ, we can assume $\varphi : L^p_+(\Omega_1, \Sigma_1, \mu_1) \to L^p_+(\Omega_1, \Sigma_1, \mu_1)$ satisfying that

$$\varphi(y) = y, \quad \text{whenever } y \text{ is a nonnegative simple function on } \Omega_1.$$

In general, let $x \in L^p_+(\Omega_1, \Sigma_1, \mu_1)$. We have

$$\|\varphi(x) - y\| = \|\varphi(x) - \varphi(y)\| = \|x - y\|,$$

whenever y is a nonnegative simple function on Ω_1. Since nonnegative simple functions are norm dense in $L^p_+(\Omega_1, \Sigma_1, \mu_1)$, we have

$$\|\varphi(x) - y\| = \|x - y\|, \quad \forall y \in L^p_+(\Omega_1, \Sigma_1, \mu_1).$$

Putting $y = x$, we see that $\varphi(x) = x$ for all x in $L^p_+(\Omega_1, \Sigma_1, \mu_1)$. Therefore, in the original setting,

$$\varphi(x) = h(x)\psi(x), \quad \forall x \in L^p_+(\Omega_1, \Sigma_1, \mu_1).$$

In the σ-finite case, we write $\Omega_1 = \bigcup_n \Omega_{1,n}$ as a countable union of disjoint measurable sets of finite measure. For each n, let $\Omega_{2,n}$ be the support of $\varphi(\mathbf{1}_{\Omega_{1,n}})$, which is a σ-finite measurable set in Σ_2. Clearly, $\Omega_2 = \bigcup_n \Omega_{2,n}$ as a countable union of disjoint σ-finite measurable sets. Let $\Sigma_{i,n}$ be the σ-algebra of all measurable subsets of $\Omega_{i,n}$ and $\mu_{i,n}$ be the measure on $\Sigma_{i,n}$ induced by μ_i for $i = 1, 2$ and $n = 1, 2, \ldots$. Then, φ induces a bijective map $\varphi_n : L^p_+(\Omega_{1,n}, \Sigma_{1,n}, \mu_{1,n}) \to L^p_+(\Omega_{2,n}, \Sigma_{2,n}, \mu_{2,n})$ preserving norm of sums for each $n = 1, 2, \ldots$.

It follows from the finite case that for each n, we have a measurable function h_n on $\Omega_{2,n}$ with $|h_n|^p = \frac{d(\mu_{1,n} \circ \Psi_n^{-1})}{d\mu_{2,n}}$, and a regular set isomorphism $\Psi_n : \Sigma_{1,n} \to \Sigma_{2,n}$, such that $\varphi_n(x_n) = h_n \psi_n(x_n)$. Here, $x_n = x\mathbf{1}_{\Omega_{1,n}} \in L^p_+(\Omega_{1,n}, \Sigma_{1,n}, \mu_{1,n})$ for any x in $L^p_+(\Omega_1, \Sigma_1, \mu_1)$ and $\psi_n : L^p(\Omega_{1,n}, \Sigma_{1,n}, \mu_{1,n}) \to L^p(\Omega_{2,n}, \Sigma_{2,n}, \mu_{2,n})$ is the linear isomorphism induced by Ψ_n for $n = 1, 2, \ldots$. Set $\Psi(A) = \bigcup_n \Psi_n(A \cap \Omega_{1,n})$ and $h(t) = h_n(t)$ whenever $t \in \Psi(\Omega_{1,n})$, and $h(t) = 0$ whenever $t \in \Omega_2 \setminus \Psi(\Omega_1)$. In this way, both (3.2) and (3.3) are satisfied.

Now, we deal with the general case. For any σ-finite set A in Σ_1, arguing as above we see that $\Psi(A)$ is also σ-finite. We can thus obtain a pair (h_A, ψ_A) of measurable function h_A on $\Psi(A)$ and linear isomorphism $\psi_A : L^p(\Omega_A, \Sigma_A, \mu_A) \to L^p(\Omega_{\Psi(A)}, \Sigma_{\Psi(A)}, \mu_{\Psi(A)})$ in a similar fashion. For any such two pairs (h_A, ψ_A) and (h_B, ψ_B), if $A \subseteq B$ then $\psi_B(x_A) = \psi_A(x_A)$ for all x in $L^p(\Omega_1, \Sigma_1, \mu_1)$, with $x_A = x\mathbf{1}_A$ and $x_B = x\mathbf{1}_B$. Define a local measurable function h on Ω_2 and a linear isomorphism $\psi : L^p(\Omega_1, \Sigma_1, \mu_1) \to L^p(\Omega_2, \Sigma_2, \mu_2)$ by union. In other words, h is determined by the condition

$$h\mathbf{1}_{\Psi(A)} = h_A, \quad \text{for all } \sigma\text{-finite } A \text{ in } \Sigma_1.$$

On the other hand, for any x in $L^p(\Omega_1, \Sigma_1, \mu_1)$ the support of x is a σ-finite measurable set A. Then

$$\psi(x) = \psi_A(x) = \psi_B(x), \quad \text{for all } \sigma\text{-finite } \mu_1\text{-measurable set } B \text{ containing } A$$

It is then routine to check the conditions (3.2) and (3.3). □

Remark 3.4. [(a)]

(1) When $p = 1$, we have a negative answer to Problem 1.1, against Theorem 2.4 for the case $1 < p \leq +\infty$, as shown in Example 4.1(a).
(2) In proving Theorem 3.3 for the case $1 < p < +\infty$, we are motivated by the approach of verifying the classical Lamperti theorem (see, e.g., [2, Theorem 3.2.5]). We choose to present all the details here due to the intension of extending it to the case of noncommutative $L^p(M)$ space associated to a von Neumann algebra in [13].

4. A concrete example for finite dimensional ℓ^p spaces

In this section, we provide a counter example to Problem 1.1 of a norm of sum preserver of the positive cone of the finite dimensional space ℓ_n^p with $p = 1$.

Example 4.1. Suppose a surjective map $\varphi : \ell^p_{n+} \mapsto \ell^p_{n+}$ preserves norm of sums, that is,

$$\|\varphi(\boldsymbol{x}) + \varphi(\boldsymbol{y})\| = \|\boldsymbol{x} + \boldsymbol{y}\|, \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \ell^p_{n+}.$$

(a) φ may not have any linear extension when $p = 1$. For example, define

$$\varphi(\boldsymbol{x}) = U_r \boldsymbol{x}, \quad \text{whenever } \|\boldsymbol{x}\|_1 = r,$$

where U_r is an $n \times n$ permutation matrix associated to $r \geq 0$. When U_r are different for different r, we have a counter example to Problem 1.1.

(b) When $1 < p < \infty$, the permutation matrix U_r defining $\varphi(\boldsymbol{x}) = U_r \boldsymbol{x}$ whenever $\|\boldsymbol{x}\| = r$ must be the same for all $r \geq 0$ by Theorem 3.3.

(1) When $p = +\infty$, we have $\varphi(\boldsymbol{x}) = U\boldsymbol{x}$ for a fixed permutation matrix U.

Proof. We verify the case when $p = \infty$ only. The proof divides into three steps.

Step 1: Suppose $\boldsymbol{x}_1 = (\lambda, 0, \cdots, 0)$, $\boldsymbol{x}_2 = (0, \lambda, \cdots, 0)$, $\boldsymbol{x}_n = (0, \cdots, 0, \lambda)$, for some $\lambda > 0$, and $\boldsymbol{y}_i = \varphi(\boldsymbol{x}_i) = (y_{i1}, y_{i2}, \cdots, y_{in})$.

We have $0 \leq y_{ik} \leq \lambda$ and $\max\limits_{1 \leq k \leq n} \{y_{ik}\} = \lambda$, because $\|\varphi(\boldsymbol{x}_i)\|_\infty = \|\boldsymbol{x}_i\|_\infty = \lambda$ for each i. Suppose that $y_{1\sigma(1)} = \lambda$. Then $y_{i\sigma(1)} = 0$ due to

$$\lambda = \|\boldsymbol{x}_1 + \boldsymbol{x}_i\|_\infty = \|\varphi(\boldsymbol{x}_1) + \varphi(\boldsymbol{x}_i)\|_\infty, \quad \forall i = 2, \cdots, n.$$

We can then assume that $y_{2\sigma(2)} = \lambda$. Argue similarly, we see that $y_{i\sigma(2)} = 0$ for all $1 \leq i \leq n$ and $i \neq 2$. By induction, we obtain a permutation σ on $\{1, 2, \cdots, n\}$ such that $\varphi(\boldsymbol{x}_i) = \boldsymbol{x}_{\sigma(i)}$.

Step 2: Suppose that $\widetilde{\boldsymbol{x}}_1 = (\mu, 0, \cdots, 0)$, $\widetilde{\boldsymbol{x}}_2 = (0, \mu, \cdots, 0)$, $\widetilde{\boldsymbol{x}}_n = (0, \cdots, 0, \mu)$, for some $\mu > 0$, and $\varphi(\widetilde{\boldsymbol{x}}_i) = \widetilde{\boldsymbol{x}}_{\widetilde{\sigma}(i)}$ for $\mu \neq \lambda$. It follows from $\|\boldsymbol{x}_i + \widetilde{\boldsymbol{x}}_i\|_\infty = \|\varphi(\boldsymbol{x}_i) + \varphi(\widetilde{\boldsymbol{x}}_i)\|_\infty$ that $\sigma = \widetilde{\sigma}$.

Step 3: Suppose that $\varphi(\boldsymbol{x}_i) = \boldsymbol{x}_{\sigma(i)}$. Set arbitrary $\boldsymbol{x}_0 = (x_{01}, x_{02}, \cdots, x_{0n})$, and $\boldsymbol{y}_0 = \varphi(\boldsymbol{x}_0) = (y_{01}, y_{02}, \cdots, y_{0n})$. Choose λ sufficiently large. It follows from $\|\boldsymbol{x}_0 + \boldsymbol{x}_i\|_\infty = \|\varphi(\boldsymbol{x}_0) + \varphi(\boldsymbol{x}_i)\|_\infty$ that $x_{0i} + \lambda = y_{0\sigma(i)} + \lambda$. Hence, $y_{0\sigma(i)} = x_{0i}$. Thus, the same permutation matrix U defines $\boldsymbol{y} = U\boldsymbol{x}$. □

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Manuscript received ,
revised ,

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