# Applications of Bregman-Opial property to Bregman nonspreading mappings in Banach spaces

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**Abstract.** The Opial property of Hilbert spaces and some other special Banach spaces is a powerful tool in establishing fixed point theorems for nonexpansive, and more generally, nonspreading mappings. Unfortunately, not every Banach space shares the Opial property. However, every Banach space has an alike Bregman-Opial property for Bregman distances. In this paper, using Bregman distances, we introduce the classes of Bregman nonspreading mappings, and investigate the Mann and Ishikawa iterations for these mappings. We establish weak and strong convergence theorems for Bregman nonspreading mappings.

**Keywords.** Bregman-Opial property; Bregman nonspreading mapping; Bregman function; fixed point.

**2000 AMS Subject Classifications.** 47H10, 37C25.

# 1 Introduction

Let $E$ be a (real) Banach space with norm $\|.\|$ and dual space $E^*$. For any $x$ in $E$, we denote the value of $x^*$ in $E^*$ at $x$ by $\langle x, x^* \rangle$. When $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in $E$, we denote the strong convergence of $\{x_n\}_{n\in\mathbb{N}}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. Let $C$ be a nonempty subset of $E$. Let $T : C \to E$ be a map. We denote by $F(T) = \{x \in C : Tx = x\}$ the set of *fixed points* of $T$. We call the map $T$

- *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y$ in $C$,
- *quasi-nonexpansive* if $F(T) \neq \emptyset$ and $\|Tx - y\| \leq \|x - y\|$ for all $x$ in $C$ and $y$ in $F(T)$.

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The nonexpansivity plays an important role in the study of the *Ishikawa iteration*, given by

$$\begin{cases} y_n = \beta_n T x_n + (1-\beta_n) x_n, \\ x_{n+1} = \gamma_n T y_n + (1-\gamma_n) x_n, \end{cases} \tag{1.1}$$

where the sequences $\{\beta_n\}_{n\in\mathbb{N}}$ and $\{\gamma_n\}_{n\in\mathbb{N}}$ satisfy some appropriate conditions. When all $\beta_n = 0$, the Ishikawa iteration (1.1) reduces to the classical Mann iteration. Construction of fixed points of nonexpansive mappings via Mann's and Ishikawa's algorithms [15] has been extensively investigated in the literature (see, for example, [20] and the references therein).

A powerful tool in deriving weak or strong convergence of iterative sequences is due to Opial [19]. A Banach space $E$ is said to satisfy the *Opial property* [19] if for any weakly convergent sequence $\{x_n\}_{n\in\mathbb{N}}$ in $E$ with weak limit $x$, we have

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y$ in $E$ with $y \neq x$. It is well known that all Hilbert spaces, all finite dimensional Banach spaces, and the Banach spaces $l^p$ $(1 \leq p < \infty)$ satisfy the Opial property. However, not every Banach space satisfies the Opial property; see, for example [6, 8].

Working with the Bregman distance $D_g$, the following Bregman Opial-like inequality holds for every Banach space $E$:

$$\limsup_{n\to\infty} D_g(x_n, x) < \limsup_{n\to\infty} D_g(x_n, y),$$

whenever $x_n \rightharpoonup x \neq y$. See Lemma 3.2 for details. The Bregman-Opial property suggests us to introduce the notions of Bregman nonexpansive-like mappings, and develop fixed point theorems and convergence results for the Ishikawa iterations for these mappings.

We recall the definition of Bregman distances. Let $g : E \to \mathbb{R}$ be a strictly convex and Gâteaux differentiable function on a Banach space $E$. The *Bregman distance* [5] (see also [1, 4]) corresponding to $g$ is the function $D_g : E \times E \to \mathbb{R}$ defined by

$$D_g(x, y) = g(x) - g(y) - \langle x - y, \nabla g(y)\rangle, \quad \forall x, y \in E. \tag{1.2}$$

It follows from the strict convexity of $g$ that $D_g(x, y) \geq 0$ for all $x, y$ in $E$. However, $D_g$ might not be symmetric and $D_g$ might not satisfy the triangular inequality.

When $E$ is a smooth Banach space, setting $g(x) = \|x\|^2$ for all $x$ in $E$, we have that $\nabla g(x) = 2Jx$ for all $x$ in $E$. Here $J$ is the normalized duality mapping from $E$ into $E^*$. Hence, $D_g(\cdot, \cdot)$ reduces to the usual map $\phi(\cdot, \cdot)$ as

$$D_g(x, y) = \phi(x, y) := \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in E. \tag{1.3}$$

If $E$ is a Hilbert space, then $D_g(x, y) = \|x - y\|^2$.

Let $g : E \to \mathbb{R}$ be strictly convex and Gâteaux differentiable, and $C \subseteq E$ be nonempty. A mapping $T : C \to E$ is said to be

- *Bregman nonexpansive* if

$$D_g(Tx, Ty) \leq D_g(x, y), \quad \forall x, y \in C;$$

- *Bregman quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$D_g(p, Tx) \leq D_g(p, x), \quad \forall x \in C, \forall p \in F(T);$$

- *Bregman skew quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$D_g(Tx, p) \leq D_g(x, p), \quad \forall x \in C, \forall p \in F(T).$$

- *Bregman nonspreading* if

$$D_g(Tx, Ty) + D_g(Ty, Tx) \leq D_g(Tx, y) + D_g(Ty, x), \quad \forall x, y \in C;$$

It is obvious that every Bregman nonspreading map $T$ with $F(T) \neq \emptyset$ is Bregman quasi-nonexpansive. Bregman nonspreading mappings include, in particular, the class of nonspreading functions studied by Takahashi and his coauthors (see, e.g., [13, 29]), which is defined with the map $\phi$ in (1.3).

Let us give an example of a Bregman nonspreading mapping with nonempty fixed point set, which is not quasi-nonexpansive.

**Example 1.1.** Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x^4$. The associated Bregman distance is given by

$$\begin{aligned} D_g(x, y) &= x^4 - y^4 - 4(x - y)y^3 \\ &= x^4 + 3y^4 - 4xy^3, \quad \forall x, y \in \mathbb{R}. \end{aligned}$$

Define $T : [0, 2] \to [0, 2]$ by

$$Tx = \begin{cases} 0 & \text{if } x \in [0, 2), \\ 1 & \text{if } x = 2. \end{cases}$$

We have $F(T) = \{0\}$. Plainly, $T$ is neither nonexpansive nor continuous.

However, $T$ is Bregman nonspreading. To see this, we define $f : [0, 2] \times [0, 2] \to \mathbb{R}$ by

$$f(x, y) = D_g(Tx, Ty) + D_g(Ty, Tx) - D_g(Tx, y) - D_g(Ty, x), \quad \forall x, y \in [0, 2].$$

Consider the following three possible cases:

*Case 1.* If $x = y = 2$, then we have $Tx = Ty = 1$ and hence

$$f(2, 2) = 0 + 0 - 17 - 17 = -34 < 0.$$

*Case 2.* If $x = 2$ and $y \in [0, 2)$, then we have $Tx = 1$, $Ty = 0$ and hence

$$f(2, y) = 1 + 3 - 1 - 3y^4 + 4y^3 - 48 = -3y^4 + 4y^3 - 45 < 0.$$

*Case 3.* If $x, y \in [0, 2)$, then we have $Tx = Ty = 0$ and hence

$$f(x, y) = -3(x^4 + y^4) \le 0.$$

Thus we have $f(x, y) \le 0$ for all $x, y$ in $[0, 2]$ and hence $T$ is a Bregman nonspreading mapping.

In Section 2, we collect and study some basic ties of Bregman distances. In Section 3, utilizing the Bregman-Opial property, we present some fixed point theorems. In Sections 4 and 5, we investigate weak and strong convergence of the Ishikawa and Bregman-Ishikawa iterations for Bregman nonspreading mappings. Our results improve and generalize some known results in the current literature; see, for example, [27].

# 2 Bregman functions and Bregman distances

Let $E$ be a (real) Banach space, and let $g : E \to \mathbb{R}$. For any $x$ in $E$, the *gradient* $\nabla g(x)$ is defined to be the linear functional in $E^*$ such that

$$\langle y, \nabla g(x) \rangle = \lim_{t \to 0} \frac{g(x + ty) - g(x)}{t}, \quad \forall y \in E.$$

The function $g$ is said to be *Gâteaux differentiable* at $x$ if $\nabla g(x)$ is well-defined, and $g$ is *Gâteaux differentiable* if it is Gâteaux differentiable everywhere on $E$. We call $g$ Fréchet differentiable at $x$ (see, for example, [2, p. 13] or [12, p. 508]) if for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$| \, g(y) - g(x) - \langle y - x, \nabla g(x) \rangle \, | \le \epsilon \|y - x\| \quad \text{whenever } \|y - x\| \le \delta.$$

The function $g$ is said to be *Fréchet differentiable* if it is Fréchet differentiable everywhere.

Let $B$ be the closed unit ball of a Banach space $E$. A function $g : E \to \mathbb{R}$ is said to be

- *strongly coercive* if
$$\lim_{\|x_n\| \to +\infty} \frac{g(x_n)}{\|x_n\|} = +\infty;$$
- *locally bounded* if $g(rB)$ is bounded for all $r > 0$;
- *locally uniformly smooth* on $E$ ([31, pp. 207, 221]) if the function $\sigma_r : [0, +\infty) \to [0, +\infty]$, defined by
$$\sigma_r(t) = \sup_{x \in rB, y \in S_E, \alpha \in (0,1)} \frac{\alpha g(x + (1-\alpha)ty) + (1-\alpha) g(x - \alpha t y) - g(x)}{\alpha(1-\alpha)},$$

satisfies

$$\lim_{t\downarrow 0}\frac{\sigma_r(t)}{t}=0,\quad \forall r>0;$$

- *locally uniformly convex* on $E$ (or *uniformly convex on bounded subsets* of $E$ ([31, pp. 203, 221])) if *the gauge* $\rho_r:[0,+\infty)\to[0,+\infty]$ *of uniform convexity of* $g$, defined by

$$\rho_r(t)=\inf_{x,y\in rB,\|x-y\|=t,\alpha\in(0,1)}\frac{\alpha g(x)+(1-\alpha)g(y)-g(\alpha x+(1-\alpha)y)}{\alpha(1-\alpha)},$$

  satisfies

$$\rho_r(t)>0,\quad \forall r,t>0;$$

For a locally uniformly convex map $g:E\to\mathbb{R}$, we have

$$g(\alpha x+(1-\alpha)y)\le \alpha g(x)+(1-\alpha)g(y)-\alpha(1-\alpha)\rho_r(\|x-y\|),\tag{2.1}$$

for all $x,y$ in $rB$ and for all $\alpha$ in $(0,1)$.

Let $E$ be a Banach space and $g:E\to\mathbb{R}$ a strictly convex and Gâteaux differentiable function. By (1.2), the Bregman distance satisfies that [5]

$$D_g(x,z)=D_g(x,y)+D_g(y,z)+\langle x-y,\nabla g(y)-\nabla g(z)\rangle,\quad \forall x,y,z\in E.\tag{2.2}$$

In particular,

$$D_g(x,y)=-D_g(y,x)+\langle y-x,\nabla g(y)-\nabla g(x)\rangle,\quad \forall x,y\in E.\tag{2.3}$$

**Lemma 2.1** ([17])**.** *Let $E$ be a Banach space and $g:E\to\mathbb{R}$ a Gâteaux differentiable function which is locally uniformly convex on $E$. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be bounded sequences in $E$. Then the following assertions are equivalent.*

*(1)* $\lim_{n\to\infty}D_g(x_n,y_n)=0$.

*(2)* $\lim_{n\to\infty}\|x_n-y_n\|=0$.

The following Bregman Opial-like inequality has been proved in [10].

**Lemma 2.2** ([10])**.** *Let $E$ be a Banach space and let $g:E\to\mathbb{R}$ be a strictly convex and Gâteaux differentiable function. Suppose $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in $E$ such that $x_n\rightharpoonup x$ for some $x$ in $E$. Then*

$$\limsup_{n\to\infty}D_g(x_n,x)<\limsup_{n\to\infty}D_g(x_n,y),$$

*for all $y$ in the interior of* $\operatorname{dom} g$ *with $y\neq x$.*

We call a function $g : E \to (-\infty, +\infty]$ *lower semicontinuous* if $\{x \in E : g(x) \leq r\}$ is closed for all $r$ in $\mathbb{R}$. For a lower semicontinuous convex function $g : E \to \mathbb{R}$, the *subdifferential* $\partial g$ of $g$ is defined by

$$\partial g(x) = \{x^* \in E^* : g(x) + \langle y - x, x^* \rangle \leq g(y), \quad \forall y \in E\}$$

for all $x$ in $E$. It is well known that $\partial g \subset E \times E^*$ is maximal monotone [22, 23]. For any lower semicontinuous convex function $g : E \to (-\infty, +\infty]$, the *conjugate function* $g^*$ of $g$ is defined by

$$g^*(x^*) = \sup_{x \in E} \{\langle x, x^* \rangle - g(x)\}, \quad \forall x^* \in E^*.$$

It is well known that

$$g(x) + g^*(x^*) \geq \langle x, x^* \rangle, \quad \forall (x, x^*) \in E \times E^*,$$

and

$$(x, x^*) \in \partial g \quad \text{is equivalent to} \quad g(x) + g^*(x^*) = \langle x, x^* \rangle. \tag{2.4}$$

We also know that if $g : E \to (-\infty, +\infty]$ is a proper lower semicontinuous convex function, then $g^* : E^* \to (-\infty, +\infty]$ is a proper weak$^*$ lower semicontinuous convex function. Here, saying $g$ is *proper* we mean that $\operatorname{dom} g := \{x \in E : g(x) < +\infty\} \neq \emptyset$.

The following definition is slightly different from that in Butnariu and Iusem [2].

**Definition 2.3** ([12])**.** Let $E$ be a Banach space. A function $g : E \to \mathbb{R}$ is said to be a *Bregman function* if the following conditions are satisfied:

(1) $g$ is continuous, strictly convex and Gâteaux differentiable;

(2) the set $\{y \in E : D_g(x, y) \leq r\}$ is bounded for all $x$ in $E$ and $r > 0$.

The following lemma follows from Butnariu and Iusem [2] and Zălinscu [31].

**Lemma 2.4.** *Let $E$ be a reflexive Banach space and $g : E \to \mathbb{R}$ a strongly coercive Bregman function. Then*

*(1)* $\nabla g : E \to E^*$ *is one-to-one, onto and norm-to-weak$^*$ continuous;*

*(2)* $\langle x - y, \nabla g(x) - \nabla g(y) \rangle = 0$ *if and only if* $x = y$*;*

*(3)* $\{x \in E : D_g(x, y) \leq r\}$ *is bounded for all $y$ in $E$ and $r > 0$;*

*(4)* $\operatorname{dom} g^* = E^*$, $g^*$ *is Gâteaux differentiable and* $\nabla g^* = (\nabla g)^{-1}$.

The following two results follow from [31, Proposition 3.6.4].

**Proposition 2.5.** *Let $E$ be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex function which is locally bounded. The following assertions are equivalent.*

*(1) $g$ is strongly coercive and locally uniformly convex on $E$;*

*(2) $\operatorname{dom} g^* = E^*$, $g^*$ is locally bounded and locally uniformly smooth on $E$;*

*(3) $\operatorname{dom} g^* = E^*$, $g^*$ is Fréchet differentiable and $\nabla g^*$ is uniformly norm-to-norm continuous on bounded subsets of $E^*$.*

**Proposition 2.6.** *Let $E$ be a reflexive Banach space and $g : E \to \mathbb{R}$ a continuous convex function which is strongly coercive. The following assertions are equivalent.*

*(1) $g$ is locally bounded and locally uniformly smooth on $E$;*

*(2) $g^*$ is Fréchet differentiable and $\nabla g^*$ is uniformly norm-to-norm continuous on bounded subsets of $E$;*

*(3) $\operatorname{dom} g^* = E^*$, $g^*$ is strongly coercive and locally uniformly convex on $E$.*

**Lemma 2.7** ([12, 3])**.** *Let $E$ be a reflexive Banach space, $g : E \to \mathbb{R}$ be a strongly coercive Bregman function and $V$ be the function defined by*

$$V(x, x^*) = g(x) - \langle x, x^* \rangle + g^*(x^*), \quad \forall x \in E, \forall x^* \in E^*.$$

*The following assertions hold.*

*(1) $D_g(x, \nabla g^*(x^*)) = V(x, x^*)$ for all $x$ in $E$ and $x^*$ in $E^*$.*

*(2) $V(x, x^*) + \langle \nabla g^*(x^*) - x, y^* \rangle \leq V(x, x^* + y^*)$ for all $x$ in $E$ and $x^*, y^*$ in $E^*$.*

It also follows from the definition that $V$ is convex in the second variable $x^*$, and

$$V(x, \nabla g(y)) = D_g(x, y).$$

Let $E$ be a Banach space and let $C$ be a nonempty convex subset of $E$. Let $g : E \to \mathbb{R}$ be a strictly convex and Gâteaux differentiable function. Then, we know from [16] that for $x$ in $E$ and $x_0$ in $C$, we have

$$D_g(x_0, x) = \min_{y \in C} D_g(y, x) \quad \text{if and only if} \quad \langle y - x_0, \nabla g(x) - \nabla g(x_0) \rangle \leq 0, \ \forall y \in C. \tag{2.5}$$

Further, if $C$ is a nonempty, closed and convex subset of a reflexive Banach space $E$ and $g : E \to \mathbb{R}$ is a strongly coercive Bregman function, then for each $x$ in $E$, there exists a unique $x_0$ in $C$ such that

$$D_g(x_0, x) = \min_{y \in C} D_g(y, x).$$

The *Bregman projection* $\mathrm{proj}_C^g$ from $E$ onto $C$ defined by $\mathrm{proj}_C^g(x) = x_0$ has the following property:

$$D_g(y, \mathrm{proj}_C^g x) + D_g(\mathrm{proj}_C^g x, x) \leq D_g(y, x), \quad \forall y \in C, \forall x \in E. \tag{2.6}$$

See [2] for details.

Let $E$ be a reflexive Banach space and $g : E \to \mathbb{R}$ be a lower-semicontinuous, strictly convex and Gâteaux differentiable function. Let $C$ be a nonempty, closed and convex subset of $E$ and $\{x_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $E$. For any $x$ in $E$, we set

$$Br(x, \{x_n\}) = \limsup_{n\to\infty} D_g(x_n, x).$$

The *Bregman asymptotic radius* of $\{x_n\}_{n\in\mathbb{N}}$ relative to $C$ is defined by

$$Br(C, \{x_n\}) = \inf\{Br(x, \{x_n\}) : x \in C\}.$$

The *Bregman asymptotic center* of $\{x_n\}_{n\in\mathbb{N}}$ relative to $C$ is the set

$$BA(C, \{x_n\}) = \{x \in C : Br(x, \{x_n\}) = Br(C, \{x_n\})\}.$$

**Proposition 2.8.** *Let $C$ be a nonempty, closed and convex subset of a reflexive Banach space $E$, and let $g : E \to \mathbb{R}$ be strictly convex, Gâteaux differentiable, and locally bounded on $E$. If $\{x_n\}_{n\in\mathbb{N}}$ is a bounded sequence of $C$, then $BA(C, \{x_n\}_{n\in\mathbb{N}})$ is a singleton.*

*Proof.* In view of the definition of Bregman asymptotic radius, we may assume that $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to $z$ in $C$. By Lemma 2.2, we conclude that $BA(C, \{x_n\}_{n\in\mathbb{N}}) = \{z\}$. □

# 3 Fixed point theorems

**Lemma 3.1** ([21])**.** *Let $C$ be a nonempty, closed and convex subset of a reflexive Banach space $E$. Let $g : E \to \mathbb{R}$ be strictly convex, continuous, strongly coercive, Gâteaux differentiable, and locally bounded on $E$. Let $T : C \to E$ be a Bregman quasi-nonexpansive mapping. Then $F(T)$ is closed and convex.*

**Lemma 3.2.** *Let $C$ be a nonempty, closed and convex subset of a reflexive Banach space $E$. Let $g : E \to \mathbb{R}$ be a strictly convex and Gâteaux differentiable function. Let $T : C \to E$ be a Bregman nonspreading mapping. Then*

$$\begin{aligned} D_g(x, Ty) \leq\ & D_g(x, y) + D_g(Tx, x) \\ & + \langle x - Tx, \nabla g(y) - \nabla g(Ty)\rangle + \langle Tx - Ty, \nabla g(x) - \nabla g(Tx)\rangle, \quad \forall x, y \in C. \end{aligned}$$

*Proof.* Let $x, y \in C$. In view of (2.2), we have

$$\begin{array}{rcl} D_g(Tx,Ty) & \leq & D_g(Tx,y) + D_g(Ty,x) - D_g(Ty,Tx) \\ & = & D_g(Tx,x) + D_g(x,y) + \langle Tx - x, \nabla g(x) - \nabla g(y)\rangle \\ & + & D_g(Ty,Tx) + D_g(Tx,x) + \langle Ty - Tx, \nabla g(Tx) - \nabla g(x)\rangle - D_g(Ty,Tx) \\ & = & D_g(x,y) + 2D_g(Tx,x) + \langle Tx - x, \nabla g(x) - \nabla g(y)\rangle \\ & + & \langle Tx - Ty, \nabla g(x) - \nabla g(Tx)\rangle. \end{array}$$

This, together with (2.2), implies that

$$\begin{array}{rcl} D_g(x,Ty) & = & D_g(x,Tx) + D_g(Tx,Ty) + \langle x - Tx, \nabla g(Tx) - \nabla g(Ty)\rangle \\ & \leq & D_g(x,Tx) + D_g(x,y) + 2D_g(Tx,x) + \langle Tx - x, \nabla g(x) - \nabla g(y)\rangle \\ & + & \langle Tx - Ty, \nabla g(x) - \nabla g(Tx)\rangle + \langle x - Tx, \nabla g(Tx) - \nabla g(Ty)\rangle \\ & = & D_g(x,y) + D_g(Tx,x) + \langle x - Tx, \nabla g(x) - \nabla g(Tx)\rangle \\ & + & \langle Tx - Ty, \nabla g(x) - \nabla g(Tx)\rangle + \langle Tx - x, \nabla g(x) - \nabla g(y)\rangle \\ & + & \langle x - Tx, \nabla g(Tx) - \nabla g(Ty)\rangle \\ & = & D_g(x,y) + D_g(Tx,x) + \langle x - Tx, \nabla g(y) - \nabla g(Ty)\rangle \\ & + & \langle Tx - Ty, \nabla g(x) - \nabla g(Tx)\rangle. \end{array}$$

□

**Proposition 3.3** (Demiclosedness Principle)**.** *Let $C$ be a nonempty subset of a reflexive Banach space $E$. Let $g : E \to \mathbb{R}$ be a strictly convex, Gâteaux differentiable and locally bounded function. Let $T : C \to E$ be a Bregman nonspreading mapping. If $x_n \rightharpoonup z$ in $C$ and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$, then $Tz = z$. That is, $I - T$ is demiclosed at zero, where $I$ is the identity mapping on $E$.*

*Proof.* Since $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to $z$ and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$, both the sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{Tx_n\}_{n\in\mathbb{N}}$ are bounded. Since $\nabla g$ is uniformly norm-to-norm continuous on bounded subsets of $E$ (see, for instance, [31]), we arrive at

$$\lim_{n\to\infty} \|\nabla g(x_n) - \nabla g(Tx_n)\| = 0.$$

In view of Lemma 2.1, we deduce that $\lim_{n\to\infty} D_g(x_n, Tx_n) = 0$. Set

$$M_1 = \sup\{\|Tx_n\|, \|Tz\|, \|\nabla g(z)\|, \|\nabla g(Tz)\| : n \in \mathbb{N}\} < +\infty.$$

By Lemma 3.2, for all $n$ in $\mathbb{N}$,

$$\begin{aligned}
& D_g(x_n, Tz) \\
\leq\ & D_g(x_n, z) + D_g(Tx_n, x_n) \\
+\ & \langle x_n - Tx_n, \nabla g(z) - \nabla g(Tz)\rangle + \langle Tx_n - Tz, \nabla g(x_n) - \nabla g(Tx_n)\rangle \\
\leq\ & D_g(x_n, z) + D_g(Tx_n, x_n) \\
+\ & \|x_n - Tx_n\| \|\nabla g(z) - \nabla g(Tz)\| + \|Tx_n - Tz\| \|\nabla g(x_n) - \nabla g(Tx_n)\| \\
\leq\ & D_g(x_n, z) + D_g(Tx_n, x_n) \\
+\ & 2M_1\|x_n - Tx_n\| + 2M_1\|\nabla g(x_n) - \nabla g(Tx_n)\|.
\end{aligned}$$

This implies

$$\limsup_{n\to\infty} D_g(x_n, Tz) \leq \limsup_{n\to\infty} D_g(x_n, z).$$

From the Bregman Opial-like property, we obtain $Tz = z$. □

Let $\ell^\infty$ be the Banach lattice of bounded real sequences with the supremum norm. It is well known that there exists a bounded linear functional $\mu$ on $\ell^\infty$ such that the following three conditions hold:

(1) If $\{t_n\}_{n\in\mathbb{N}} \in \ell^\infty$ and $t_n \geq 0$ for every $n$ in $\mathbb{N}$, then $\mu(\{t_n\}) \geq 0$;

(2) If $t_n = 1$ for every $n$ in $\mathbb{N}$, then $\mu(\{t_n\}) = 1$;

(3) $\mu(\{t_{n+1}\}) = \mu(\{t_n\})$ for all $\{t_n\}_{n\in\mathbb{N}}$ in $\ell^\infty$.

Here, $\{t_{n+1}\}$ denotes the sequence $(t_2, t_3, t_4, \ldots, t_{n+1}, \ldots)$ in $\ell^\infty$. Such a functional $\mu$ is called a *Banach limit* and the value of $\mu$ at $\{t_n\}_{n\in\mathbb{N}}$ in $\ell^\infty$ is denoted by $\mu_n t_n$. Therefore, condition (3) means $\mu_n t_n = \mu_n t_{n+1}$. If $\mu$ satisfies conditions (1) and (2), we call $\mu$ a *mean* on $\ell^\infty$. See, for example [26].

To see some examples of those mappings $T$ satisfying all the stated hypotheses in the following result, we refer the reader to [11].

**Theorem 3.4** ([11])**.** *Let $C$ be a nonempty, closed and convex subset of a reflexive Banach space $E$. Let $g : E \to \mathbb{R}$ be strictly convex, continuous, strongly coercive, Gâteaux differentiable, locally bounded and locally uniformly convex on $E$. Let $T : C \to C$ be a mapping. Let $\{x_n\}_{n\in\mathbb{N}}$ be a bounded sequence of $C$ and let $\mu$ be a mean on $\ell^\infty$. Suppose that*

$$\mu_n D_g(x_n, Ty) \leq \mu_n D_g(x_n, y), \forall y \in C.$$

*Then $T$ has a fixed point in $C$.*

**Corollary 3.5.** *Let $C$ be a nonempty, bounded, closed and convex subset of a reflexive Banach space $E$. Let $g : E \to \mathbb{R}$ be strictly convex, continuous, strongly coercive, Gâteaux differentiable function, locally bounded and locally uniformly convex on $E$. Let $T : C \to C$ be a Bregman nonspreading mapping. Then $T$ has a fixed point.*

*Proof.* Let $\mu$ a Banach limit on $\ell^\infty$ and $x \in C$ be such that $\{T^n x\}_{n\in\mathbb{N}}$ is bounded. For any $n$ in $\mathbb{N}$ we have

$$D_g(T^n x, Ty) + D_g(Ty, T^n x) \leq D_g(T^n x, y) + D_g(Ty, T^{n-1}x), \quad \forall y \in C.$$

This implies that

$$\mu_n D_g(T^n x, Ty) + \mu_n D_g(Ty, T^n x) \leq \mu_n D_g(T^n x, y) + \mu_n D_g(Ty, T^{n-1}x), \quad \forall y \in C.$$

Thus we have

$$\mu_n D_g(T^n x, Ty) \leq \mu_n D_g(T^n x, y), \quad \forall y \in C.$$

It follows from Theorem 3.4 that $F(T) \neq \emptyset$. □

# 4 Weak and strong convergence theorems for Bregman nonspreading mappings

In this section, we prove weak and strong convergence theorems concerning Bregman nonspreading mappings in a reflexive Banach space.

**Lemma 4.1.** *Let $C$ be a nonempty, closed and convex subset of a reflexive Banach space $E$. Let $g : E \to \mathbb{R}$ be a strictly convex and Gâteaux differentiable function. Let $T : C \to C$ be a Bregman skew quasi-nonexpansive mapping with a nonempty fixed point set $F(T)$. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be two sequences defined by (1.1) such that $\{\beta_n\}_{n\in\mathbb{N}}$ and $\{\gamma_n\}_{n\in\mathbb{N}}$ are arbitrary sequences in $[0,1]$. Then the following assertions hold:*

*(1)* $\max\{D_g(x_{n+1}, z), D_g(y_n, z)\} \leq D_g(x_n, z)$ *for all $z$ in $F(T)$ and $n = 1, 2, \ldots$.*

*(2)* $\lim_{n\to\infty} D_g(x_n, z)$ *exists for any $z$ in $F(T)$.*

*Proof.* Let $z \in F(T)$. In view of (2.1), we have

$$\begin{aligned} D_g(y_n, z) &= D_g(\beta_n T x_n + (1-\beta_n)x_n, z) \\ &\leq \beta_n D_g(Tx_n, z) + (1-\beta_n)D_g(x_n, z) \\ &\leq \beta_n D_g(x_n, z) + (1-\beta_n)D_g(x_n, z) \\ &= D_g(x_n, z). \end{aligned}$$

Consequently,

$$\begin{aligned} D_g(x_{n+1}, z) &= D_g(\gamma_n T y_n + (1-\gamma_n)x_n, z) \\ &\leq \gamma_n D_g(Ty_n, z) + (1-\gamma_n)D_g(x_n, z) \\ &\leq \gamma_n D_g(y_n, z) + (1-\gamma_n)D_g(x_n, z) \\ &\leq \gamma_n D_g(x_n, z) + (1-\gamma_n)D_g(x_n, z) \\ &= D_g(x_n, z). \end{aligned}$$

This implies that $\{D_g(x_n, z)\}_{n\in\mathbb{N}}$ is a bounded and nonincreasing sequence for all $z$ in $F(T)$. Thus we have $\lim_{n\to\infty} D_g(x_n, z)$ exists for any $z$ in $F(T)$. □

**Theorem 4.2.** *Let $C$ be a nonempty, closed and convex subset of a reflexive Banach space $E$. Let $g : E \to \mathbb{R}$ be strictly convex,Gâteaux differentiable, locally bounded and locally uniformly convex on $E$. Let $T : C \to C$ a Bregman nonspreading and Bregman skew quasi-nonexpansive mapping. Let $\{\beta_n\}_{n\in\mathbb{N}}$ and $\{\gamma_n\}_{n\in\mathbb{N}}$ be sequences in $[0, 1]$, and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence with $x_1$ in $C$ defined by (1.1) .*

*(a) If $\{x_n\}_{n\in\mathbb{N}}$ is bounded and $\liminf\limits_{n\to\infty} \|Tx_n - x_n\| = 0$, then the fixed point set $F(T) \neq \emptyset$.*

*(b) Assume $F(T) \neq \emptyset$. Then $\{x_n\}_{n\in\mathbb{N}}$ is bounded.*

- *i.* $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$ *when* $\liminf_{n\to\infty} \gamma_n(1-\gamma_n) > 0$ *and* $\lim_{n\to\infty} \beta_n = 1$.
- *ii.* $\liminf_{n\to\infty} \|Tx_n - x_n\| = 0$ *when either*
  - $\limsup_{n\to\infty} \gamma_n(1-\gamma_n) > 0$ *and* $\lim_{n\to\infty} \beta_n = 1$, *or*
  - $\liminf_{n\to\infty} \gamma_n(1-\gamma_n) > 0$ *and* $\limsup_{n\to\infty} \beta_n = 1$.

*Proof.* Assume that $\{x_n\}_{n\in\mathbb{N}}$ is bounded and $\liminf\limits_{n\to\infty} \|Tx_n - x_n\| = 0$. Consequently, there is a bounded subsequence $\{Tx_{n_k}\}_{k\in\mathbb{N}}$ of $\{Tx_n\}_{n\in\mathbb{N}}$ such that $\lim_{k\to\infty} \|Tx_{n_k} - x_{n_k}\| = 0$. Since $\nabla g$ is uniformly norm-to-norm continuous on bounded subsets of $E$ (see, for example, [31]),

$$\lim_{k\to\infty} \|\nabla g(Tx_{n_k}) - \nabla g(x_{n_k})\| = 0.$$

In view of Proposition 2.8, we conclude that $BA(C, \{x_{n_k}\}) = \{z\}$ for some $z$ in $C$. Let

$$M_2 = \sup\{\|T(z)\|, \|Tx_{n_k}\|, \|\nabla g(z)\|, \|\nabla g(Tz)\| : k \in \mathbb{N}\} < +\infty.$$

It follows from Lemma 3.2 that

$$\begin{aligned} & D_g(x_{n_k}, Tz) \\ \leq\; & D_g(x_{n_k}, z) + D_g(Tx_{n_k}, x_{n_k}) \\ +\; & \langle x_{n_k} - Tx_{n_k}, \nabla g(z) - \nabla g(Tz)\rangle + \langle Tx_{n_k} - Tz, \nabla g(x_{n_k}) - \nabla g(Tx_{n_k})\rangle \\ \leq\; & D_g(x_{n_k}, z) + D_g(Tx_{n_k}, x_{n_k}) \\ +\; & \|x_{n_k} - Tx_{n_k}\|\|\nabla g(z) - \nabla g(Tz)\| + \|Tx_{n_k} - Tz\|\|\nabla g(x_{n_k}) - \nabla g(Tx_{n_k})\| \\ \leq\; & D_g(x_{n_k}, z) + D_g(Tx_{n_k}, x_{n_k}) \\ +\; & 2M_2\|x_{n_k} - Tx_{n_k}\| + 2M_2\|\nabla g(x_{n_k}) - \nabla g(Tx_{n_k})\|, \quad k = 1, 2, \ldots. \end{aligned}$$

This implies

$$\limsup_{k\to\infty} D_g(x_{n_k}, Tz) \leq \limsup_{k\to\infty} D_g(x_{n_k}, z).$$

From the Bregman Opial-like property, we obtain $Tz = z$.

Let $F(T) \neq \emptyset$ and let $z \in F(T)$. It follows from Lemma 4.1 that $\lim_{n\to\infty} \|x_n - z\|$ exists and hence $\{x_n\}_{n\in\mathbb{N}}$ is bounded. This implies that the sequence $\{Ty_n\}_{n\in\mathbb{N}}$ is bounded too. Let $s_1 = \sup\{\|x_n\|, \|Ty_n\| : n \in \mathbb{N}\} < \infty$. In view of (2.1), we obtain a continuous, strictly increasing and convex function $\rho_{s_1} : [0, +\infty) \to [0, +\infty)$ with $\rho_{s_1}(0) = 0$ such that

$$\begin{aligned} D_g(x_{n+1}, z) &= D_g(\gamma_n Ty_n + (1-\gamma_n)x_n, z) \\ &\leq \gamma_n D_g(Ty_n, z) + (1-\gamma_n) D_g(x_n, z) - \gamma_n(1-\gamma_n)\rho_{s_1}(\|Ty_n - x_n\|) \\ &\leq \gamma_n D_g(y_n, z) + (1-\gamma_n) D_g(x_n, z) - \gamma_n(1-\gamma_n)\rho_{s_1}(\|Ty_n - x_n\|) \\ &\leq \gamma_n D_g(x_n, z) + (1-\gamma_n) D_g(x_n, z) - \gamma_n(1-\gamma_n)\rho_{s_1}(\|Ty_n - x_n\|) \\ &= D_g(x_n, z) - \gamma_n(1-\gamma_n)\rho_{s_1}(\|Ty_n - x_n\|). \end{aligned}$$

Consequently, we conclude that

$$\begin{aligned} \gamma_n(1-\gamma_n)\rho_{s_1}(\|Ty_n - x_n\|) &\leq D_g(x_n, z) - D_g(x_{n+1}, z) \\ &\to 0, \quad \text{as } n \to \infty. \end{aligned}$$

It follows that

$$\liminf_{n\to\infty} \rho_{s_1}(\|Ty_n - x_n\|) = 0 \quad \text{whenever} \quad \limsup_{n\to\infty} \gamma_n(1-\gamma_n) > 0.$$

From the property of $\rho_{s_1}$ we deduce that

$$\liminf_{n\to\infty} \|Ty_n - x_n\| = 0 \quad \text{whenever} \quad \limsup_{n\to\infty} \gamma_n(1-\gamma_n) > 0. \tag{4.1}$$

In the same manner, we also obtain that

$$\lim_{n\to\infty} \|Ty_n - x_n\| = 0 \quad \text{whenever} \quad \liminf_{n\to\infty} \gamma_n(1-\gamma_n) > 0. \tag{4.2}$$

Since $\nabla g$ is uniformly norm-to-norm continuous on bounded subsets of $E$ (see, for instance, [31]), we arrive at

$$\lim_{n\to\infty} \|\nabla g(Ty_n) - \nabla g(x_n)\| = 0.$$

On the other hand, from (1.1) we get

$$Tx_n - y_n = (1-\beta_n)(Tx_n - x_n), \quad x_n - y_n = \beta_n(x_n - Tx_n). \tag{4.3}$$

Assuming first $\liminf\limits_{n\to\infty} \gamma_n(1-\gamma_n) > 0$. By (4.2) we see that

$$M_3 := \sup\{\|\nabla g(x_n)\|, \|\nabla g(Tx_n)\|, \|\nabla g(Ty_n\|) : n \in \mathbb{N}\} < +\infty.$$

Since $T$ is Bregman nonspreading, in view of (2.2), (2.3) and (4.3), we obtain

$$\begin{aligned}
& D_g(x_n, Tx_n) \\
= \quad & D_g(x_n, Ty_n) + D_g(Ty_n, Tx_n) + \langle x_n - Ty_n, \nabla g(Ty_n) - \nabla g(Tx_n)\rangle \\
\leq \quad & D_g(x_n, Ty_n) + [D_g(Ty_n, x_n) + D_g(Tx_n, y_n) - D_g(Tx_n, Ty_n)] \\
+ \quad & \|x_n - Ty_n\|\|\nabla g(Ty_n) - \nabla g(Tx_n)\| \\
\leq \quad & D_g(x_n, Ty_n) + [-D_g(x_n, Ty_n) + \langle x_n - Ty_n, \nabla g(x_n) - \nabla g(Ty_n)\rangle] \\
+ \quad & [-D_g(y_n, Tx_n) + \langle y_n - Tx_n, \nabla g(y_n) - \nabla g(Tx_n)\rangle] \\
+ \quad & \|x_n - Ty_n\|\|\nabla g(Ty_n) - \nabla g(Tx_n)\| \\
\leq \quad & \|x_n - Ty_n\|\|\nabla g(x_n) - \nabla g(Ty_n)\| + \|y_n - Tx_n\|\|\nabla g(y_n) - \nabla g(Tx_n)\| \\
+ \quad & \|x_n - Ty_n\|\|\nabla g(Ty_n) - \nabla g(Tx_n)\| \\
= \quad & (1-\beta_n)\|x_n - Tx_n\|\|\nabla g(y_n) - \nabla g(Tx_n)\| \\
+ \quad & \|x_n - Ty_n\|[\|\nabla g(x_n) - \nabla g(Ty_n) + \|\nabla g(Ty_n) - \nabla g(Tx_n)\|] \\
\leq \quad & 2(1-\beta_n)M_3\|x_n - Tx_n\| + 4M_3\|x_n - Ty_n\|.
\end{aligned}$$

When $\lim_{n\to\infty}\beta_n = 1$, we conclude that

$$\lim_{n\to\infty} D_g(x_n, Tx_n) = 0.$$

In view of Lemma 2.1, we have that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{4.4}$$

Finally, we assume $\limsup_{n\to\infty}\gamma_n(1-\gamma_n) > 0$ and $\lim_{n\to\infty}\beta_n = 1$ instead. By (4.1) we have subsequences $\{x_{n_k}\}_{k\in\mathbb{N}}$ and $\{y_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$, respectively, such that

$$\lim_{k\to\infty} \|Ty_{n_k} - x_{n_k}\| = 0.$$

Replacing $M_3$ with the finite number $\sup\{\|\nabla g(x_{n_k})\|, \|\nabla g(Tx_{n_k})\|, \|\nabla g(Ty_{n_k})\| : k \in \mathbb{N}\} < +\infty$, and dealing with the subsequences $\{x_{n_k}\}_{k\in\mathbb{N}}$ and $\{y_{n_k}\}_{k\in\mathbb{N}}$ in (4.2) and (4.3). Passing to a further subsequence if necessary, we will arrive at the desired conclusion with (4.4) that $\lim_{k\to\infty}\|Tx_{n_k} - x_{n_k}\| = 0$. Hence, $\liminf_{n\to\infty}\|Tx_n - x_n\| = 0$. The other case can be argued similarly. □

**Theorem 4.3.** *Let $C$ be a nonempty, closed and convex subset of a reflexive Banach space $E$. Let $g : E \to \mathbb{R}$ be strictly convex, Gâteaux differentiable, locally bounded and locally uniformly convex on $E$. Let $T : C \to C$ be a Bregman nonspreading and Bregman skew quasi-nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{\beta_n\}_{n\in\mathbb{N}}$ and $\{\gamma_n\}_{n\in\mathbb{N}}$ be sequences in $[0,1]$, and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence with $x_1$ in $C$ defined by (1.1). Assume that $\liminf_{n\to\infty}\gamma_n(1-\gamma_n) > 0$ and $\lim_{n\to\infty}\beta_n = 1$. Then $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to a fixed point of $T$.*

*Proof.* It follows from Theorem 4.2 that $\{x_n\}_{n\in\mathbb{N}}$ is bounded and $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$. Since $E$ is reflexive, then there exists a subsequence $\{x_{n_i}\}_{i\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $x_{n_i} \rightharpoonup p \in C$ as $i \to \infty$. By Proposition 3.3, $p \in F(T)$. We claim that $x_n \rightharpoonup p$ as $n \to \infty$. If not, then there exists a subsequence $\{x_{n_j}\}_{j\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $\{x_{n_j}\}_{j\in\mathbb{N}}$ converges weakly to some $q$ in $C$ with $p \neq q$. In view of Proposition 3.3 again, we conclude that $q \in F(T)$. By Lemma 4.1, $\lim_{n\to\infty} D_g(x_n, z)$ exists for all $z$ in $F(T)$. Thus we obtain by the Bregman Opial-like property that

$$\begin{aligned}\lim_{n\to\infty} D_g(x_n,p) &= \lim_{i\to\infty} D_g(x_{n_i},p) &&< \lim_{i\to\infty} D_g(x_{n_i},q)\\ &= \lim_{n\to\infty} D_g(x_n,q) &&= \lim_{j\to\infty} D_g(x_{n_j},q)\\ &< \lim_{j\to\infty} D_g(x_{n_j},p) &&= \lim_{n\to\infty} D_g(x_n,p).\end{aligned}$$

This is a contradiction. Thus we have $p = q$, and the desired assertion follows. □

**Theorem 4.4.** *Let $C$ be a nonempty, compact and convex subset of a reflexive Banach space $E$. Let $g : E \to \mathbb{R}$ be strictly convex, Gâteaux differentiable, locally bounded and uniformly convex on bounded sets. Let $T : C \to C$ be a Bregman nonspreading and Bregman skew quasi-nonexpansive mapping. Let $\{\beta_n\}_{n\in\mathbb{N}}$ and $\{\gamma_n\}_{n\in\mathbb{N}}$ be sequences in $[0,1]$. Assume that either $\limsup_{n\to\infty}\gamma_n(1-\gamma_n) > 0$ and $\lim_{n\to\infty}\beta_n = 1$, or $\liminf_{n\to\infty}\gamma_n(1-\gamma_n) > 0$ and $\limsup_{n\to\infty}\beta_n = 1$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence with $x_1$ in $C$ defined by (1.1). Then $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to a fixed point $z$ of $T$.*

*Proof.* By Corollary 3.5, we see that the fixed point set $F(T)$ of $T$ is nonempty. In view of Theorem 4.2, we obtain that $\{x_n\}_{n\in\mathbb{N}}$ is bounded and $\liminf\limits_{n\to\infty}\|Tx_n - x_n\| = 0$. By the compactness of $C$, there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $\{x_{n_k}\}_{k\in\mathbb{N}}$ converges strongly to some $z$ in $C$. In view of Lemma 2.1 we deduce that $\lim_{k\to\infty} D_g(x_{n_k}, z) = 0$. We can even assume that $\lim_{k\to\infty}\|Tx_{n_k} - x_{n_k}\| = 0$, and in particular, $\{Tx_{n_k}\}_{k\in\mathbb{N}}$ is bounded. Since $\nabla g$ is uniformly norm-to-norm continuous on bounded subsets of $E$ (see, for example, [31]),

$$\lim_{k\to\infty}\|\nabla g(Tx_{n_k}) - \nabla g(x_{n_k})\| = 0.$$

Let $M_4 = \sup\{\|Tz\|, \|Tx_{n_k}\|, \|\nabla g(z)\|, \|\nabla g(Tz)\| : k \in \mathbb{N}\} < +\infty$. In view of Lemma 3.2, we obtain

$$\begin{aligned}D_g(x_{n_k}, Tz) &\leq D_g(x_{n_k}, z) + D_g(Tx_{n_k}, x_{n_k})\\ &\quad + \langle x_{n_k} - Tx_{n_k}, \nabla g(z) - \nabla g(Tz)\rangle\\ &\quad + \langle Tx_{n_k} - Tz, \nabla g(x_{n_k}) - \nabla g(Tx_{n_k})\rangle\\ &\leq D_g(x_{n_k}, z) + D_g(Tx_{n_k}, x_{n_k})\\ &\quad + 2M_4[\|x_{n_k} - Tx_{n_k}\| + \|\nabla g(x_{n_k}) - \nabla g(Tx_{n_k})\|]\end{aligned}$$

for all $k$ in $\mathbb{N}$.

It follows $\lim_{k\to\infty}\|x_{n_k} - Tz\| = 0$. Thus we have $Tz = z$. In view of Lemmas 4.1 and 2.1, we conclude that $\lim_{n\to\infty}\|x_n - z\| = 0$. Therefore, $z$ is the strong limit of the sequence $\{x_n\}_{n\in\mathbb{N}}$. □

## 5 Bregman Ishikawa's type iteration for Bregman nonspreading mappings

We propose the following Bregman Ishikawa's type iteration. Let $E$ be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a strictly convex and Gâteaux differentiable function. Let $C$ be a nonempty, closed and convex subset of $E$. Let $T : C \to C$ be a Bregman nonspreading mapping such that the fixed point set $F(T)$ is nonempty. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be two sequences defined by

$$\begin{cases} y_n = \nabla g^*[\beta_n \nabla g(Tx_n) + (1-\beta_n)\nabla g(x_n)], \\ x_{n+1} = \mathrm{proj}_C^g(\nabla g^*[\gamma_n \nabla g(Ty_n) + (1-\gamma_n)\nabla g(x_n)]), \end{cases} \tag{5.1}$$

where $\{\beta_n\}_{n\in\mathbb{N}}$ and $\{\gamma_n\}_{n\in\mathbb{N}}$ are arbitrary sequences in $[0,1]$.

**Lemma 5.1.** *Let $C$ be a nonempty, closed and convex subset of a reflexive Banach space $E$. Let $g : E \to \mathbb{R}$ be a strongly coercive Bregman function. Let $T : C \to C$ be a Bregman quasi-nonexpansive mapping. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be two sequences defined by* (5.1) *such that $\{\beta_n\}_{n\in\mathbb{N}}$ and $\{\gamma_n\}_{n\in\mathbb{N}}$ are arbitrary sequences in $[0,1]$. Then the following assertions hold:*
*(1) $\max\{D_g(z,x_{n+1}), D_g(z,y_n)\} \le D_g(z,x_n)$ for all $z$ in $F(T)$ and $n = 1,2,\ldots$.*
*(2) $\lim_{n\to\infty} D_g(z,x_n)$ exists for any $z$ in $F(T)$.*

*Proof.* Let $z \in F(T)$. In view of Lemma 2.7 and (5.1), we conclude that

$$\begin{aligned} D_g(z,y_n) &= D_g(z, \nabla g^*[\beta_n \nabla g(Tx_n) + (1-\beta_n)\nabla g(x_n)]) \\ &= V(z, \beta_n \nabla g(Tx_n) + (1-\beta_n)\nabla g(x_n)) \\ &\le \beta_n V(z, \nabla g(Tx_n)) + (1-\beta_n) V(z, \nabla g(x_n)) \\ &= \beta_n D_g(z, Tx_n) + (1-\beta_n) D_g(z,x_n) \\ &\le \beta_n D_g(z,x_n) + (1-\beta_n) D_g(z,x_n) \\ &= D_g(z,x_n). \end{aligned}$$

Consequently, using (2.6) we have

$$\begin{aligned} D_g(z,x_{n+1}) &= D_g(z, \mathrm{proj}_C^g(\nabla g^*[\gamma_n \nabla g(Ty_n) + (1-\gamma_n)\nabla g(x_n)])) \\ &\le D_g(z, \nabla g^*[\gamma_n \nabla g(Ty_n) + (1-\gamma_n)\nabla g(x_n)]) \\ &= V(z, \gamma_n \nabla g(Ty_n) + (1-\gamma_n)\nabla g(x_n)) \\ &\le \gamma_n V(z, \nabla g(Ty_n)) + (1-\gamma_n) V(z, \nabla g(x_n)) \\ &= \gamma_n D_g(z, Ty_n) + (1-\gamma_n) D_g(z,x_n) \\ &\le \gamma_n D_g(z,y_n) + (1-\gamma_n) D_g(z,x_n) \\ &\le \gamma_n D_g(z,x_n) + (1-\gamma_n) D_g(z,x_n) \\ &= D_g(z,x_n). \end{aligned}$$

This implies that $\{D_g(z,x_n)\}_{n\in\mathbb{N}}$ is a bounded and nonincreasing sequence for all $z$ in $F(T)$. Thus we have $\lim_{n\to\infty} D_g(z,x_n)$ exists for any $z$ in $F(T)$. □

**Theorem 5.2.** *Let $C$ be a nonempty, closed and convex subset of a reflexive Banach space $E$. Let $g : E \to \mathbb{R}$ be a strongly coercive Bregman function which is locally bounded, locally uniformly convex and locally uniformly smooth on $E$. Let $T : C \to C$ be a Bregman nonspreading mapping. Let $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ be two sequence in $[0, 1]$ satisfying the control condition:*

$$\sum_{n=1}^{\infty} \gamma_n \beta_n (1 - \beta_n) = +\infty. \tag{5.2}$$

*Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by the algorithm (5.1). Then the following are equivalent.*

*(1) There exists a bounded sequence $\{x_n\}_{n\in\mathbb{N}} \subset C$ such that $\liminf_{n\to\infty} \|Tx_n - x_n\| = 0$.*

*(2) The fixed point set $F(T) \neq \emptyset$.*

*Proof.* The implication (1) $\Longrightarrow$ (2) follows similarly as in the first part of the proof of Theorem 4.2.

For the implication (2) $\Longrightarrow$ (1), we assume $F(T) \neq \emptyset$. The boundedness of the sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ follows from Lemma 5.1 and Definition 2.3. Since $T$ is a Bregman quasi-nonexpansive mapping, for any $q$ in $F(T)$ we have

$$D_g(q, Tx_n) \leq D_g(q, x_n), \quad \forall n \in \mathbb{N}.$$

This, together with Definition 2.3 and the boundedness of $\{x_n\}_{n\in\mathbb{N}}$, implies that $\{Tx_n\}_{n\in\mathbb{N}}$ is bounded.

The function $g$ is bounded on bounded subsets of $E$ and therefore $\nabla g$ is also bounded on bounded subsets of $E^*$ (see, for example, [2, Proposition 1.1.11] for more details). This implies the sequences $\{\nabla g(x_n)\}_{n\in\mathbb{N}}$, $\{\nabla g(y_n)\}_{n\in\mathbb{N}}$, $\{\nabla g(Ty_n)\}_{n\in\mathbb{N}}$ and $\{\nabla g(Tx_n)\}_{n\in\mathbb{N}}$ are bounded in $E^*$.

In view of Proposition 2.6, we have that dom $g^* = E^*$ and $g^*$ is strongly coercive and uniformly convex on bounded subsets of $E^*$. Let $s_2 = \sup\{\|\nabla g(x_n)\|, \|\nabla g(Tx_n)\| : n \in \mathbb{N}\} < \infty$ and let $\rho^*_{s_2} : E^* \to \mathbb{R}$ be the gauge of uniform convexity of the conjugate function $g^*$.

**Claim.** For any $p$ in $F(T)$ and $n$ in $\mathbb{N}$,

$$D_g(p, y_n) \leq D_g(p, x_n) - \beta_n(1 - \beta_n)\rho^*_{s_2}(\|\nabla g(x_n) - \nabla g(Tx_n)\|). \tag{5.3}$$

Let $p \in F(T)$. For each $n$ in $\mathbb{N}$, it follows from the definition of Bregman distance (1.2), Lemma 2.7, (2.1) and (5.1) that

$$\begin{aligned}
D_g(p, y_n) &= g(p) - g(y_n) - \langle p - y_n, \nabla g(y_n)\rangle \\
&= g(p) + g^*(\nabla g(y_n)) - \langle y_n, \nabla g(y_n)\rangle - \langle p, \nabla g(y_n)\rangle + \langle y_n, \nabla g(y_n)\rangle \\
&= g(p) + g^*((1-\beta_n)\nabla g(x_n) + \beta_n \nabla g(Tx_n)) \\
&\quad - \langle p, (1-\beta_n)\nabla g(x_n) + \beta_n \nabla g(Tx_n))\rangle \\
&\leq (1-\beta_n)g(p) + \beta_n g(p) + (1-\beta_n)g^*(\nabla g(x_n)) + \beta_n g^*(\nabla g(Tx_n)) \\
&\quad - \beta_n(1-\beta_n)\rho^*_{s_2}(\|\nabla g(x_n) - \nabla g(Tx_n)\|) \\
&\quad - (1-\beta_n)\langle p, \nabla g(x_n)\rangle - \beta_n\langle p, \nabla g(Tx_n)\rangle \\
&= (1-\beta_n)[g(p) + g^*(\nabla g(x_n)) - \langle p, \nabla g(x_n)\rangle] \\
&\quad + \beta_n[g(p) + g^*(\nabla g(Tx_n)) - \langle p, \nabla g(Tx_n)\rangle] \\
&\quad - \beta_n(1-\beta_n)\rho^*_{s_2}(\|\nabla g(x_n) - \nabla g(Tx_n)\|) \\
&= (1-\beta_n)[g(p) - g(x_n) + \langle x_n, \nabla g(x_n)\rangle - \langle p, \nabla g(x_n)\rangle] \\
&\quad + \beta_n[g(p) - g(Tx_n) + \langle Tx_n, \nabla g(Tx_n)\rangle - \langle p, \nabla g(Tx_n)\rangle] \\
&\quad - \beta_n(1-\beta_n)\rho^*_{s_2}(\|\nabla g(x_n) - \nabla g(Tx_n)\|) \\
&= (1-\beta_n)D(p, x_n) + \beta_n D(p, Tx_n) - \beta_n(1-\beta_n)\rho^*_{s_2}(\|\nabla g(x_n) - \nabla g(Tx_n)\|) \\
&\leq (1-\beta_n)D(p, x_n) + \beta_n D(p, x_n) - \beta_n(1-\beta_n)\rho^*_{s_2}(\|\nabla g(x_n) - \nabla g(Tx_n)\|) \\
&= D(p, x_n) - \beta_n(1-\beta_n)\rho^*_{s_2}(\|\nabla g(x_n) - \nabla g(Tx_n)\|).
\end{aligned}$$

In view of Lemma 2.7 and (5.3), we obtain

$$\begin{aligned}
D_g(p, x_{n+1}) &= D_g(p, \nabla g^*[\gamma_n \nabla g(Ty_n) + (1-\gamma_n)\nabla g(x_n)]) \\
&= V(p, \gamma_n \nabla g(Ty_n) + (1-\gamma_n)\nabla g(x_n)) \\
&\leq \gamma_n V(p, \nabla g(Ty_n)) + (1-\gamma_n)V(p, \nabla g(x_n)) \\
&= \gamma_n D_g(p, Ty_n) + (1-\gamma_n)D_g(p, x_n) \\
&\leq \gamma_n D_g(p, y_n) + (1-\gamma_n)D_g(p, x_n) \\
&\leq D_g(p, x_n) - \gamma_n\beta_n(1-\beta_n)\rho^*_{s_2}(\|\nabla g(x_n) - \nabla g(Tx_n)\|)].
\end{aligned}$$

Thus we have

$$\gamma_n\beta_n(1-\beta_n)\rho^*_{s_2}(\|\nabla g(x_n) - \nabla g(T_n x_n)\|) \leq D_g(p, x_n) - D_g(p, x_{n+1}). \tag{5.4}$$

Since $\{D_g(p, x_n)\}_{n\in\mathbb{N}}$ converges, together with the control condition (5.2), we have

$$\liminf_{n\to\infty} \rho^*_{s_2}(\|\nabla g(x_n) - \nabla g(Tx_n)\|) = 0.$$

Therefore, from the property of $\rho^*_{s_2}$ we deduce that

$$\liminf_{n\to\infty} \|\nabla g(x_n) - \nabla g(Tx_n)\| = 0.$$

Since $\nabla g^*$ is uniformly norm-to-norm continuous on bounded subsets of $E^*$ (see, for example, [31]), we arrive at

$$\liminf_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{5.5}$$

□

**Theorem 5.3.** *Let $C$ be a nonempty, closed and convex subset of a reflexive Banach space $E$. Let $g : E \to \mathbb{R}$ be a strongly coercive Bregman function which is locally bounded, locally uniformly convex and locally uniformly smooth on $E$. Let $T : C \to C$ be a Bregman nonspreading mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ be two sequence in $[0,1]$ satisfying the control conditions $\sum_{n=1}^{\infty} \gamma_n\beta_n(1-\beta_n) = +\infty$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by the algorithm (5.1). Then, there exists a subsequence $\{x_{n_i}\}_{i\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ which converges weakly to a fixed point of $T$ as $i \to \infty$.*

*Proof.* It follows from Theorem 5.2 that $\{x_n\}_{n\in\mathbb{N}}$ is bounded and $\liminf_{n\to\infty} \|Tx_n - x_n\| = 0$. Since $E$ is reflexive, then there exists a subsequence $\{x_{n_i}\}_{i\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $x_{n_i} \rightharpoonup p \in C$ as $i \to \infty$. In view of Proposition 3.3, we conclude that $p \in F(T)$ and the desired conclusion follows. □

The construction of fixed points of nonexpansive mappings via Halpern's algorithm [9] has been extensively investigated recently in the current literature (see, for example, [20] and the references therein). Numerous results have been proved on Halpern's iterations for nonexpansive mappings in Hilbert and Banach spaces (see, e.g., [18, 25, 27]).

Before dealing with the strong convergence of a Halpern-type iterative algorithm, we need the following lemmas.

**Lemma 5.4** ([14])**.** *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ with a subsequence $\{a_{n_i}\}_{i\in\mathbb{N}}$ such that $a_{n_i} < a_{n_i+1}$ for all $i$ in $\mathbb{N}$. Then there exists another subsequence $\{a_{m_k}\}_{k\in\mathbb{N}}$ such that for all (sufficiently large) number $k$ we have*

$$a_{m_k} \leq a_{m_k+1} \quad \text{and} \quad a_k \leq a_{m_k+1}.$$

*In fact, we can set $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.*

**Lemma 5.5** ([30])**.** *Let $\{s_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1-\gamma_n)s_n + \gamma_n\delta_n, \quad \forall n \geq 1,$$

*where $\{\gamma_n\}_{n\in\mathbb{N}}$ and $\{\delta_n\}_{n\in\mathbb{N}}$ satisfy the conditions:*

*(i)* $\{\gamma_n\}_{n\in\mathbb{N}} \subset [0,1]$ *and* $\sum_{n=1}^{\infty} \gamma_n = +\infty$*, or equivalently,* $\Pi_{n=1}^{\infty}(1-\gamma_n) = 0$*;*

*(ii)* $\limsup_{n\to\infty} \delta_n \leq 0$*, or*

*(ii)'* $\sum_{n=1}^{\infty} \gamma_n\delta_n < \infty$*.*

*Then,* $\lim_{n\to\infty} s_n = 0$*.*

**Theorem 5.6.** *Let $C$ be a nonempty, closed and convex subset of a reflexive Banach space $E$. Let $g : E \to \mathbb{R}$ be a strongly coercive Bregman function which is locally bounded, locally uniformly convex and locally uniformly smooth on $E$. Let $T : C \to C$ be a Bregman nonspreading mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ be two sequences in $[0, 1]$ satisfying the following control conditions:*

*(a)* $\lim_{n\to\infty} \alpha_n = 0$;

*(b)* $\sum_{n=1}^{\infty} \alpha_n = +\infty$;

*(c)* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

*Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by*

$$\begin{cases} u \in C, \ x_1 \in C \quad \text{chosen arbitrarily,} \\ y_n = \nabla g^*[\beta_n \nabla g(x_n) + (1 - \beta_n)\nabla g(Tx_n)], \\ x_{n+1} = \operatorname{proj}_C^g(\nabla g^*[\alpha_n \nabla g(u) + (1 - \alpha_n)\nabla g(y_n)]) \text{ for } n \text{ in } \mathbb{N}, \end{cases} \tag{5.6}$$

*Then the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined in (5.6) converges strongly to $\operatorname{proj}_{F(T)}^g u$ as $n \to \infty$.*

*Proof.* We divide the proof into several steps. In view of Lemma 3.1, we conclude that $F(T)$ is closed and convex. Set

$$z = \operatorname{proj}_{F(T)}^g u.$$

**Step 1.** We prove that $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are bounded sequences in $C$.
We first show that $\{x_n\}_{n\in\mathbb{N}}$ is bounded. Let $p \in F(T)$ be fixed. In view of Lemma 2.7 and (5.6), we have

$$\begin{aligned} D_g(p, y_n) &= D_g(p, \nabla g^*[(1 - \beta_n)\nabla g(x_n) + \beta_n \nabla g(Tx_n)) \\ &= V(p, (1 - \beta_n)\nabla g(x_n) + \beta_n \nabla g(Tx_n)]) \\ &\leq (1 - \beta_n)V(p, \nabla g(x_n)) + \beta_n V(p, \nabla g(Tx_n)) \\ &= (1 - \beta_n)D_g(p, x_n) + \beta_n D_g(p, Tx_n) \\ &\leq (1 - \beta_n)D_g(p, x_n) + \beta_n D_g(p, x_n) \\ &= D_g(p, x_n). \end{aligned}$$

This, together with (5.1), implies that

$$\begin{aligned} D_g(p, x_{n+1}) &= D_g(p, \operatorname{proj}_C^g(\nabla g^*[\alpha_n \nabla g(u) + (1 - \alpha_n)\nabla g(y_n)])) \\ &\leq D_g(p, \nabla g^*[\alpha_n \nabla g(u) + (1 - \alpha_n)\nabla g(y_n)]) \\ &= V(p, \alpha_n \nabla g(u) + (1 - \alpha_n)\nabla g(y_n)) \\ &\leq \alpha_n V(p, \nabla g(u)) + (1 - \alpha_n)V(p, \nabla g(y_n)) \\ &= \alpha_n D_g(p, u) + (1 - \alpha_n)D_g(p, y_n) \\ &\leq \alpha_n D_g(p, u) + (1 - \alpha_n)D_g(p, y_n) \\ &\leq \alpha_n D_g(p, u) + (1 - \alpha_n)D_g(p, x_n) \\ &\leq \max\{D_g(p, u), D_g(p, x_n)\}. \end{aligned}$$

By induction, we obtain

$$D_g(p, x_{n+1}) \leq \max\{D_g(p, u), D_g(p, x_1)\} \tag{5.7}$$

for all $n$ in $\mathbb{N}$. It follows from (5.7) that the sequence $\{D_g(p, x_n)\}_{n\in\mathbb{N}}$ is bounded and hence there exists $M_7 > 0$ such that

$$D_g(p, x_n) \leq M_7, \quad \forall n \in \mathbb{N}. \tag{5.8}$$

In view of Definition 2.3, we deduce that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is bounded. Since $T$ is a Bregman quasi-nonexpansive mapping from $C$ into itself, we conclude that

$$D_g(p, Tx_n) \leq D_g(p, x_n), \quad \forall n \in \mathbb{N}. \tag{5.9}$$

This, together with Definition 2.3 and the boundedness of $\{x_n\}_{n\in\mathbb{N}}$, implies that $\{Tx_n\}_{n\in\mathbb{N}}$ is bounded. The function $g$ is bounded on bounded subsets of $E$ and therefore $\nabla g$ is also bounded on bounded subsets of $E^*$ (see, for example, [2, Proposition 1.1.11] for more details). This, together with Step 1, implies that the sequences $\{\nabla g(x_n)\}_{n\in\mathbb{N}}$, $\{\nabla g(y_n)\}_{n\in\mathbb{N}}$ and $\{\nabla g(Tx_n)\}_{n\in\mathbb{N}}$ are bounded in $E^*$. In view of Proposition 2.6, we obtain that $\operatorname{dom} g^* = E^*$ and $g^*$ is strongly coercive and uniformly convex on bounded subsets of $E$. Let $s_3 = \sup\{\|\nabla g(x_n)\|, \|\nabla g(Tx_n)\| : n \in \mathbb{N}\}$ and let $\rho^*_{s_3} : E^* \to \mathbb{R}$ be the gauge of uniform convexity of the conjugate function $g^*$.

**Step 2.** We prove that

$$D_g(z, y_n) \leq D_g(z, x_n) - \beta_n(1-\beta_n)\rho^*_{s_3}(\|\nabla g(x_n) - \nabla g(Tx_n)\|), \quad \forall n \in \mathbb{N}. \tag{5.10}$$

For each $n$ in $\mathbb{N}$, in view of the definition of Bregman distance ((1.2)), Lemma 2.7 and (2.4), we obtain

$$\begin{aligned}
D_g(z, y_n) &= g(z) - g(y_n) - \langle z - y_n, \nabla g(y_n)\rangle \\
&= g(z) + g^*(\nabla g(y_n)) - \langle y_n, \nabla g(y_n)\rangle - \langle z, \nabla g(y_n)\rangle + \langle y_n, \nabla g(y_n)\rangle \\
&= g(z) + g^*((1-\beta_n)\nabla g(x_n) + \beta_n \nabla g(Tx_n)) \\
&\quad - \langle z, (1-\beta_n)\nabla g(x_n) + \beta_n \nabla g(Tx_n))\rangle \\
&\leq (1-\beta_n)g(z) + \beta_n g(z) + (1-\beta_n)g^*(\nabla g(x_n)) + \beta_n g^*(\nabla g(Tx_n)) \\
&\quad - \beta_n(1-\beta_n)\rho^*_{s_3}(\|\nabla g(x_n) - \nabla g(Tx_n)\|) \\
&\quad - (1-\beta_n)\langle z, \nabla g(x_n)\rangle - \beta_n\langle z, \nabla g(Tx_n)\rangle \\
&= (1-\beta_n)[g(z) + g^*(\nabla g(x_n)) - \langle z, \nabla g(x_n)\rangle] \\
&\quad + \beta_n[g(z) + g^*(\nabla g(Tx_n)) - \langle z, \nabla g(Tx_n)\rangle] \\
&\quad - \beta_n(1-\beta_n)\rho^*_{s_3}(\|\nabla g(x_n) - \nabla g(Tx_n)\|) \\
&= (1-\beta_n)[g(z) - g(x_n) + \langle x_n, \nabla g(x_n)\rangle - \langle z, \nabla g(x_n)\rangle] \\
&\quad + \beta_n[g(z) - g(Tx_n) + \langle Tx_n, \nabla g(Tx_n)\rangle - \langle z, \nabla g(Tx_n)\rangle] \\
&\quad - \beta_n(1-\beta_n)\rho^*_{s_3}(\|\nabla g(x_n) - \nabla g(T_{r_n}x_n)\|) \\
&= (1-\beta_n)D(z, x_n) + \beta_n D(z, Tx_n) \\
&\quad - \beta_n(1-\beta_n)\rho^*_{s_3}(\|\nabla g(x_n) - \nabla g(Tx_n)\|) \\
&\leq (1-\beta_n)D_g(z, x_n) + \beta_n D_g(z, x_n) \\
&\quad - \beta_n(1-\beta_n)\rho^*_{s_3}(\|\nabla g(x_n) - \nabla g(Tx_n)\|) \\
&= D(z, x_n) - \beta_n(1-\beta_n)\rho^*_{s_3}(\|\nabla g(x_n) - \nabla g(Tx_n)\|).
\end{aligned}$$

In view of Lemma 2.7 and (5.10), we obtain

$$\begin{aligned}
D_g(z, x_{n+1}) &= D_g(z, \mathrm{proj}_C^g(\nabla g^*[\alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n)])) \\
&\leq D_g(z, \nabla g^*[\alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n)]) \\
&= V(z, \alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n)) \\
&\leq \alpha_n V(z, \nabla g(u)) + (1-\alpha_n) V(z, \nabla g(y_n)) \\
&= \alpha_n D_g(z, u) + (1-\alpha_n) D_g(z, y_n) \\
&\leq \alpha_n D_g(z, u) \\
&\quad + (1-\alpha_n)[D_g(z, x_n) - \beta_n(1-\beta_n)\rho_{s_3}^*(\|\nabla g(x_n) - \nabla g(Tx_n)\|)].
\end{aligned} \tag{5.11}$$

Let

$$M_8 := \sup\{|D_g(z, u) - D_g(z, x_n)| + \beta_n(1-\beta_n)\rho_{s_3}^*(\|\nabla g(x_n) - \nabla g(Tx_n)\|) : n \in \mathbb{N}\}.$$

It follows from (5.11) that

$$\beta_n(1-\beta_n)\rho_{s_3}^*(\|\nabla g(x_n) - \nabla g(Tx_n)\|) \leq D_g(z, x_n) - D_g(z, x_{n+1}) + \alpha_n M_8. \tag{5.12}$$

Let

$$z_n = \nabla g^*[\alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n)].$$

Then $x_{n+1} = \mathrm{proj}_C^g(z_n)$ for all $n$ in $\mathbb{N}$. In view of Lemma 2.7 and (5.10) we obtain

$$\begin{aligned}
D_g(z, x_{n+1}) &= D_g(z, \mathrm{proj}_C^g(\nabla g^*[\alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n)])) \\
&\leq D_g(z, \nabla g^*[\alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n)]) \\
&= V(z, \alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n)) \\
&\leq V(z, \alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n) - \alpha_n(\nabla g(u) - \nabla g(z))) \\
&\quad - \langle \nabla g^*[\alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n)] - z, -\alpha_n(\nabla g(u) - \nabla g(z))\rangle \\
&= V(z, \alpha_n \nabla g(z) + (1-\alpha_n)\nabla g(y_n)) + \alpha_n\langle z_n - z, \nabla g(u) - \nabla g(z)\rangle \\
&\leq \alpha_n V(z, \nabla g(z)) + (1-\alpha_n)V(z, \nabla g(y_n)) + \alpha_n\langle z_n - z, \nabla g(u) - \nabla g(z)\rangle \\
&= \alpha_n D_g(z, z) + (1-\alpha_n)D_g(z, y_n) + \alpha_n\langle z_n - z, \nabla g(u) - \nabla g(z)\rangle \\
&= (1-\alpha_n)D_g(z, x_n) + \alpha_n\langle z_n - z, \nabla g(u) - \nabla g(z)\rangle.
\end{aligned} \tag{5.13}$$

**Step 3.** We show that $x_n \to z$ as $n \to \infty$.

*Case 1.* If there exists $n_0$ in $\mathbb{N}$ such that $\{D_g(z, x_n)\}_{n=n_0}^{\infty}$ is non-increasing, then $\{D_g(z, x_n)\}_{n\in\mathbb{N}}$ is convergent. Thus, we have $D_g(z, x_n) - D_g(z, x_{n+1}) \to 0$ as $n \to \infty$. This, together with (5.12) and conditions (a) and (c), implies that

$$\lim_{n\to\infty} \rho_{s_3}^*(\|\nabla g(x_n) - \nabla g(Tx_n)\|) = 0.$$

Therefore, from the property of $\rho_{s_3}^*$ we deduce that

$$\lim_{n\to\infty} \|\nabla g(x_n) - \nabla g(Tx_n)\| = 0. \tag{5.14}$$

Since $\nabla g^* = (\nabla g)^{-1}$ (Lemma 2.4) is uniformly norm-to-norm continuous on bounded subsets of $E^*$ (see, for example, [31]), we arrive at

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{5.15}$$

On the other hand, we have

$$\begin{aligned} D_g(Tx_n, y_n) &= D_g(Tx_n, \nabla g^*[\beta_n \nabla g(x_n) + (1-\beta_n)\nabla g(Tx_n)]) \\ &= V(Tx_n, \beta_n \nabla g(x_n) + (1-\beta_n)\nabla g(Tx_n)) \\ &\le \beta_n V(Tx_n, \nabla g(x_n)) + (1-\beta_n)V(Tx_n, \nabla g(Tx_n)) \\ &= \beta_n D_g(Tx_n, x_n) + (1-\beta_n)D_g(Tx_n, Tx_n) \\ &= \beta_n D_g(Tx_n, x_n). \end{aligned}$$

This, together with Lemma 2.1 and (5.15), implies that

$$\lim_{n\to\infty} D_g(Tx_n, y_n) = 0.$$

Similarly, we have

$$D_g(y_n, z_n) \le \alpha_n D_g(y_n, u) + (1-\alpha_n)D_g(y_n, y_n) = \alpha_n D_g(y_n, u) \to 0 \quad \text{as } n \to \infty.$$

In view of Lemma 2.1 and (5.15), we conclude that

$$\lim_{n\to\infty} \|y_n - Tx_n\| = 0 \ \text{ and } \ \lim_{n\to\infty} \|z_n - x_n\| = 0.$$

Since $\{x_n\}_{n\in\mathbb{N}}$ is bounded, together with (2.5) we can assume there exists a subsequence $\{x_{n_i}\}_{i\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $x_{n_i} \rightharpoonup y \in F(T)$ (Proposition 3.3) and

$$\begin{aligned} \limsup_{n\to\infty} \langle x_n - z, \nabla g(u) - \nabla g(z)\rangle &= \lim_{i\to\infty} \langle x_{n_i} - z, \nabla g(u) - \nabla g(z)\rangle \\ &= \langle y - z, \nabla g(u) - \nabla g(z)\rangle \le 0. \end{aligned}$$

We thus conclude

$$\limsup_{n\to\infty} \langle z_n - z, \nabla g(u) - \nabla g(z)\rangle = \limsup_{n\to\infty} \langle x_n - z, \nabla g(u) - \nabla g(z)\rangle \le 0.$$

The desired result follows from Lemmas 2.1 and 5.5 and (5.13).

*Case 2.* Suppose there exists a subsequence $\{n_i\}_{i\in\mathbb{N}}$ of $\{n\}_{n\in\mathbb{N}}$ such that

$$D_g(z, x_{n_i}) < D_g(z, x_{n_i+1})$$

for all $i$ in $\mathbb{N}$. By Lemma 5.4, there exists a non-decreasing sequence $\{m_k\}_{k\in\mathbb{N}}$ of positive integers such that $m_k \to \infty$,

$$D_g(z, x_{m_k}) < D_g(z, x_{m_k+1}) \quad \text{and} \quad D_g(z, x_k) \le D_g(z, x_{m_k+1}), \quad \forall k \in \mathbb{N}.$$

This, together with (5.12), implies that

$$\beta_{m_k}(1-\beta_{m_k})\rho_{s_3}^*(\|\nabla g(x_{m_k})-\nabla g(Tx_{m_k})\|) \leq D_g(z,x_{m_k})-D_g(z,x_{m_k+1})+\alpha_{m_k}M_8 \leq \alpha_{m_k}M_8, \quad \forall k \in \mathbb{N}.$$

Then, by conditions (a) and (c), we get

$$\lim_{k\to\infty} \rho_{s_3}^*(\|\nabla g(x_{m_k}) - \nabla g(Tx_{m_k})\|) = 0.$$

By the same argument, as in Case 1, we arrive at

$$\limsup_{k\to\infty}\langle z_{m_k} - z, \nabla g(u) - \nabla g(z)\rangle = \limsup_{k\to\infty}\langle x_{m_k} - z, \nabla g(u) - \nabla g(z)\rangle \leq 0. \tag{5.16}$$

It follows from (5.13) that

$$D_g(z, x_{m_k+1}) \leq (1 - \alpha_{m_k})D_g(z, x_{m_k}) + \alpha_{m_k}\langle z_{m_k} - z, \nabla g(u) - \nabla g(z)\rangle. \tag{5.17}$$

Since $D_g(z, x_{m_k}) \leq D_g(z, x_{m_k+1})$, we have that

$$\begin{aligned} \alpha_{m_k}D_g(z, x_{m_k}) &\leq D_g(z, x_{m_k}) - D_g(z, x_{m_k+1}) + \alpha_{m_k}\langle z_{m_k} - z, \nabla g(u) - \nabla g(z)\rangle \\ &\leq \alpha_{m_k}\langle z_{m_k} - z, \nabla g(u) - \nabla g(z)\rangle. \end{aligned}$$

In particular, since $\alpha_{m_k} > 0$, we obtain

$$D_g(z, x_{m_k}) \leq \langle z_{m_k} - z, \nabla g(u) - \nabla g(z)\rangle.$$

In view of (5.16), we deduce that

$$\lim_{k\to\infty} D_g(z, x_{m_k}) = 0.$$

This, together with (5.17), implies

$$\lim_{k\to\infty} D_g(z, x_{m_k+1}) = 0.$$

On the other hand, we have $D_g(z, x_k) \leq D_g(z, x_{m_k+1})$ for all $k$ in $\mathbb{N}$. This ensures that $x_k \to z$ as $k \to \infty$ by Lemma 2.1.

□

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