Weak Sequential Completeness of Spaces of Homogeneous Polynomials

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Abstract. Let $\mathcal{P}_w(^nE;F)$ be the space of all continuous n-homogeneous polynomials from a Banach space E into another F, that are weakly continuous on bounded sets. We give sufficient conditions for the weak sequential completeness of $\mathcal{P}_w(^nE;F)$. These sufficient conditions are also necessary if both E^* and F have the bounded compact approximation property. We also show that the weak sequential completeness and the reflexivity of $\mathcal{P}_w(^nE;F)$ are equivalent whenever both E and F are reflexive.

1. Introduction

For Banach spaces E and F, let $\mathcal{P}(^nE;F)$ be the space of all continuous n-homogeneous polynomials from E into F. After the pioneer work of Ryan [26], several authors (e.g. see [1, 2, 19, 24, 25]) have searched for necessary and sufficient conditions for the reflexivity of $\mathcal{P}(^nE;F)$. Among them, Alencar [1] gave necessary and sufficient conditions for the reflexivity of $\mathcal{P}(^nE;\mathbb{C})$ under the hypothesis of the approximation property of E, and Mujica [24] gave necessary and sufficient conditions for the reflexivity of $\mathcal{P}(^nE;F)$ under the hypothesis of the compact approximation property of E.

A property closely related to the reflexivity is the weak sequential completeness. In section 3 of this paper, we give sufficient conditions for the weak sequential completeness of $\mathcal{P}_w(^nE;F)$, the subspace of all P in $\mathcal{P}(^nE;F)$ that are weakly continuous on bounded sets. We show that these sufficient conditions are also necessary when both E^* and F have the bounded compact approximation property.

In section 4, we show that the weak sequential completeness and the reflexivity of $\mathcal{P}_w(^nE;F)$ coincide whenever both E and F are reflexive. As a consequence, a result of Mujica [24] about the reflexivity of $\mathcal{P}(^nE;F)$ is obtained.

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Mujica [23] showed that the (bounded) approximation property is inherited by the symmetric projective tensor products. In section 5, we show that the (bounded) compact approximation property is also inherited by the (symmetric) projective tensor products. However, we note that Aron and Schottenloher's counter-example [7] shows that the (bounded) compact approximation property is not inherited by the spaces of homogeneous polynomials in general.

2. Preliminaries

Throughout the paper, E and F are Banach spaces over the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. Denote by $\mathcal{L}(E;F)$, $\mathcal{K}(E;F)$, and $\mathcal{W}(E;F)$, respectively, the spaces of all bounded, all compact, and all weakly compact linear operators from E into F. For a bounded linear operator $T : E \to F$, let $T[E]$ denote the image of T and let $T^* : F^* \to E^*$ denote the adjoint operator (i.e., the dual map) of T.

Let n be a positive integer. A map $P : E \to F$ is said to be a continuous n-homogeneous polynomial if there is a continuous symmetric n-linear map T from $E \times \cdots \times E$ (a product of n copies of E) into F such that $P(x) = T(x, \ldots, x)$. Indeed, the symmetric n-linear operator $T_P : E \times \cdots \times E \to F$ associated to P can be given by the *Polarization Formula*:

$$T_P(x_1, \ldots, x_n) = \frac{1}{2^n n!} \sum_{\epsilon_i = \pm 1} \epsilon_1 \cdots \epsilon_n P\Big(\sum_{i=1}^{n} \epsilon_i x_i\Big), \quad \forall\, x_1, \ldots, x_n \in E.$$

Let $\mathcal{P}(^nE;F)$, $\mathcal{P}_w(^nE;F)$, and $\mathcal{P}_{wsc}(^nE;F)$, respectively, denote the space of all continuous n-homogeneous polynomials from E into F, the subspace of all P in $\mathcal{P}(^nE;F)$ that are weakly continuous on bounded sets, and the subspace of all P in $\mathcal{P}(^nE;F)$ that are weakly sequentially continuous. In particular, if $F = \mathbb{R}$ or $\mathbb{C}$, then $\mathcal{P}(^nE;F)$, $\mathcal{P}_w(^nE;F)$, and $\mathcal{P}_{wsc}(^nE;F)$ are simply denoted by $\mathcal{P}(^nE)$, $\mathcal{P}_w(^nE)$, and $\mathcal{P}_{wsc}(^nE)$, respectively. It is known that

$$\mathcal{P}_w(^nE;F) \subseteq \mathcal{P}_{wsc}(^nE;F) \subseteq \mathcal{P}(^nE;F) \tag{2.1}$$

and that $\mathcal{P}_w(^nE;F) = \mathcal{P}_{wsc}(^nE;F)$ for any $n \in \mathbb{N}$ if and only if E contains no copy of ℓ_1 (see [5, Prop. 2.12], also see [14, p.116, Prop. 2.36]).

Let $\otimes_n E$ denote the *n-fold algebraic tensor product* of E. For $x_1 \otimes \cdots \otimes x_n \in \otimes_n E$, let $x_1 \otimes_s \cdots \otimes_s x_n$ denote its symmetrization, that is,

$$x_1 \otimes_s \cdots \otimes_s x_n = \frac{1}{n!} \sum_{\sigma \in \pi(n)} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)},$$

where $\pi(n)$ is the group of permutations of $\{1, \cdots, n\}$. Let $\otimes_{n,s}E$ denote the *n-fold symmetric algebraic tensor product* of E, that is, the linear span of $\{x_1 \otimes_s \cdots \otimes_s x_n : x_1, \ldots, x_n \in E\}$ in $\otimes_n E$. Let $\hat{\otimes}_{n,s,\pi}E$ denote the *n-fold symmetric projective tensor product* of E, that is, the completion of $\otimes_{n,s}E$ under the *symmetric projective tensor norm* on $\otimes_{n,s}E$ defined by

$$\|u\| = \inf\left\{\sum_{k=1}^{m} |\lambda_k| \cdot \|x_k\|^n : x_k \in E, u = \sum_{k=1}^{m} \lambda_k x_k \otimes \cdots \otimes x_k\right\}, \quad u \in \otimes_{n,s}E.$$

Define $\theta_n : E \to \hat{\otimes}_{n,s,\pi}E$ by $\theta_n(x) = x \otimes \cdots \otimes x$ for every $x \in E$. Then $\theta_n \in \mathcal{P}(^nE; \hat{\otimes}_{n,s,\pi}E)$. For every $P \in \mathcal{P}(^nE;F)$, let $A_P \in \mathcal{L}(\hat{\otimes}_{n,s,\pi}E;F)$ denote its linearization, that is, $P = A_P \circ \theta_n$. Then under the isometry: $P \to A_P$, the Banach space $\mathcal{P}(^nE;F)$ is isometrically isomorphic to $\mathcal{L}(\hat{\otimes}_{n,s,\pi}E;F)$. This implies that $\mathcal{P}(^nE) = \mathcal{P}_{wsc}(^nE)$ if and only if $\theta_n : E \to \hat{\otimes}_{n,s,\pi}E$ is sequentially continuous with respect to the weak topology of E and the weak topology of $\hat{\otimes}_{n,s,\pi}E$.

A polynomial $P \in \mathcal{P}(^nE;F)$ is called *compact* (resp. *weakly compact*) if P takes bounded subsets in E into relatively (resp. weakly) compact subsets in F. Equivalently, P is compact (resp. weakly compact) if and only if its linearization A_P is compact (resp. weakly compact) (see [26] or [23, Prop. 3.4]). Let $\mathcal{P}_K(^nE;F)$ (resp. $\mathcal{P}_{wK}(^nE;F)$) denote the space of all compact (resp. weakly compact) n-homogeneous polynomials from E into F. Then through the isometry: $P \to A_P$ we have

$$\mathcal{P}_K(^nE;F) = \mathcal{K}(\hat{\otimes}_{n,s,\pi}E;F), \qquad \mathcal{P}_{wK}(^nE;F) = \mathcal{W}(\hat{\otimes}_{n,s,\pi}E;F). \tag{2.2}$$

It follows from [6, Lemma 2.2 and Prop. 2.5] (also see [14, p.88, Prop. 2.6]) that

$$\mathcal{P}_w(^nE;F) \subseteq \mathcal{P}_K(^nE;F) \subseteq \mathcal{P}_{wK}(^nE;F). \tag{2.3}$$

Moreover, we have the following.

Lemma 2.1. *Assume $\mathcal{P}(^nE) = \mathcal{P}_{wsc}(^nE)$. Then for any Banach space F, we have*

$$\mathcal{P}_w(^nE;F) \subseteq \mathcal{P}_K(^nE;F) \subseteq \mathcal{P}_{wsc}(^nE;F). \tag{2.4}$$

Proof. Take any $P \in \mathcal{P}_K(^nE;F)$. Then $A_P \in \mathcal{K}(\hat{\otimes}_{n,s,\pi}E;F)$ and hence, A_P is a completely continuous linear operator. Since $\mathcal{P}(^nE) = \mathcal{P}_{wsc}(^nE)$, it follows that $\theta_n : E \to \hat{\otimes}_{n,s,\pi}E$ is sequentially continuous with respect to the weak topology of E and the weak topology of $\hat{\otimes}_{n,s,\pi}E$. Note that $P = A_P \circ \theta_n$. Thus P takes weakly convergent sequences in E into norm convergent sequences in F, and so $P \in \mathcal{P}_{wsc}(^nE;F)$. ☐

For the basic knowledge about homogeneous polynomials and symmetric projective tensor products, readers are referred to [14, 15, 22, 26].

3. Weak Sequential Completeness

For every $P \in \mathcal{P}({}^nE)$, let $\widetilde{P} \in \mathcal{P}({}^nE^{**})$ denote the *Aron-Berner extension* of P (see [4, 13]). The following lemma is a special case of [17, Corollary 5].

Lemma 3.1[17]. *Let $P_k, P \in \mathcal{P}_w({}^nE)$ for each $k \in \mathbb{N}$. Then $\lim_k P_k = P$ weakly in $\mathcal{P}_w({}^nE)$ if and only if $\lim_k \widetilde{P}_k(z) = \widetilde{P}(z)$ for every $z \in E^{**}$.*

Next we will give sufficient conditions to ensure the weak sequential completeness of $\mathcal{P}({}^nE)$.

Theorem 3.2. *If E^* is weakly sequentially complete and $\mathcal{P}({}^nE) = \mathcal{P}_w({}^nE)$, then $\mathcal{P}_w({}^nE)$ is weakly sequentially complete.*

Proof. Take a weakly Cauchy sequence $\{P_k\}_1^\infty$ in $\mathcal{P}_w({}^nE)$. Then $\{P_k(x)\}_1^\infty$ is a scalar-valued Cauchy sequence for every $x \in E$. Define a scalar-valued polynomial P on E by $P(x) = \lim_k P_k(x)$ for every $x \in E$. Then $P \in \mathcal{P}({}^nE) = \mathcal{P}_w({}^nE)$. It follows from the Polarization Formula that for every $x_1, \ldots, x_n \in E$, we have

$$\lim_k T_{P_k}(x_1, \ldots, x_n) = T_P(x_1, \ldots, x_n). \tag{3.1}$$

Next we show that $\lim_k P_k = P$ weakly in $\mathcal{P}_w({}^nE)$. For every $z, z_1, \ldots, z_n \in E^{**}$, by Lemma 3.1, $\{\widetilde{P}_k(z)\}_1^\infty$ is a scalar-valued Cauchy sequence, and then by the Polarization Formula, $\{T_{\widetilde{P}_k}(z_1, \ldots, z_n)\}_{k=1}^\infty$ is also a scalar-valued Cauchy sequence. For every fixed $x_2, \ldots, x_n \in E$, define $\phi_k(x) = T_{\widetilde{P}_k}(x, x_2, \ldots, x_n)$ for every $x \in E$. Then $\phi_k \in E^*$ and $\langle \phi_k, z_1 \rangle = T_{\widetilde{P}_k}(z_1, x_2, \ldots, x_n)$ for every $z_1 \in E^{**}$. Thus $\{\phi_k\}_{k=1}^\infty$ is a weakly Cauchy sequence in E^* and hence,

$$\text{weak-}\lim_k T_{\widetilde{P}_k}(\cdot, x_2, \ldots, x_n) = \text{weak-}\lim_k \phi_k \text{ exists in } E^*. \tag{3.2}$$

Note that $T_{\widetilde{P}}(\cdot, x_2, \ldots, x_n) \in E^*$ and (3.1) implies that

$$\text{weak}^*\text{-}\lim_k T_{\widetilde{P}_k}(\cdot, x_2, \ldots, x_n) = T_{\widetilde{P}}(\cdot, x_2, \ldots, x_n). \tag{3.3}$$

Combining (3.2) and (3.3) we have that for every $z_1 \in E^{**}$ and every $x_2, \ldots, x_n \in E$,

$$\lim_k T_{\widetilde{P}_k}(z_1, x_2, \ldots, x_n) = T_{\widetilde{P}}(z_1, x_2, \ldots, x_n).$$

Inductively, we can verify that for every $z_1, z_2, \ldots, z_n \in E^{**}$,

$$\lim_k T_{\widetilde{P}_k}(z_1, z_2, \ldots, z_n) = T_{\widetilde{P}}(z_1, z_2, \ldots, z_n).$$

In particular, $\lim_k \widetilde{P}_k(z) = \widetilde{P}(z)$ for every $z \in E^{**}$. It follows from Lemma 3.1 that $\lim_k P_k = P$ weakly in $\mathcal{P}_w(^nE)$. $\square$

The following lemma is straightforward from Eberlein-Šmulian's Theorem and Rosenthal's ℓ_1-Theorem.

Lemma 3.3. *If X and Y are Banach spaces such that X contains no copy of ℓ_1 and Y is weakly sequentially complete, then every continuous linear operator from X to Y is weakly compact.*

Lemma 3.4. *If $n \geqslant 2$ and $\mathcal{P}_w(^nE)$ is weakly sequentially complete, then E contains no copy of ℓ_1.*

Proof. Assume that E contains a copy of ℓ_1. In the proof of [14, p.116, Prop. 2.36], there exist continuous linear operators $U : E \to L^\infty[0,1]$ and $j : L^\infty[0,1] \to \ell_2$ such that the following diagram commutes:

$$\begin{array}{ccc} \ell_1 & \xrightarrow{\ i\ } & \ell_2 \\ {\scriptstyle k}\downarrow & & \uparrow{\scriptstyle j} \\ E & \xrightarrow[U]{} & L^\infty[0,1] \end{array}$$

where i is the inclusion of ℓ_1 into ℓ_2 and k is the inclusion of ℓ_1 into E.

Define P and P_k $(k \geqslant 1)$ on ℓ_2 by

$$P((x_i)_i) = \sum_{i=1}^{\infty} x_i^n \quad \text{and} \quad P_k((x_i)_i) = \sum_{i=1}^{k} x_i^n, \quad \forall\, (x_i)_i \in \ell_2.$$

Then $P \in \mathcal{P}(^n\ell_2)$ and $P_k \in \mathcal{P}_w(^n\ell_2)$. Let $Q := P \circ j \circ U$ and $Q_k := P_k \circ j \circ U$. We have $Q \in \mathcal{P}(^nE)$ and $Q_k \in \mathcal{P}_w(^nE)$. Note that $\widetilde{Q} = P \circ j^{**} \circ U^{**} \in \mathcal{P}(^nE^{**})$ and $\widetilde{Q}_k = P_k \circ j^{**} \circ U^{**} \in \mathcal{P}(^nE^{**})$. Also note that $\lim_k P_k((x_i)_i) = P((x_i)_i)$ for every $(x_i)_i \in \ell_2$. Thus $\lim_k \widetilde{Q}_k(z) = \widetilde{Q}(z)$ for every $z \in E^{**}$. It follows that $\{Q_k\}_1^\infty$ is a weakly Cauchy sequence in $\mathcal{P}_w(^nE)$ and hence by Lemma 3.1, $Q = \text{weak-}\lim_k Q_k \in \mathcal{P}_w(^nE)$. However, Dineen showed in the proof of [14, p.116, Prop. 2.36] that $Q \notin \mathcal{P}_w(^nE)$. This contradiction shows that E can not contain a copy of ℓ_1. $\square$

To ensure that the sufficient conditions for the weak sequential completeness of $\mathcal{P}_w(^nE)$ in Theorem 3.2 are also necessary, we need the bounded compact approximation property. Recall that a Banach space X is said to have the *compact approximation property* (CAP in short) (see [12, p. 308]) if for every compact subset C of X and for every $\varepsilon > 0$ there is $T \in \mathcal{K}(X, X)$ such that $\|T(x) - x\| \leqslant \varepsilon$ for all $x \in C$. A Banach space X is said to have the *bounded compact approximation property* (BCAP in short) (see [12, p. 308]) if there exists $\lambda \geqslant 1$ so that for every compact subset C of X and for every $\varepsilon > 0$ there is $T \in \mathcal{K}(X, X)$ such that $\|T\| \leqslant \lambda$ and $\|T(x) - x\| \leqslant \varepsilon$ for all $x \in C$. Clearly, the (bounded) approximation property implies the (B)CAP, but the converse is not true (see [27] or see [12, p. 309]).

Theorem 3.5. *If E^* has the BCAP, then $\mathcal{P}_w(^nE)$ is weakly sequentially complete if and only if E^* is weakly sequentially complete and $\mathcal{P}_w(^nE) = \mathcal{P}(^nE)$.*

Proof. Note that E^* is isomorphic to a closed subspace of $\mathcal{P}_w(^nE)$. By Theorem 3.2, we only need to show the assertion

(*): the weak sequential completeness of $\mathcal{P}_w(^nE)$ implies that $\mathcal{P}_w(^nE) = \mathcal{P}(^nE)$.

It is trivial that the assertion (*) holds for $n = 1$. Using the induction, we assume that the assertion (*) holds for $n - 1$ and we will show that the assertion (*) holds for n, where $n \geqslant 2$. To do this, we suppose that $\mathcal{P}_w(^nE)$ is weakly sequentially complete. By [7, Prop. 5.3] or [8, Prop. 5], $\mathcal{P}(^{n-1}E)$ is isomorphic to a (complemented) subspace of $\mathcal{P}(^nE)$ and hence, $\mathcal{P}_w(^{n-1}E)$ is isomorphic to a (closed) subspace of $\mathcal{P}_w(^nE)$, which implies that $\mathcal{P}_w(^{n-1}E)$ is also weakly sequentially complete. It follows from the induction hypothesis that $\mathcal{P}_w(^{n-1}E) = \mathcal{P}(^{n-1}E)$. Moreover, by Lemma 3.4, E contains no copy of ℓ_1 and hence, $\mathcal{P}_w(^iE) = \mathcal{P}_{wsc}(^iE)$ for all $i \in \mathbb{N}$. Next we show that $\mathcal{P}_w(^nE) = \mathcal{P}(^nE)$.

Take any $P \in \mathcal{P}(^nE)$. To show that $P \in \mathcal{P}_w(^nE) = \mathcal{P}_{wsc}(^nE)$, we only need to show that $\lim_k P(t_k) = P(t_0)$ whenever $t_0, t_1, t_2, \ldots$ are in E such that $\lim_k t_k = t_0$ weakly in E. Define $L_P : E \to \mathcal{P}(^{n-1}E)$ by

$$L_P(x)(y) = T_P(x, y, \ldots, y), \quad \forall\, x, y \in E. \tag{3.4}$$

Then L_P is a continuous linear operator. Since $\mathcal{P}(^{n-1}E) = \mathcal{P}_w(^{n-1}E)$ is weakly sequentially complete, it follows from Lemma 3.3 that L_P is weakly compact and hence, $L_P^* : \mathcal{P}(^{n-1}E)^* \to E^*$ is weakly compact. Thus the space $L_P^*[\mathcal{P}(^{n-1}E)^*]$ is weakly compact generated. By [3, p.43], there is a norm one projection u of $L_P^*[\mathcal{P}(^{n-1}E)^*]$ onto a closed separable subspace Y of $L_P^*[\mathcal{P}(^{n-1}E)^*]$ that contains the closed linear span of $\{L_P^*(\theta_{n-1}(t_k))\}_{k=0}^\infty$, where $\theta_{n-1}(t_k) = t_k \otimes \cdots \otimes t_k \in \otimes_{n-1,s}E \subseteq \mathcal{P}(^{n-1}E)^*$.

Let $\{y_i\}_1^\infty$ be a dense sequence in Y. Since E^* has the BCAP, there exist $\lambda \geqslant 1$ and a sequence $\{u_k\}_1^\infty$ of compact linear operators from E^* to E^* such that for each $k \in \mathbb{N}$,

$$\|u_k\| \leqslant \lambda \quad \text{and} \quad \|u_k(y_i) - y_i\| < \frac{1}{k},\ i = 1, \ldots, k.$$

It follows that $\lim_k u_k(y_i) = y_i$ in E^* for each $i \in \mathbb{N}$. Now for any $y \in Y$ and any $i, k \in \mathbb{N}$ with $i < k$,

$$\begin{aligned}\|u_k(y) - y\| &\leqslant \|u_k(y) - u_k(y_i)\| + \|u_k(y_i) - y_i\| + \|y_i - y\| \\ &\leqslant \|u_k(y_i) - y_i\| + (\lambda + 1)\|y_i - y\|,\end{aligned}$$

which implies that

$$\lim_k u_k(y) = y \quad \text{in} \quad E^*, \quad \forall\, y \in Y. \tag{3.5}$$

Define $T, T_k : E \times \cdots \times E \to \mathbb{R}$ or $\mathbb{C}$ by

$$T(x_1, \ldots, x_n) = \frac{1}{n}\sum_{i=1}^{n} \langle (u \circ L_P^*)(\delta(x_i)), x_i\rangle, \quad \forall\, x_1, \ldots, x_n \in E$$

and

$$T_k(x_1, \ldots, x_n) = \frac{1}{n}\sum_{i=1}^{n} \langle (u_k \circ u \circ L_P^*)(\delta(x_i)), x_i\rangle, \quad \forall\, x_1, \ldots, x_n \in E$$

respectively, where

$$\delta(x_i) := x_1 \otimes_s \cdots \otimes_s x_{i-1} \otimes_s x_{i+1} \otimes_s \cdots \otimes_s x_n \in \otimes_{n-1,s} E \subseteq \mathcal{P}({}^{n-1}E)^*.$$

(In particular, if $x_1 = \cdots = x_n = x$ then $\delta(x) = \theta_{n-1}(x)$.) Then T and T_k are symmetric n-linear operators and hence, there exist $Q, P_k \in \mathcal{P}({}^nE)$ such that

$$Q(x) = T(x, \ldots, x) = \langle (u \circ L_P^*)(\theta_{n-1}(x)), x\rangle, \quad \forall\, x \in E \tag{3.6}$$

and

$$P_k(x) = T_k(x, \ldots, x) = \langle (u_k \circ u \circ L_P^*)(\theta_{n-1}(x)), x\rangle, \quad \forall\, x \in E. \tag{3.7}$$

Next we show that $P_k \in \mathcal{P}_w({}^nE)$ for each $k \in \mathbb{N}$.

Take $x, x_i \in E$ for each $i \in \mathbb{N}$ such that $c = \sup\{\|x_i\| : i \in \mathbb{N}\} < \infty$ and $\lim_i x_i = x$ weakly in E. Then for any $k, i \in \mathbb{N}$, we have

$$\begin{aligned}|P_k(x_i) - P_k(x)| &= |\langle (u_k \circ u \circ L_P^*)(\theta_{n-1}(x_i)), x_i\rangle - \langle (u_k \circ u \circ L_P^*)(\theta_{n-1}(x)), x\rangle| \\ &= |\langle L_P^*(\theta_{n-1}(x_i)), (u_k \circ u)^*(x_i)\rangle - \langle L_P^*(\theta_{n-1}(x)), (u_k \circ u)^*(x)\rangle| \\ &= |\langle L_P^*(\theta_{n-1}(x_i)), (u_k \circ u)^*(x_i - x)\rangle \\ &\quad + \langle L_P^*(\theta_{n-1}(x_i) - \theta_{n-1}(x)), (u_k \circ u)^*(x)\rangle| \\ &\leqslant c^{n-1} \cdot \|L_P^*\|\,\|\theta_{n-1}\|\,\|(u_k \circ u)^*(x_i - x)\| \\ &\quad + |\langle \theta_{n-1}(x_i) - \theta_{n-1}(x), L_P^{**} \circ (u_k \circ u)^*(x)\rangle|.\end{aligned} \tag{3.8}$$

Note that $(u_k \circ u)^*$ is compact and hence, completely continuous. Thus for each $k \in \mathbb{N}$, we have

$$\|(u_k \circ u)^*(x_i - x)\| \to 0, \quad \text{as} \quad i \to \infty. \tag{3.9}$$

Note that $\mathcal{P}({}^{n-1}E) = \mathcal{P}_w({}^{n-1}E) = \mathcal{P}_{wsc}({}^{n-1}E)$. Thus θ_{n-1} is sequentially continuous from the weak topology in E to the weak topology in $\hat{\otimes}_{n-1,s,\pi}E$. Note that L_P is weakly compact implies that $L_P^{**} \circ (u_k \circ u)^*(x) \in \mathcal{P}({}^{n-1}E) = (\hat{\otimes}_{n-1,s,\pi}E)^*$. Thus for each $k \in \mathbb{N}$, we have

$$\langle \theta_{n-1}(x_i) - \theta_{n-1}(x), L_P^{**} \circ (u_k \circ u)^*(x)\rangle \to 0 \quad \text{as} \quad i \to \infty. \tag{3.10}$$

Combining (3.8) with (3.9) and (3.10) yields that $\lim_i P_k(x_i) = P_k(x)$ and hence, $P_k \in \mathcal{P}_{wsc}({}^nE) = \mathcal{P}_w({}^nE)$ for each $k \in \mathbb{N}$.

Now for every $z \in E^{**}$, since $\theta_{n-1}(z) \in \otimes_{n-1,s}E^{**} \subseteq \mathcal{P}({}^{n-1}E)^*$, it follows from (3.7) that

$$\begin{aligned} \widetilde{P}_k(z) &= \langle (u_k \circ u \circ L_P^*)^{**}(\theta_{n-1}(z)), z\rangle = \langle (u_k \circ u)^{**} \circ L_P^{***}(\theta_{n-1}(z)), z\rangle \\ &= \langle (u_k \circ u)^{**} \circ L_P^*(\theta_{n-1}(z)), z\rangle = \langle (u_k \circ u \circ L_P^*)(\theta_{n-1}(z)), z\rangle. \end{aligned}$$

Similarly, it follows from (3.6) that

$$\widetilde{Q}(z) = \langle (u \circ L_P^*)(\theta_{n-1}(z)), z\rangle.$$

Note that $(u \circ L_P^*)(\theta_{n-1}(z)) \in Y$. It follows from (3.5) that $\lim_k \widetilde{P}_k(z) = \widetilde{Q}(z)$ and hence, $\{P_k\}_1^\infty$ is a weakly Cauchy sequence in $\mathcal{P}_w({}^nE)$ by Lemma 3.1. Thus $Q = \text{weak-}\lim_k P_k \in \mathcal{P}_w({}^nE) = \mathcal{P}_{wsc}({}^nE)$. Since $\lim_k t_k = t_0$ weakly in E, it follows that $\lim_k Q(t_k) = Q(t_0)$. By (3.4) and (3.6), for $k = 0, 1, 2, \ldots$, we have

$$\begin{aligned} Q(t_k) &= \langle (u \circ L_P^*)(\theta_{n-1}(t_k)), t_k\rangle = \langle L_P^*(\theta_{n-1}(t_k)), t_k\rangle \\ &= \langle \theta_{n-1}(t_k), L_P(t_k)\rangle = L_P(t_k)(t_k) = P(t_k). \end{aligned}$$

Therefore, $\lim_k P(t_k) = \lim_k Q(t_k) = Q(t_0) = P(t_0)$. □

Next we will consider the weak sequential completeness of the space of vector-valued homogeneous polynomials.

Theorem 3.6. *Assume that both E^* and F are weakly sequentially complete.*

(i) If $\mathcal{P}_w({}^nE;F) = \mathcal{P}_{wK}({}^nE;F)$, then $\mathcal{P}_w({}^nE;F)$ is weakly sequentially complete.

(ii) If both E^* and F have the BCAP, then $\mathcal{P}_w({}^nE;F)$ is weakly sequentially complete if and only if $\mathcal{P}_w({}^nE;F) = \mathcal{P}_{wK}({}^nE;F)$.

Proof. (i) Suppose that $\mathcal{P}_w({}^nE;F) = \mathcal{P}_{wK}({}^nE;F)$. By (2.3) and then by (2.2),

$$\mathcal{K}(\hat{\otimes}_{n,s,\pi}E;F) = \mathcal{W}(\hat{\otimes}_{n,s,\pi}E;F).$$

Moreover, Theorem 3.2 implies that $(\hat{\otimes}_{n,s,\pi}E)^* = \mathcal{P}(^nE)$ is weakly sequentially complete. It follows from [11, Theorem 2.2] that $\mathcal{W}(\hat{\otimes}_{n,s,\pi}E;F)$ is weakly sequentially complete and then by (2.2), $\mathcal{P}_w(^nE;F)$ is weakly sequentially complete.

(ii) We only need to show that the weak sequential completeness of $\mathcal{P}_w(^nE;F)$ implies that $\mathcal{P}_w(^nE;F) = \mathcal{P}_{wK}(^nE;F)$. In the case that $n = 1$, the assertion follows from [11, Theorem 2.3]. Now assume that $n \geqslant 2$. It follows that $\mathcal{P}_w(^nE)$ is also weakly sequentially complete. By Lemma 3.4, E contains no copy of ℓ_1 and hence, $\mathcal{P}_w(^nE;F) = \mathcal{P}_{wsc}(^nE;F)$. Moreover, Theorem 3.5 implies that $\mathcal{P}(^nE) = \mathcal{P}_w(^nE)$ and then Lemma 2.1 implies that $\mathcal{P}_K(^nE;F) \subseteq \mathcal{P}_{wsc}(^nE;F)$. Thus we have the following:

$$\mathcal{P}_{wsc}(^nE;F) = \mathcal{P}_w(^nE;F) \subseteq \mathcal{P}_K(^nE;F) \subseteq \mathcal{P}_{wsc}(^nE;F),$$

which implies that $\mathcal{P}_K(^nE;F) = \mathcal{P}_w(^nE;F)$ is weakly sequentially complete and hence, $\mathcal{K}(\hat{\otimes}_{n,s,\pi}E;F)$ is weakly sequentially complete. It follows from [11, Theorem 2.3] that $\mathcal{W}(\hat{\otimes}_{n,s,\pi}E;F) = \mathcal{K}(\hat{\otimes}_{n,s,\pi}E;F)$ and hence, $\mathcal{P}_{wK}(^nE;F) = \mathcal{P}_K(^nE;F) = \mathcal{P}_w(^nE;F)$ as well. □

Under the hypothesis of the BCAP, in the linear operator case, Theorem 2.3 in [11] ensures that the weak sequential completeness of $\mathcal{L}(E;F)$ implies that all T in $\mathcal{W}(E;F)$ are in $\mathcal{K}(E;F)$. It is much better in the polynomial case as we will see that the following corollary ensures that the weak sequential completeness of $\mathcal{P}(^nE;F)$ implies that all P in $\mathcal{P}(^nE;F)$ are in $\mathcal{P}_w(^nE;F)$.

Corollary 3.7. *Assume that $n \geqslant 2$ and that both E^* and F are weakly sequentially complete. If both E^* and F have the BCAP, then $\mathcal{P}_w(^nE;F)$ is weakly sequentially complete if and only if $\mathcal{P}_w(^nE;F) = \mathcal{P}(^nE;F)$.*

Proof. By Theorem 3.6, we only need to show that the weak sequential completeness of $\mathcal{P}_w(^nE;F)$ implies that $\mathcal{P}(^nE;F) \subseteq \mathcal{P}_{wK}(^nE;F)$. Assume that $\mathcal{P}_w(^nE;F)$ is weakly sequentially complete. Then $\mathcal{P}_w(^nE)$ is weakly sequentially complete and by Lemma 3.4, E contains no copy of ℓ_1. Moreover, Theorem 3.5 implies that $\mathcal{P}_w(^nE) = \mathcal{P}(^nE)$. It follows from [9, Corollary 3.9] that $\hat{\otimes}_{n,s,\pi}E$ contains no copy of ℓ_1. Now take any $P \in \mathcal{P}(^nE;F)$. By Lemma 3.3, $A_P \in \mathcal{W}(\hat{\otimes}_{n,s,\pi}E;F)$ and hence, $P \in \mathcal{P}_{wK}(^nE;F)$. □

Remark 3.8. It was proved in [11, Theorem 2.3] that if either E^* or F has the BCAP then the weak sequential completeness of $\mathcal{W}(E;F)$ implies that $\mathcal{W}(E;F) = \mathcal{K}(E;F)$. In Theorem 3.6 and Corollary 3.7, we may not weaken the condition that both E^* and F have the BCAP to the condition that either E^* or F has the BCAP. Indeed, in the proof of

Theorem 3.6, we apply [11, Theorem 2.3] to the space $\mathcal{W}(\hat{\otimes}_{n,s,\pi}E;F)$. If F does not have the BCAP, then we must assume that $(\hat{\otimes}_{n,s,\pi}E)^* = \mathcal{P}(^nE)$ have the BCAP. However, the BCAP is not inherited by $\mathcal{P}(^nE)$ from E^* in general. See Remark 5.6 in section 5 below.

4. Reflexivity

Before we present the main result of this section, we need the following lemma, which is a special case of [17, Corollary 5].

Lemma 4.1[17]. *Suppose that E and F are reflexive Banach spaces. Let $P_k, P \in \mathcal{P}_w(^nE;F)$ for each $k \in \mathbb{N}$. Then $\lim_k P_k = P$ weakly in $\mathcal{P}_w(^nE;F)$ if and only if $\lim_k \langle P_k(x), y^*\rangle = \langle P(x), y^*\rangle$ for every $x \in E$ and every $y^* \in F^*$.*

Theorem 4.2. *If E and F are reflexive, then $\mathcal{P}_w(^nE;F)$ is weakly sequentially complete if and only if $\mathcal{P}_w(^nE;F)$ is reflexive.*

Proof. It follows from [11, Theorem 2.5] that the theorem holds for $n = 1$. Using the induction, we assume that the theorem holds for $n-1$ and we will show that the theorem holds for n, where $n \geqslant 2$.

To do this, we suppose that $\mathcal{P}_w(^nE;F)$ is weakly sequentially complete. We want to show that $\mathcal{P}_w(^nE;F)$ is reflexive. It follows from [8, Prop. 5] that $\mathcal{P}(^{n-1}E;F)$ is isomorphic to a (complemented) subspace of $\mathcal{P}(^nE;F)$ and hence, $\mathcal{P}_w(^{n-1}E;F)$ is isomorphic to a subspace of $\mathcal{P}_w(^nE;F)$. Thus $\mathcal{P}_w(^{n-1}E;F)$ is also weakly sequentially complete. By the induction hypothesis, $\mathcal{P}_w(^{n-1}E;F)$ is reflexive.

To show that $\mathcal{P}_w(^nE;F)$ is reflexive, we only need to show that every bounded sequence in $\mathcal{P}_w(^nE;F)$ has a weakly Cauchy subsequence. Take any bounded sequence $\{P_k\}_1^\infty$ in $\mathcal{P}_w(^nE;F)$. For each $k \in \mathbb{N}$, define $\hat{d}^{n-1}P_k : E \to \mathcal{P}(^{n-1}E;F)$, see [14, p.13], by

$$\hat{d}^{n-1}P_k(x)(y) = T_{P_k}(x, y, \ldots, y), \quad \forall\, x, y \in E.$$

Then $\hat{d}^{n-1}P_k \in \mathcal{K}(E;\mathcal{P}(^{n-1}E;F))$ by [14, p.88, Prop. 2.6]. Since $P_k \in \mathcal{P}_w(^nE;F)$, it follows that $\hat{d}^{n-1}P_k(x) \in \mathcal{P}_w(^{n-1}E;F)$ for every $x \in E$, and hence, $\hat{d}^{n-1}P_k \in \mathcal{K}(E;\mathcal{P}_w(^{n-1}E;F))$. Note that E and $\mathcal{P}_w(^{n-1}E;F)$ are reflexive and note that $\{\hat{d}^{n-1}P_k\}_1^\infty$ is a bounded sequence in $\mathcal{K}(E;\mathcal{P}_w(^{n-1}E;F))$. It follows from [11, Lemma 2.4] that $\{\hat{d}^{n-1}P_k\}_1^\infty$ has a weakly Cauchy subsequence, without loss of generality, say $\{\hat{d}^{n-1}P_k\}_1^\infty$.

For every $x \in E$ and every $y^* \in F^*$, define a linear functional ϕ_{x,y^*} on $\mathcal{P}_w(^{n-1}E;F)$ by $\phi_{x,y^*}(P) = \langle P(x), y^*\rangle$ for every $P \in \mathcal{P}_w(^{n-1}E;F)$. Then $\phi_{x,y^*} \in \mathcal{P}_w(^{n-1}E;F)^*$.

Since $\{\hat{d}^{n-1}P_k\}_1^\infty$ is a weakly Cauchy sequence in $\mathcal{K}(E;\mathcal{P}_w({}^{n-1}E;F))$, it follows that the scalar-valued sequence $\{\langle \hat{d}^{n-1}P_k(x), \phi_{x,y^*}\rangle\}_1^\infty$ is Cauchy. Note that $\langle \hat{d}^{n-1}P_k(x), \phi_{x,y^*}\rangle = \langle P_k(x), y^*\rangle$. Thus $\{\langle P_k(x), y^*\rangle\}_1^\infty$ is a scalar-valued Cauchy sequence. By Lemma 4.1, $\{P_k\}_1^\infty$ is a weakly Cauchy sequence in $\mathcal{P}_w({}^nE;F)$. □

Note that if a reflexive Banach space has the CAP then its dual space has the BCAP (see [16, Corollary 1.6]). Thus Theorems 3.2, 3.5 and 4.2 yield the following corollary, which was obtained by Mujica and Valdivia in [25].

Corollary 4.3. *Assume that E is reflexive.*

(i) If $\mathcal{P}_w({}^nE) = \mathcal{P}({}^nE)$, then $\mathcal{P}_w({}^nE)$ is reflexive.

(ii) If E has the CAP, then $\mathcal{P}_w({}^nE)$ is reflexive if and only if $\mathcal{P}_w({}^nE) = \mathcal{P}({}^nE)$.

In the scalar-valued case, Alencar [1] proved that if E is a reflexive Banach space with the approximation property, then $\mathcal{P}({}^nE)$ is reflexive if and only if $\mathcal{P}({}^nE) = \mathcal{P}_w({}^nE)$. Mujica and Valdivia [25] improved this result by weakening the hypothesis of the approximation property of E to the hypothesis of the compact approximation property of E. In the vector-valued case, Alencar [2] proved that if E and F are reflexive Banach spaces with the approximation property, then $\mathcal{P}({}^nE;F)$ is reflexive if and only if $\mathcal{P}({}^nE;F) = \mathcal{P}_w({}^nE;F)$; while Jaramillo and Moraes [19] obtained the same conclusion when only E is assumed to have the approximation property. Moreover, Mujica [24] improved this result by weakening the hypothesis of the approximation property of E to the hypothesis of the compact approximation property of E. As a consequence of Theorems 3.6 and 4.2, we rediscover the Mujica's result above as the following corollary.

Corollary 4.4. *Assume that both E and F are reflexive.* (i) *If $\mathcal{P}_w({}^nE;F) = \mathcal{P}({}^nE;F)$, then $\mathcal{P}_w({}^nE;F)$ is reflexive.* (ii) *If E has the CAP, then $\mathcal{P}_w({}^nE;F)$ is reflexive if and only if $\mathcal{P}_w({}^nE;F) = \mathcal{P}({}^nE;F)$.*

Proof. (i) follows from Theorem 3.6(i) and Theorem 4.2. To prove (ii), suppose that $\mathcal{P}_w({}^nE;F)$ is reflexive. By Corollary 4.3(ii), $\mathcal{P}({}^nE) = \mathcal{P}_w({}^nE)$ and then by Proposition 5.5 in the section 5 below, $(\hat{\otimes}_{n,s,\pi}E)^* = \mathcal{P}({}^nE)$ has the BCAP. Note that in the proof of Theorem 3.6(ii), we do not need the BCAP of F if $(\hat{\otimes}_{n,s,\pi}E)^*$ has the BCAP (see Remark 3.8). Thus Theorem 3.6(ii) implies that $\mathcal{P}_w({}^nE;F) = \mathcal{P}_{wK}({}^nE;F) = \mathcal{P}({}^nE;F)$. □

At the end of this section, we give examples of spaces of homogeneous polynomials that are weakly sequentially complete but not reflexive. It is well known that $\mathcal{P}_w({}^nc_0) = \mathcal{P}({}^nc_0)$ for all $n \in \mathbb{N}$. By Theorem 3.2, we have one example that $\mathcal{P}({}^nc_0)$ is weakly sequentially complete for all $n \in \mathbb{N}$. Moreover, $\mathcal{P}(^n c_0)$ does not contain a copy of ℓ_∞. It follows from [14, p.119, Prop. 2.38] that $\mathcal{P}_K(^n c_0; \ell_p) = \mathcal{P}(^n c_0; \ell_p)$ for all $1 \leqslant p < \infty$. By Lemma 2.1, $\mathcal{P}_K(^n c_0; \ell_p) \subseteq \mathcal{P}_{wsc}(^n c_0; \ell_p)$ and hence, $\mathcal{P}_w(^n c_0; \ell_p) = \mathcal{P}_{wsc}(^n c_0; \ell_p) = \mathcal{P}(^n c_0; \ell_p)$. Thus by Theorem 3.6(i), we have another example that $\mathcal{P}(^n c_0; \ell_p)$ is weakly sequentially complete for all $n \in \mathbb{N}$ and all $1 \leqslant p < \infty$.

5. BCAP and CAP for Projective Tensor Products

For Banach spaces X and Y, let B_X denote the closed unit ball of X and $X \hat{\otimes}_\pi Y$ denote the projective tensor product of X and Y. For a subset C of X and a subset D of Y, let $C \otimes D := \{x \otimes y : x \in C, y \in D\}$. It is easy to see that if C and D are relatively compact subsets of X and Y respectively, then $C \otimes D$ is a relatively compact subset of $X \hat{\otimes}_\pi Y$. The converse is the following lemma due to Grothendieck [18].

Lemma 5.1. *If A is a compact subset of $X \hat{\otimes}_\pi Y$, then there are a compact subset C of X and a compact subset D of Y such that $A \subseteq \overline{co}(C \otimes D)$.*

The following proposition was proved in [10]. We give a proof here for completeness.

Proposition 5.2. *If X and Y have the CAP (resp. BCAP), then $X \hat{\otimes}_\pi Y$ has the CAP (resp. BCAP).*

Proof. Take any compact subset A of $X \hat{\otimes}_\pi Y$ and any $\varepsilon > 0$. By Lemma 5.1, there are a compact subset C of X and a compact subset D of Y such that $A \subseteq \overline{co}(C \otimes D)$. Let

$$c_1 = \sup\{\|x\| : x \in C\} \quad \text{and} \quad c_2 = \sup\{\|y\| : y \in D\}.$$

Suppose X, Y have the CAP. Then there exist compact operators $T : X \to X$ and $S : Y \to Y$ such that

$$\|T(x) - x\| \leqslant \frac{\varepsilon}{4c_2}, \quad \forall\, x \in C,$$

and

$$\|S(y) - y\| \leqslant \frac{\varepsilon}{4\|T\|c_1}, \quad \forall\, y \in D.$$

Thus for every $x \otimes y \in C \otimes D$, we have

$$\begin{aligned} \|(T \otimes S)(x \otimes y) - (x \otimes y)\| &= \|T(x) \otimes S(y) - x \otimes y\| \\ &\leqslant \|T(x) \otimes (S(y) - y)\| + \|(T(x) - x) \otimes y\| \\ &\leqslant \|T\|c_1 \cdot \frac{\varepsilon}{4\|T\|c_1} + c_2 \cdot \frac{\varepsilon}{4c_2} = \frac{\varepsilon}{2}. \end{aligned}$$

Now for every $u \in A \subseteq \overline{co}(C \otimes D)$, there is $v \in co(C \otimes D)$ such that

$$\|u - v\| \leqslant \frac{\varepsilon}{2(1 + \|T\|\|S\|)}.$$

Write $v = \sum_{i=1}^{n} t_i(x_i \otimes y_i)$, where $x_i \otimes y_i \in C \otimes D$ and $\sum_{i=1}^{n} |t_i| \leqslant 1$. Then

$$\|(T \otimes S)(v) - v\| \leqslant \sum_{i=1}^{n} |t_i| \cdot \|(T \otimes S)(x_i \otimes y_i) - (x_i \otimes y_i)\| \leqslant \frac{\varepsilon}{2},$$

which implies that

$$\begin{aligned} \|(T \otimes S)(u) - u\| &\leqslant \|(T \otimes S)(u - v)\| + \|(T \otimes S)(v) - v\| + \|v - u\| \\ &\leqslant (\|T\| \cdot \|S\| + 1) \cdot \|u - v\| + \frac{\varepsilon}{2} \leqslant \varepsilon. \end{aligned}$$

Clearly, $T \otimes S$ is compact with $\|T \otimes S\| \leqslant \|T\| \cdot \|S\|$.

If both X, Y have the BCAP, then we can further assume that $\|T\| \leqslant \lambda_1$, $\|S\| \leqslant \lambda_2$, and thus $\|T \otimes S\| \leqslant \lambda_1 \lambda_2$ for two universal constants λ_1, λ_2, independent of A, C, and D. The proof is complete. □

For Banach spaces $X_1, X_2, \ldots, X_n$, let $X_1 \hat{\otimes}_\pi X_2 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_n$ denote the projective tensor product of $X_1, X_2, \ldots, X_n$. Note that $X_1 \hat{\otimes}_\pi X_2 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_n = X_1 \hat{\otimes}_\pi (X_2 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_n)$. By Proposition 5.2 and using the induction, we have the following proposition.

Proposition 5.3. *If $X_1, \ldots, X_n$ have the BCAP (resp. CAP), then $X_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_n$ has the BCAP (resp. CAP).*

In particular, if $X_1 = \cdots = X_n = E$, let $\hat{\otimes}_{n,\pi} E := X_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_n$. Note that $\hat{\otimes}_{n,s,\pi} E$ is isomorphic to a complemented subspace of $\hat{\otimes}_{n,\pi} E$ (see [14, p. 21]). Thus Proposition 5.3 yields the following proposition.

Proposition 5.4. *If a Banach space E has the BCAP (resp. CAP), then both $\hat{\otimes}_{n,\pi} E$ and $\hat{\otimes}_{n,s,\pi} E$ have the BCAP (resp. CAP).*

Proposition 5.5. *If E is a reflexive Banach space with the CAP and if $\mathcal{P}(^n E) = \mathcal{P}_w(^n E)$, then $\mathcal{P}(^n E)$ has the BCAP.*

Proof. By Proposition 5.4, $\hat{\otimes}_{n,s,\pi} E$ has the CAP. Since E is reflexive and $\mathcal{P}(^n E) = \mathcal{P}_w(^n E)$, Corollary 4.3 implies that $\mathcal{P}(^n E)$ and hence, $\hat{\otimes}_{n,s,\pi} E$ is reflexive. Note that if a reflexive Banach space has the CAP then its dual space has the BCAP (see [16, Corollary 1.6]). Thus $\mathcal{P}(^n E)$ has the BCAP. □

Remark 5.6. (i) Note that in the proof of Proposition 5.2, if $T : X \to X$ and $S : Y \to Y$ are finite rank operators, then $T \otimes S$ is also finite rank. Thus we have all same results of Propositions 5.2–5.5 for the approximation property and the bounded approximation property. It is worthwhile to mention that Mujica in [23, Corollary 5.5 and Corollary 5.8] proved that if a Banach space E has the (bounded) approximation property then $\hat{\otimes}_{n,s,\pi}E$ has the (bounded) approximation property.

(ii) Aron and Schottenloher in [7, Prop. 5.2] constructed a reflexive Banach space E with a basis such that $\mathcal{P}(^2E)$ does not have the approximation property. Actually, $\mathcal{P}(^2E)$ does not have the CAP, either. The explanation is as follows.

Johnson [20] constructed a Banach space C_1 such that for every separable Banach space Y, its dual space Y^* is isometric to a norm one complemented subspace of C_1^* (also see [12, p. 280]). Note that each ℓ_p $(1 \leqslant p < 2)$ contains a closed subspace without the CAP (see [21, p. 107]). Thus C_1^* does not have the CAP. Aron and Schottenloher in [7, Prop. 5.2] constructed a reflexive Banach space E with a basis such that C_1^* is a complemented subspace of $\mathcal{P}(^2E)$. Therefore, $\mathcal{P}(^2E)$ does not have the CAP.

(iii) Aron and Schottenloher's counter-example tells us that the BCAP (or CAP) is not inherited by $\mathcal{P}(^nE)$ in general. However, it is inherited by $\mathcal{P}(^nE)$ in some special circumstances (see Proposition 5.5). For instance, it is well known that $\mathcal{P}_w(^n\mathcal{T}) = \mathcal{P}(^n\mathcal{T})$ for all $n \in \mathbb{N}$, where $\mathcal{T}$ is the original Tsirelson space. By Proposition 5.5, we have one example that $\mathcal{P}(^n\mathcal{T})$ has the BCAP for every $n \in \mathbb{N}$. We have another example that $\mathcal{P}(^n\ell_1)$ has the BCAP for every $n \in \mathbb{N}$ since $\mathcal{P}(^n\ell_1)$ is isomorphic to ℓ_∞ by [7, Prop. 5.1].

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