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The linking von Neumann algebras of W^*-TROs

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In this note, we show that a von Neumann algebra can be written as the linking von Neumann algebra of a W^*-TRO if and only if it contains no abelian direct summand. We also provide some new characterizations of nuclear TROs and W^*-exact TROs in terms of the properties of their linking algebras.

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1. Introduction

A *ternary ring of operators* (or simply *TRO*) is a norm closed subspace V of the Banach space $B(K, H)$ of bounded linear operators between Hilbert spaces K and H, which is closed under the triple product

$$(x, y, z) \in V \times V \times V \mapsto xy^*z \in V.$$

TROs were first introduced by Hestenes ([13]), and pursued by many others. In [18], it is proved that TROs form a special class of concrete operator spaces and characterized TROs in terms of the operator space theoretic properties. The interconnections between TROs and JC*-triples are studied in [3].

When V is a TRO, $V^\sharp = \{x^* \in B(H, K) : x \in V\}$ is also a TRO. We assume that V is *non-degenerate* in this note, in the sense that VK and $V^\sharp H$ are norm dense in H and K, respectively. A TRO V of $B(K, H)$ is called a W^*-TRO if it is closed in the strongly operator topology (SOT), or equivalently, closed in the weak operator topology, or the weak* topology of $B(K, H)$ ([24]).

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A fundamental tool to study TROs is the construction of the linking algebra, that is, a particular C*-algebra containing the related TRO as a corner. For example, an operator space is injective if and only if it is completely isometric to a ternary corner of an injective C*-algebra (see, e.g., [1]). Let $VV^\sharp$ and $V^\sharp V$ be the linear span of vw^* and v^*w for all $v, w \in V$ respectively. Clearly, $VV^\sharp$ and $V^\sharp V$ are $*$-subalgebras of $B(H)$ and $B(K)$. Let

$$C(V) = \overline{VV^\sharp}^{\|\cdot\|} \quad \text{and} \quad D(V) = \overline{V^\sharp V}^{\|\cdot\|}$$

denote the C^*-algebras generated by $VV^\sharp$ and $V^\sharp V$ respectively. The *linking C^*-algebra* $A(V)$ of V is defined by

$$A(V) = \begin{pmatrix} C(V) & V \\ V^\sharp & D(V) \end{pmatrix}.$$

When V is a W^*-TRO, let

$$M(V) = \overline{VV^\sharp}^{\mathrm{SOT}} \quad \text{and} \quad N(V) = \overline{V^\sharp V}^{\mathrm{SOT}}$$

denote the von Neumann algebras generated by $VV^\sharp$ and $V^\sharp V$ respectively. The *linking von Neumann algebra* $R(V)$ of V is defined by

$$R(V) = \begin{pmatrix} M(V) & V \\ V^\sharp & N(V) \end{pmatrix} = A(V)'',$$

the double commutant of $A(V)$ in $B(H \oplus K)$ (see, e.g., [7, Proposition 2.3]).

TROs and their associated linking algebras share many common properties, wherefore the application of operator algebraic methods simplifies the study of TROs that are not algebras themselves. Basic properties and most recent results of TROs are discussed in, e.g., [6, 9, 13, 12, 15, 16, 10, 11, 7, 17, 20, 22, 24] and references therein.

In this note, we show that a von Neumann algebra $\mathcal{M}$ can be written as the linking von Neumann algebra $R(V)$ of a W^*-TRO V if and only if $\mathcal{M}$ does not contain an abelian direct summand. We also provide new characterizations of a TRO being nuclear or W^*-exact in term of its linking algebra. These results generalize, in particular, [17, Theorem 6.5] and [4, Theorem 4.1].

2. The results

2.1. Conditions to be a linking von Neumann algebra of a W^*-TRO

Theorem 1. *Let $\mathcal{M}$ be a von Neumann algebra. The following conditions are equivalent.*

(a) There is a W^*-TRO V such that its linking von Neumann algebra $R(V) = \mathcal{M}$.
(b) There exists a projection e in $\mathcal{M}$ with central covers $C_e = C_{I-e} = I$.
(c) $\mathcal{M}$ has no abelian direct summand.

Proof. For the implication (a) $\implies$ (b), suppose $\mathcal{M} = R(V)$ is the linking algebra of W^*-TRO V as in (1). Then $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is a projection in $\mathcal{M}$ with $C_e = C_{I-e} = I$.

For the implication (b) $\implies$ (a), suppose there exists a projection e in $\mathcal{M}$ with central covers $C_e = C_{I-e} = I$. Let $V = e\mathcal{M}(I-e)$. Then V is a W^*-TRO with

$$R(V) = \left(\begin{array}{cc} M(V) & V \\ V^\sharp & N(V) \end{array} \right). \tag{1}$$

We claim that $R(V) = \mathcal{M}$.

To see $M(V) = e\mathcal{M}e$, it suffices to show that $e\mathcal{M}_+e \subset M(V)$. Since

$$I = C_{I-e} = \bigvee \{u(I-e)u^* : u \text{ is a unitary in } \mathcal{M}\},$$

there is a net $\{F_i\}_i$ of increasing finite subsets of the unitary group of $\mathcal{M}$ such that

$$\bigvee_{u \in F_i} u(I-e)u^* \to I$$

in the strong operator topology. Given $x \in \mathcal{M}_+$. Since $ex^{1/2}u(I-e)u^*x^{1/2}e$ belongs to the von Neumann algebra $M(V)$, we have

$$exe = \text{SOT-}\lim_i ex^{1/2} \left(\bigvee_{u \in F_i} u(I-e)u^* \right) x^{1/2}e \in M(V).$$

Similarly, we see that $N(V) = (I-e)\mathcal{M}(1-e)$. It follows $\mathcal{M} = R(V)$.

For the implication (c) $\implies$ (b), suppose $\mathcal{M}$ has no abelian direct summand. We show that there is a projection e in $\mathcal{M}$ such that the central covers $C_e = C_{I-e} = I$. Indeed, the assertion is contained in [14, Exercise 6.19]. For completeness, we present a short proof below.

Write $\mathcal{M} = \mathcal{M}_d \oplus \mathcal{M}_c$ as the direct sum of its discrete part $\mathcal{M}_d$ and continuous part $\mathcal{M}_c$, with identity elements $I_{\mathcal{M}_d}$ and $I_{\mathcal{M}_c}$, respectively. For a discrete summand $M_n(\mathbb{C}) \otimes L_\infty(\mu_n)$ (here $n \geq 2$), let $e_n = E_{11}^n \otimes I_{L_\infty(\mu_n)}$, where E_{11}^n is the matrix unit in $M_n(\mathbb{C})$ with the $(1,1)$ entry being 1 and all others 0. Let $e_d = \sum_{n\geq 2} e_n$. Then $C_{e_d} = C_{I_{\mathcal{M}_d}-e_d} = I_{\mathcal{M}_d}$. For the continuous part, we have a projection $e_c \in \mathcal{M}_c$ such that e_c and $I_{\mathcal{M}_c} - e_c$ are both unitarily equivalent to $I_{\mathcal{M}_c}$. In particular, $C_{e_c} = C_{I_{\mathcal{M}_c}-e_c} = I_{\mathcal{M}_c}$. Consequently, $e = e_d + e_c$ finishes our task.

Finally, we verify the implication (b) $\implies$ (c). Let e be a projection in $\mathcal{M}$ such that $C_e = C_{I-e} = I$. Let z be any abelian central projection in $\mathcal{M}$. Since

$$ze(u^*(1-e)u)ez = zez(zuz)^*(1-e)uez = (zuz)^*(zez)(1-e)u = 0$$

for any unitary u in $\mathcal{M}$, and $C_{I-e} = I$, we see that $ze = 0$. Similarly, we see that $z(1-e) = 0$, and thus $z = 0$. It follows that $\mathcal{M}$ has no direct abelian summand. □

2.2. Characterization of nuclearity

Nuclear C^*-algebras play an important role in the study of operator algebras. Analogously, nuclear TROs are also characterized in [17]. Our aim is to give some new characterizations of nuclear TROs.

Recall that a C^*-algebra A (resp. TRO V) is said to be *nuclear* (resp. *Lance nuclear*) if for every C^*-algebra B, there is a unique C^*-algebra tensor norm on $A \otimes B$ (resp. a unique TRO tensor norm on $V \otimes B$).

Proposition 2. *If V is a TRO in $B(K,H)$ such that $C(V)$ and $D(V)$ are nuclear, then $A(V)$ is nuclear and V is Lance nuclear.*

Proof. There is a projection e in $B(H \oplus K)$ such that $C(V) = eA(V)e$ and $D(V) = (I-e)A(V)(I-e)$. Hence $C(V)^{**} = eA(V)^{**}e$ and $D(V)^{**} = (I-e)A(V)^{**}(I-e)$. Since $C(V)$ and $D(V)$ are nuclear, $C(V)^{**}$ and $D(V)^{**}$ are hyperfinite. We have $A(V)^{**}$ is hyperfinite by [23, Lemma 2.8]. Hence $A(V)$ is nuclear, and therefore V is Lance nuclear by [17, Theorem 6.1]. □

A TRO (respectively W^*-TRO) $V \subset B(K,H)$ carries a natural operator space structure ([8]; see also [21]) with matrix norms arising from identifying

$M_n(V)$ with a TRO (respectively W^*-TRO) in $M_n(B(K,H)) = B(K^n, H^n)$ for $n = 1, 2, \ldots$. In general, an operator space X is said to be *injective* if for any operator spaces $W_1 \subseteq W_2$, every complete contraction $\phi : W_1 \to X$ has a completely contractive extension $\hat{\phi} : W_2 \to X$. On the other hand, X is said to be 1-*nuclear* if the identity operator I_X on X can be factorized approximately through matrix spaces $M_{n_\alpha}(\mathbb{C})$ in the sense that $I_X = \lim_\alpha \phi_\alpha \circ \psi_\alpha$ in the point-norm topology with completely bounded maps $\psi_\alpha : X \to M_{n_\alpha}(\mathbb{C})$ and $\phi_\alpha : M_{n_\alpha}(\mathbb{C}) \to X$ such that $\|\phi_\alpha\|_{\mathrm{cb}} \|\psi_\alpha\|_{\mathrm{cb}} \leq 1$.

Arguing as in Proposition 2, we have a similar result held for W^*-TROs.

Proposition 3. *If V is a W^*-TRO such that $M(V)$ and $N(V)$ are injective von Neumann algebras, then $R(V)$ is an injective von Neumann algebra and therefore V is an injective TRO.*

It follows from Proposition 2 the following characterization of nuclear TROs, which adds condition (6) to the list in [17, Theorem 6.5]

Theorem 4. *Let V be a TRO. The following are equivalent:*

(1) V is Lance nuclear;
(2) V is 1-nuclear;
(3) V^{**} is injective;
(4) $A(V)^{**}$ is injective;
(5) $A(V)$ is nuclear;
(6) $C(V)$ and $D(V)$ are nuclear.

2.3. Characterization of exactness

Recall that a von Neumann algebra $\mathcal{M}$ is said to be *weakly exact* ([19]) if for any unital C^*-algebra A with a closed two-sided ideal J and any left normal $*$-representation $\pi : \mathcal{M} \otimes_{\min} A \to B(H)$ with $\pi(\mathcal{M} \otimes J) = 0$, the induced $*$-representation $\hat{\pi} : \mathcal{M} \otimes (A/J) \to B(H)$ is continuous with respect to the minimum tensor norm.

Lemma 5 ([2, Corollary 14.1.15])**.** *If $\mathcal{M}$ is a weakly exact von Neumann algebra, then $\mathcal{M} \overline{\otimes} B(H)$ is weakly exact.*

Dong and Ruan studied the connection between weak* exact W^*-TROs and their linking von Neumann algebras in [4]. In view of [4, Theorem 3.3], a dual operator space X is *weak* exact* if for any operator space W and any finite rank complete contraction $\phi : W \to X$, there exists a net of weak* continuous finite rank complete contractions $\phi_\alpha : W \to V$ converging to ϕ in the point-weak* topology.

Proposition 6 ([4, Lemma 3.4]). *Let $\mathcal{M}$ be a von Neumann algebra. Then $\mathcal{M}$ is weak* exact if and only if $\mathcal{M}$ is weakly exact.*

Lemma 7 ([4, Lemma 3.1]). *Assume that X_0 is a weak*-closed subspace of a dual operator space X such that there exists a weak* continuous completely contractive projection $P : X \to X_0$. If X is weak* exact, so is X_0.*

The following result connects the weak exactness of a von Neumann algebras with its "diagonals".

Lemma 8. *Let $\mathcal{M}$ be a von Neumann algebra. Suppose $\{Q_j : j \in \mathbb{J}\}$ is a family of mutually equivalent and mutually orthogonal projections in $\mathcal{M}$ such that $Q_j\mathcal{M}Q_j$ is a weakly exact von Neumann algebra for each j and $\sum_{j\in\mathbb{J}} Q_j = I$. Then $\mathcal{M}$ is a weakly exact von Neumann algebra.*

Proof. Fix an index j_0 in $\mathbb{J}$. For any finite subset F of $\mathbb{J}$ of n elements, we see that

$$(\textstyle\sum_{j\in F} Q_j)\mathcal{M}(\sum_{j\in F} Q_j) \text{ is isomorphic to } (Q_{j_0}\mathcal{M}Q_{j_0}) \otimes M_n(\mathbb{C}).$$

But the latter von Neumann algebra is weakly exact by Lemma 5. Let $P_F(x) = (\sum_{j\in F} Q_j)x(\sum_{j\in F} Q_j)$. Then as F runs over all finite subsets of $\mathbb{J}$, we have a net of normal completely positive contractions $P_F : \mathcal{M} \to P_F\mathcal{M}P_F$ which converges to the identity on $\mathcal{M}$ in the point ultraweak topology. It follows from [2, Proposition 14.1.4] that $\mathcal{M}$ is a weakly exact von Neumann algebra. □

Lemma 9 ([5, Theorem 3.1]; see also [23, Lemma 2.7]). *Let P be a projection in a von Neumann algebra $\mathcal{M}$. There is a family $\{Q_\alpha\}_\alpha$ of subprojections of P in $\mathcal{M}$ such that $C_P = \sum_\alpha C_{Q_\alpha}$ is a sum of mutually orthogonal central projections. Moreover, each $C_{Q_\alpha} = Q_\alpha + \sum_i Q_\alpha^i$ is a sum of mutually orthogonal and mutually equivalent projections.*

Proof. We sketch the proof in [5] for completeness. We first claim that for any nonzero projection P in $\mathcal{M}$ there is a subprojection $Q \leq P$ and a family of mutually orthogonal projections $Q_\alpha \sim Q$ such that the central cover $C_Q = Q + \sum_\alpha Q_\alpha$. The assertion will follow from the claim and a Zorn's Lemma argument.

To prove the claim, we might assume $C_P = I$. Suppose $\mathcal{M}$ has type III. There are subprojections P_1, P_2 of P such that $P = P_1 + P_2$ and $P_1 \sim P_2$. Enlarge $\{P_1, P_2\}$ to a maximal family $\mathcal{P}$ of mutually orthogonal projections P_α such that $P_\alpha \sim P_1$ for all indices α. If $\sum_\alpha P_\alpha = I$, then we can let $Q = P_1$ and $\{Q_\alpha\}_\alpha = \mathcal{P} \setminus \{P_1\}$. Otherwise, $P_1 \not\precsim (1 - \sum_{\alpha\neq 1} P_\alpha)$ by the maximality of $\mathcal{P}$. Hence there is a nonzero central projection E of $\mathcal{M}$ such that $E(1 - \sum_{\lambda\neq 1} P_\lambda) \prec EP_1$. Then $Q = EP_1$ and $Q_\alpha = EP_\alpha$ for $\alpha \neq 1$ will do the job.

If $\mathcal{M}$ is semifinite, there will be a nonzero central projection $E = \sum_{\alpha\in\Lambda} E_\alpha$ written as an orthogonal sum of mutually equivalent finite projections E_α. Since $C_P = I$, we can replace P by $EP \neq 0$ and assume $I = E = \sum_{\alpha\in\Lambda} E_\alpha$. Letting $E_{\alpha\beta} = E^*_{\beta\alpha}$ be the partial isometry in $\mathcal{M}$ such that $E_\alpha = E_{\alpha\beta}E^*_{\alpha\beta}$ and $E_\beta = E^*_{\alpha\beta}E_{\alpha\beta}$, we get a family $\{E_{\alpha\beta} : \alpha, \beta \in \Lambda\}$ of matrix units in $\mathcal{M}$ with $E_{\alpha\alpha} = E_\alpha$ for each α in Λ. Then $U_{\alpha\beta} = E_{\alpha\beta} + E_{\beta\alpha} + \sum_{\gamma\neq\alpha,\beta} E_{\gamma\gamma} = U_{\beta\alpha}$ is a self-adjoint unitary in $\mathcal{M}$ such that $U_{\alpha\beta}E_\alpha = E_\beta$, $U_{\alpha\beta}E_\beta = E_\alpha$ and $U_{\alpha\beta}E_\gamma = E_\gamma$ for $\gamma \neq \alpha, \beta$.

If there is a nonzero central projection F in $\mathcal{M}$ such that $FE_\alpha \precsim FP$ for some (and thus all) α, then let $Q \leq FP$ such that $Q \sim FE_\alpha$. Since FE_α is a finite projection, there is a unitary U in $\mathcal{M}$ such that $Q = U^*FE_\alpha U$. Letting $Q_\beta = U^*FE_\beta U$ for all $\beta \neq \alpha$ in Λ, we have $C_Q = F = Q + \sum_{\beta\neq\alpha} Q_\beta$ as claimed.

If there is no such nonzero central projection F, then $P \prec E_\alpha$ for all α in Λ. In particular, $P \leq U^*E_\alpha U$ for some index α and a unitary U in $\mathcal{M}$. Replacing all E_β with $U^*E_\beta U$, we can assume $U = I$, and thus P is a projection in the finite von Neumann algebra $E_\alpha\mathcal{M}E_\alpha$. By [14, Proposition 8.2.1], P contains a nonzero monic subprojection Q; namely, there are mutually orthogonal subprojections $Q_1 = Q, Q_2 \dots, Q_k$ of E_α, each of them is equivalent to Q such that $E_\alpha C_Q E_\alpha = Q_1 + \cdots + Q_k$. Consequently,

$$\begin{aligned} C_Q = \sum_\beta E_\beta C_Q E_\beta &= E_\alpha C_Q E_\alpha + \sum_{\beta\neq\alpha} E_\beta C_Q E_\beta \\ &= (Q_1 + \cdots + Q_k) + \sum_{\beta\neq\alpha} U_{\alpha\beta}(Q_1 + \cdots + Q_k)U_{\alpha\beta}. \end{aligned}$$

Because each $U_{\alpha\beta}Q_jU_{\alpha\beta}$ is equivalent to Q, we establish the claim.

Since every von Neumann algebra is a direct sum of its semifinite summand and its type III summand, the claim is verified. □

Proposition 10. *Let $\mathcal{M}$ be a von Neumann algebra and $e \in \mathcal{M}$ a nonzero projection with $C_e = I$. Then*

$$\mathcal{M} \text{ is weak}^*\text{-exact} \iff e\mathcal{M}e \text{ is weak}^*\text{-exact.}$$

Proof. ($\Rightarrow$) It follows from Lemma 7 that if $\mathcal{M}$ is weak*-exact, then $e\mathcal{M}e$ and $(I-e)\mathcal{M}(I-e)$ are weak*-exact. This is because the maps $P(x) = exe$ and $Q(x) = (I-e)x(I-e)$ are weak*-continuous completely contractive projections from $\mathcal{M}$ onto $e\mathcal{M}e$ and $(I-e)\mathcal{M}(I-e)$, respectively.

($\Leftarrow$) Assume that $e\mathcal{M}e$ is weak*-exact. By Lemma 9, there is a family $\{Q_\alpha : \alpha \in \Gamma\}$ of mutually disjoint subprojections of e such that $\sum_\alpha C_{Q_\alpha} = 1$, and $C_{Q_\alpha} = Q_\alpha + \sum_i Q_\alpha^i$ where all Q_α^i are equivalent to Q_α for each α. By Lemmas 6 and 7, we have $Q_\alpha \mathcal{M} Q_\alpha$ is weakly exact for each $\alpha \in \Gamma$. It follows from Lemma 8 that $C_{Q_\alpha} \mathcal{M} C_{Q_\alpha}$ is a weakly exact von Neumann algebra. Since the direct sum of weakly exact von Neumann algebras is again weakly exact, $\mathcal{M} = \sum_{\alpha \in \Gamma} C_{Q_\alpha} \mathcal{M}$ is a weakly exact von Neumann algebra. This completes the proof. □

It follows from Proposition 10 that for a W^*-TRO V, the linking W^*-algebra $R(V)$ is weak*-exact if and only if $M(V)$ or $N(V)$ is a weakly exact von Neumann algebra. The following result is immediate and it adds condition (2') to the list in [4, Theorem 4.1].

Theorem 11. *Let V be a W^*-TRO. Then the following are equivalent.*

(1) V is weak-exact.*
(2) $M(V)$ and $N(V)$ are weak exact.*
(2') $M(V)$ or $N(V)$ is weak exact.*
(3) $R(V)$ is weak exact.*

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