NONLINEAR ERGODIC THEOREM FOR COMMUTATIVE FAMILIES OF POSITIVELY HOMOGENEOUS NONEXPANSIVE MAPPINGS IN BANACH SPACES AND APPLICATIONS

WATARU TAKAHASHI, NGAI-CHING WONG, AND JEN-CHIH YAO

ABSTRACT. Recently, two retractions (projections) which are different from the metric projection and the sunny nonexpansive retraction in a Banach space were found. In this paper, using nonlinear analytic methods and new retractions, we prove a nonlinear ergodic theorem for a commutative family of positively homogeneous and nonexpansive mappings in a uniformly convex Banach space. The limit points are characterized by using new retractions. In the proof, we use the theory of invariant means essentially. We apply our nonlinear ergodic theorem to get some nonlinear ergodic theorems in Banach spaces.

1. INTRODUCTION

Let E be a real Banach space and let C be a nonempty subset of E. Let T be a mapping of C into C. Then we denote by $F(T)$ the set of fixed points of T. A mapping $T : C \to C$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. Let C be a closed convex cone of E. A mapping $T : C \to C$ is called *positively homogeneous* if $T(\alpha x) = \alpha T(x)$ for all $x \in C$ and $\alpha \geq 0$. Baillon [2] proved the first nonlinear ergodic theorem for nonexpansive mappings in a Hilbert space.

Theorem 1.1 (Baillon [2]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Then, for any $x \in C$, the Cesàro means*

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converge weakly as $n \to \infty$ to $z \in F(T)$.

Bruck [5] extended Baillon's result to Banach spaces as follows:

Theorem 1.2 (Bruck [5]). *Let E be a uniformly convex Banach space whose norm is a Fréchet differentiable and let C be a nonempty, closed and convex subset of E. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Then, for any $x \in C$, the Cesàro means $S_n x$ converge weakly as $n \to \infty$ to $z \in F(T)$.*

Hirano, Kido and Takahashi [10] extended Bruck's theorem to commutative families of nonexpansive mappings. However, the limit points $z \in F(T)$ in 1.2 and [10] are not characterized. Recently, Ibaraki and Takahashi [12] found a new nonlinear projection called a sunny generalized nonexpansive retraction which is different

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from the metric projection, the sunny nonexpansive retraction and the generalized projection in a Banach space. By using this retraction, Takahashi, Wong and Yao [27] proved the following theorem.

Theorem 1.3. *Let E be a uniformly convex and smooth Banach space. Let $T : E \to E$ be a positively homogeneous nonexpansive mapping. Then for any $x \in E$, the Cesàro means $S_n x$ converges weakly to $z_0 \in F(T)$. Additionally, if the norm of E is a Fréchet differentiable, then $z_0 = \lim_{n\to\infty} R_{F(T)} T^n x$, where $R_{F(T)}$ is the sunny generalized nonexpansive retraction of E onto $F(T)$.*

In this paper, using nonlinear analytic methods and new retractions which were found recently, we prove a nonlinear ergodic theorem for a commutative family of positively homogeneous and nonexpansive mappings in a uniformly convex Banach space. The limit points are characteralized by new retractions. In the proof, we use the theory of invariant means essentially. We apply our nonlinear ergodic theorem to get some nonlinear ergodic theorems in Banach spaces.

2. Preliminaries

Let E be a real Banach space and let E^* be the dual space of E. For a sequence $\{x_n\}$ of E and a point $x \in E$, the weak convergence of $\{x_n\}$ to x and the strong convergence of $\{x_n\}$ to x are denoted by $x_n \rightharpoonup x$ and $x_n \to x$, respectively. Let A be a nonempty subset of E. We denote by $\overline{co}A$ the closure of the convex hull of A. The *duality mapping* J from E into E^* is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad \forall x \in E.$$

Let $S(E)$ be the unit sphere centered at the origin of E. Then the space E is said to be *smooth* if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all $x, y \in S(E)$. The norm of E is said to be *Fréchet differentiable* if for each $x \in S(E)$, the limit is attained uniformly for $y \in S(E)$. A Banach space E is said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$ whenever $x, y \in S(E)$ and $x \neq y$. It is said to be *uniformly convex* if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that $\|\frac{x+y}{2}\| \leq 1 - \delta$ whenever $x, y \in S(E)$ and $\|x - y\| \geq \varepsilon$. Furthermore, we know from [24] that

(i) if E is smooth, then J is single-valued;
(ii) if E is reflexive, then J is onto;
(iii) if E is strictly convex, then J is one-to-one;
(iv) if E is strictly convex, then J is strictly monotone, i.e.,

$$\langle x - y, Jx - Jy \rangle > 0, \quad \forall x, y \in E,\ x \neq y;$$

(v) if E has a Fréchet differentiable norm, then J is norm-to-norm continuous.

Let E be a smooth Banach space and let J be the duality mapping on E. Throughout this paper, let $\mathbb{N}$ and $\mathbb{R}$ be the sets of positive integers and real numbers, respectively. Define the function $\phi : E \times E \to \mathbb{R}$ by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E.$$

Observe that, in a Hilbert space H, $\phi(x, y) = \|x - y\|^2$ for all $x, y \in H$. We also know that for each $x, y, z, w \in E$,

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2; \tag{2.1}$$

$$\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x - z, Jz - Jy\rangle; \tag{2.2}$$

$$2\langle x - y, Jz - Jw\rangle = \phi(x,w) + \phi(y,z) - \phi(x,z) - \phi(y,w). \tag{2.3}$$

If E is additionally assumed to be strictly convex, then

$$\phi(x,y) = 0 \text{ if and only if } x = y. \tag{2.4}$$

The following result was proved by Kamimura and Takahashi [18].

Lemma 2.1 (Kamimura and Takahashi [18]). *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Let E be a Banach space and let C be a nonempty subset of E. A mapping $T : C \to C$ is *quasi-nonexpansive* if $F(T) \neq \emptyset$ and $\|Tx - y\| \leq \|x - y\|$ for all $x \in C$ and $y \in F(T)$. We know the following result.

Lemma 2.2 (Itoh and Takahashi [17]). *Let E be a strictly convex Banach space and let C be a nonempty, closed and convex subset of E. Let T be a quasi-nonexpansive mapping of C into itself. Then $F(T)$ is closed and convex.*

Let E be a smooth Banach space and let C be a nonempty subset of E. A mapping $T : C \to C$ is called *generalized nonexpansive* [12] if $F(T) \neq \emptyset$ and

$$\phi(Tx, y) \leq \phi(x, y), \quad \forall x \in C,\ y \in F(T).$$

Takahashi and Yao [29] obtained the following result by using the Hahn-Banach theorem.

Lemma 2.3 (Takahashi and Yao [29]). *Let E be a Banach space and let C be a closed and convex cone of E. Let $T : C \to C$ be a positively homogeneous nonexpansive mapping. Then, for any $x \in C$ and $m \in F(T)$, there exists $j \in Jm$ such that*

$$\langle x - Tx, j\rangle \leq 0,$$

where J is the duality mapping of E into E^*.

Using Lemma 2.3, Takahashi and Yao [29] proved the following result.

Lemma 2.4 (Takahashi and Yao [29]). *Let E be a smooth Banach space and let C be a closed and convex cone of E. Let $T : C \to C$ be a positively homogeneous nonexpansive mapping. Then, T is a generalized nonexpansive mapping.*

Let D be a nonempty subset of a Banach space E. A mapping $R : E \to D$ is said to be *sunny* if

$$R(Rx + t(x - Rx)) = Rx, \quad \forall x \in E,\ t \geq 0.$$

A mapping $R : E \to D$ is said to be a *retraction* or a *projection* if $Rx = x$ for all $x \in D$. A nonempty subset D of a smooth Banach space E is said to be a *generalized nonexpansive retract* (resp. *sunny generalized nonexpansive retract*) of E if there exists a generalized nonexpansive retraction (resp. sunny generalized nonexpansive retraction) R from E onto D; see [11, 13, 12] for more details. The following results are in Ibaraki and Takahashi [12].

Lemma 2.5 (Ibaraki and Takahashi [12]). *Let C be a nonempty closed sunny generalized nonexpansive retract of a smooth and strictly convex Banach space E. Then the sunny generalized nonexpansive retraction from E onto C is uniquely determined.*

Lemma 2.6 (Ibaraki and Takahashi [12]). *Let C be a nonempty closed subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto C and let $(x,z) \in E \times C$. Then the following hold:*

(i) $z = Rx$ *if and only if* $\langle x - z, Jy - Jz\rangle \le 0$ *for all* $y \in C$;
(ii) $\phi(Rx, z) + \phi(x, Rx) \le \phi(x, z)$.

In 2007, Kohsaka and Takahashi [19] also proved the following results:

Lemma 2.7 (Kohsaka and Takahashi [19]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed subset of E. Then the following are equivalent:*

(a) *C is a sunny generalized nonexpansive retract of E;*
(b) *C is a generalized nonexpansive retract of E;*
(c) *JC is closed and convex.*

Lemma 2.8 (Kohsaka and Takahashi [19]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed sunny generalized nonexpansive retract of E. Let R be the sunny generalized nonexpansive retraction from E onto C and let $(x,z) \in E \times C$. Then the following are equivalent:*

(i) *$z = Rx$;*
(ii) $\phi(x, z) = \min_{y \in C} \phi(x, y)$.

Inthakon, Dhompongsa and Takahashi [16] obtained the following result concerning the set of fixed points of a generalized nonexpansive mapping in a Banach space; see also Ibaraki and Takahashi [14, 15].

Lemma 2.9 (Inthakon, Dhompongsa and Takahashi [16]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a closed subset of E such that $J(C)$ is closed and convex. Let T be a generalized nonexpansive mapping from C into itself. Then, $F(T)$ is closed and $JF(T)$ is closed and convex.*

The following is a direct consequence of Lemmas 2.7 and 2.9.

Lemma 2.10 (Inthakon, Dhompongsa and Takahashi [16]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a closed subset of E such that $J(C)$ is closed and convex. Let T be a generalized nonexpansive mapping from C into itself. Then, $F(T)$ is a sunny generalized nonexpansive retract of E.*

3. Semitopological Semigroups and Invariant Means

Let S be a semitopological semigroup, i.e., S is a semigroup with a Hausdorff topology such that for each $a \in S$ the mappings $s \mapsto a \cdot s$ and $s \mapsto s \cdot a$ from S to S are continuous. In the case when S is commutative, we denote st by $s + t$. Let $B(S)$ be the Banach space of all bounded real valued functions on S with supremum norm and let $C(S)$ be the subspace of $B(S)$ of all bounded real valued continuous functions on S. Let μ be an element of $C(S)^*$ (the dual space of $C(S)$). We denote by $\mu(f)$ the value of μ at $f \in C(S)$. Sometimes, we denote by $\mu_t(f(t))$ or $\mu_t f(t)$ the value $\mu(f)$. For each $s \in S$ and $f \in C(S)$, we define two functions $l_s f$ and $r_s f$ as follows:

$$(l_s f)(t) = f(st) \quad \text{and} \quad (r_s f)(t) = f(ts)$$

for all $t \in S$. An element μ of $C(S)^*$ is called a *mean* on $C(S)$ if $\mu(e) = \|\mu\| = 1$, where $e(s) = 1$ for all $s \in S$. We know that $\mu \in C(S)^*$ is a mean on $C(S)$ if and only if

$$\inf_{s \in S} f(s) \le \mu(f) \le \sup_{s \in S} f(s), \quad \forall f \in C(S).$$

A mean μ on $C(S)$ is called *left invariant* if $\mu(l_s f) = \mu(f)$ for all $f \in C(S)$ and $s \in S$. Similarly, a mean μ on $C(S)$ is called *right invariant* if $\mu(r_s f) = \mu(f)$ for all $f \in C(S)$ and $s \in S$. A left and right invariant invariant mean on $C(S)$ is called an *invariant* mean on $C(S)$. The following theorem is in [24, Theorem 1.4.5].

Theorem 3.1 ([24])**.** *Let S be a commutative semitopological semigroup. Then there exists an invariant mean on $C(S)$, i.e., there exists an element $\mu \in C(S)^*$ such that $\mu(e) = \|\mu\| = 1$ and $\mu(r_s f) = \mu(f)$ for all $f \in C(S)$ and $s \in S$.*

Theorem 3.2 ([24])**.** *Let S be a semitopological semigroup. Let μ be a right invariant mean on $C(S)$. Then*

$$\sup_s \inf_t f(ts) \le \mu(f) \le \inf_s \sup_t f(ts), \quad \forall f \in C(S).$$

Similarly, let μ be a left invariant mean on $C(S)$. Then

$$\sup_s \inf_t f(st) \le \mu(f) \le \inf_s \sup_t f(st), \quad \forall f \in C(S).$$

Let S be a semitopological semigroup. For any $f \in C(S)$ and $c \in \mathbb{R}$, we write

$$f(s) \to c, \quad \text{as} \quad s \to \infty_R$$

if for each $\varepsilon > 0$ there exists an $\omega \in S$ such that

$$|f(tw) - c| < \varepsilon, \quad \forall t \in S.$$

We denote $f(s) \to c$, as $s \to \infty_R$ by

$$\lim_{s \to \infty_R} f(s) = c.$$

When S is commutative, we also denote $s \to \infty_R$ by $s \to \infty$.

Theorem 3.3 ([24])**.** *Let $f \in C(S)$ and $c \in \mathbb{R}$. If*

$$f(s) \to c, \quad \text{as} \quad s \to \infty_R,$$

then $\mu(f) = c$ for all right invariant mean μ on $C(S)$.

Theorem 3.4 ([24])**.** *If $f \in C(S)$ fulfills*

$$f(ts) \le f(s), \quad \forall t, s \in S,$$

then

$$f(t) \to \inf_{w \in S} f(w), \quad \text{as} \quad t \to \infty_R.$$

Theorem 3.5 ([24])**.** *Let S be a commutative semitopological semigroup and let $f \in C(S)$. Then the following are equivalent:*

(i) $f(s) \to c, \quad \text{as} \quad s \to \infty$;
(ii) $\sup_w \inf_t f(t+w) = \inf_w \sup_t f(t+w) = c$.

Let E be a Banach space and let C be a nonempty, closed and convex subset of E. Let S be a semitopological semigroup and let $\mathcal{S} = \{T_s : s \in S\}$ be a family of nonexpansive mappings of C into itself. Then $\mathcal{S} = \{T_s : s \in S\}$ is called a *continuous representation* of S as nonexpansive mappings on C if $T_{st} = T_s T_t$ for all $s, t \in S$ and $s \mapsto T_s x$ is continuous for each $x \in C$. The following definition [22] is crucial in the nonlinear ergodic theory of abstract semigroups. Let S be a topological space and Let $C(S)$ be the Banach space of all bounded real valued continuous functions on S with supremum norm. Let E be a reflexive Banach space. Let $u : S \to E$ be a continuous function such that $\{u(s) : s \in S\}$ is bounded and let μ be a mean on $C(S)$. Then there exists a unique element z_0 of E such that

$$\mu_s \langle u(s), x^* \rangle = \langle z_0, x^* \rangle, \quad \forall x^* \in E^*.$$

We call such z_0 the *mean vector* of u for μ and denote by $\tau(\mu)u$, i.e., $\tau(\mu)u = z_0$. In particular, if $\mathcal{S} = \{T_s : s \in S\}$ is a continuous representation of S as nonexpansive mappings on C and $u(s) = T_s x$ for all $s \in S$, then there exists $z_0 \in C$ such tat

$$\mu_s \langle T_s x, x^* \rangle = \langle z_0, x^* \rangle, \quad \forall x^* \in E^*.$$

We denote such z_0 by $T_\mu x$. The following result is in Hirano, Kido and Takahashi [10].

Lemma 3.6. *Let S be a commutative semitopological semigroup. Let E be a uniformly convex Banach space and let C be a nonempty, closed and convex subset of E. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as nonexpansive mappings on C. Let μ be an invariant mean on $C(S)$. Then for any $x \in C$, the mean vector $T_\mu x$ of $\{T_s x : s \in S\}$ for μ is in $F(\mathcal{S})$.*

The following lemma plays an important role for proving our main theorem.

Lemma 3.7. *Let E be a smooth, strictly convex and reflexive Banach space and let D be a nonempty, closed and convex subset of E. Let S be a semitopological semigroup and let $u : S \to D$ be a continuous function such that $\{u(s) : s \in S\}$ is bounded. Let μ be a mean on $C(S)$. If $g : D \to \mathbb{R}$ is defined by*

$$g(z) = \mu_s \phi(u(s), z), \quad \forall z \in D,$$

then the mean vector $z_0 = \tau(\mu)u$ of u for μ is a unique minimizer in D such that

$$g(z_0) = \min\{g(z) : z \in D\}.$$

Proof. For a continuous function $u : S \to D$ such that $\{u(s) : s \in S\}$ is bounded and a mean μ on $C_(S)$, we know that a function $g : D \to \mathbb{R}$ defined by

$$g(z) = \mu_s \phi(u(s), z), \quad \forall z \in D$$

is well-defined. We also know that there exists the mean vector $z_0 = \tau(\mu)u$ of u for μ, that is, there exists $z_0 \in \overline{co}\{u(s) : s \in S\}$ such that

$$\mu_s \langle u(s), y^* \rangle = \langle z_0, y^* \rangle, \quad \forall y^* \in E^*.$$

Since D is closed and convex and $\{u(s) : s \in S\} \subset D$, we have $z_0 \in D$. Furthermore we have from (2.2) and (2.3) that for any $z \in D$,

$$\begin{aligned} g(z) - g(z_0) &= \mu_s \phi(u(s), z) - \mu_s \phi(u(s), z_0) \\ &= \mu_s \big(\phi(u(s), z) - \phi(u(s), z_0)\big) \\ &= \mu_s \big(\phi(u(s), z) - \phi(u(s), z) - \phi(z, z_0) - 2\langle u(s) - z, Jz - Jz_0\rangle\big) \\ &= \mu_s \big(-\phi(z, z_0) - 2\langle u(s) - z, Jz - Jz_0\rangle\big) \\ &= -\phi(z, z_0) - 2\langle z_0 - z, Jz - Jz_0\rangle \\ &= -\phi(z, z_0) - \phi(z_0, z_0) - \phi(z, z) + \phi(z_0, z) + \phi(z, z_0) \\ &= \phi(z_0, z). \end{aligned}$$

Then we have that

$$g(z) = g(z_0) + \phi(z_0, z), \quad \forall z \in D. \tag{3.1}$$

This implies that $z_0 \in D$ is a minimizer in D such that $g(z_0) = \min\{g(z) : z \in D\}$. Furthermore, if $v \in D$ satisfies $g(v) = g(z_0)$, then we have from (3.1) that $\phi(z_0, v) = 0$. Since E is strictly convex, we have that $z_0 = v$ and hence z_0 is a unique minimizer in D such that

$$g(z_0) = \min\{g(z) : z \in D\}.$$

This completes the proof. $\square$

4. Nonlinear Ergodic Theorem

In the section, we prove a nonlinear ergodic theorem for a commutative family of positively homogeneous nonexpansive mappings in a Banach space. Before proving it, we need the following two lemmas. Using Lemma 3.7, we can first obtain the following result.

Lemma 4.1. *Let S be a commutative semitopological semigroup. Let E be a uniformly convex and smooth Banach space and let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as positively homogeneous nonexpansive mappings of E into itself. Then for any $x \in C$, $\{T_s x : s \in S\}$ is bounded and the set*

$$\cap_{s \in S} \overline{co}\{T_{t+s} x : t \in S\} \cap F(\mathcal{S})$$

consists of one point z_0, where z_0 is a unique minimizer of $F(\mathcal{S})$ such that

$$\lim_{s \to \infty} \phi(T_s x, z_0) = \min\{\lim_{s \to \infty} \phi(T_s x, z) : z \in F(\mathcal{S})\}.$$

Proof. For each $s \in S$, $T_s : E \to E$ is positively homogeneous and nonexpansive. It follows from Lemma 2.4 that T_s is generalized nonexpansive. Thus we have that for any $z \in F(\mathcal{S})$ and $x \in C$,

$$\phi(T_{t+s} x, z) \leq \phi(T_s x, z) \leq \cdots \leq \phi(x, z), \quad \forall s, t \in S.$$

Then $\{T_s x : s \in S\}$ is bounded. Let μ be an innvariant mean on $C(S)$. From Lemma 3.7, the mean vector $z_0 \in E$ of $\{T_s x\}$ for μ is a unique minimizer $z_0 \in E$ such that

$$\mu_s \phi(T_s x, z_0) = \min\{\mu_s \phi(T_s x, y) : \ y \in E\}.$$

We also know from Lemma 3.6 that $z_0 \in F(\mathcal{S})$. Furthermore, this $z_0 \in F(\mathcal{S})$ satisfies that

$$\mu_s \phi(T_s x, z_0) = \min\{\mu_s \phi(T_s x, y) : \ y \in F(\mathcal{S})\}.$$

Let us show that $z_0 \in \cap_{s\in S}\overline{co}\{T_{t+s}x : t \in S\}$. If not, there exists some $s_0 \in S$ such that $z_0 \notin \overline{co}\{T_{t+s_0}x : t \in S\}$. By the separation theorem, there exists $y_0^* \in E^*$ such that

$$\langle z_0, y_0^*\rangle < \inf\big\{\langle z, y_0^*\rangle : z \in \overline{co}\{T_{t+s}x : t \in S\}\big\}.$$

Using the property of the invariant mean μ, we have that

$$\begin{aligned}\langle z_0, y_0^*\rangle &< \inf\big\{\langle z, y_0^*\rangle : z \in \overline{co}\{T_{t+s_0}x : t \in S\}\big\}\\ &\le \inf\{\langle T_{t+s_0}x, y_0^*\rangle : t \in S\}\\ &\le \mu_t\langle T_{t+s_0}x, y_0^*\rangle\\ &= \mu_t\langle T_t x, y_0^*\rangle\\ &= \langle z_0, y_0^*\rangle.\end{aligned}$$

This is a contradiction. Thus we have that $z_0 \in \cap_{s\in S}\overline{co}\{T_{t+s}x : t \in S\}$. Next we show that $\cap_{s\in S}\overline{co}\{T_{t+s}x : t \in S\} \cap F(\mathcal{S})$ consists of one point z_0. Assume that $z_1 \in \cap_{s\in S}\overline{co}\{T_{t+s}x : t \in S\} \cap F(\mathcal{S})$. Since $z_1 \in F(\mathcal{S})$, we have that

$$\phi(T_{t+s}x, z_1) \le \phi(T_s x, z_1), \quad \forall t, s \in S.$$

Then $\lim_s \phi(T_s x, z_1)$ exists. Furthermore, we know from the property of an invariant mean μ (Theorem 3.3) that

$$\mu_t \phi(T_t x, z_1) = \lim_{t\to\infty} \phi(T_t x, z_1).$$

In general, since $\lim_t \phi(T_t x, z)$ exists for every $z \in F(\mathcal{S})$, we define a function $g : F(\mathcal{S}) \to \mathbb{R}$ as follows:

$$g(z) = \lim_{t\to\infty} \phi(T_t x, z), \quad \forall z \in F(\mathcal{S}).$$

Since

$$\phi(z_0, z_1) = \phi(T_t x, z_1) - \phi(T_t x, z_0) - 2\langle T_t x - z_0, Jz_0 - Jz_1\rangle$$

for every $t \in S$, we have

$$\begin{aligned}&\phi(z_0, z_1) + 2\lim_{t\to\infty}\langle T_t x - z_0, Jz_0 - Jz_1\rangle\\ &\qquad = \lim_{t\to\infty}\phi(T_t x, z_1) - \lim_{t\to\infty}\phi(T_t x, z_0)\\ &\qquad \ge 0.\end{aligned}$$

Let $\epsilon > 0$. Then we have that

$$2\lim_{t\to\infty}\langle T_t x - z_0, Jz_0 - Jz_1\rangle > -\phi(z_0, z_1) - \epsilon.$$

Hence there exists $s_0 \in \mathbb{N}$ such that

$$2\langle T_{t+s_0}x - z_0, Jz_0 - Jz_1\rangle > -\phi(z_0, z_1) - \epsilon$$

for every $t \in S$. Since $z_1 \in \cap_s\overline{co}\{T_{t+s}x : t \in S\} \subset \overline{co}\{T_{t+s_0}x : t \in S\}$, we have

$$2\langle z_1 - z_0, Jz_0 - Jz_1\rangle \ge -\phi(z_0, z_1) - \epsilon.$$

We have from (2.3) that

$$\phi(z_1, z_1) + \phi(z_0, z_0) - \phi(z_1, z_0) - \phi(z_0, z_1) \ge -\phi(z_0, z_1) - \epsilon$$

and hence $\phi(z_1, z_0) \le \epsilon$. Since $\epsilon > 0$ is arbitrary, we have $\phi(z_1, z_0) = 0$. Since E is strictly convex, we have $z_0 = z_1$. Therefore

$$\{z_0\} = \cap_s\overline{co}\{T_{t+s}x : t \in S\} \cap F(\mathcal{S}).$$

This completes the proof. $\square$

For proving our main theorem (Theorem 4.3), we also need the following lemma.

Lemma 4.2. *Let S be a commutative semitopological semigroup. Let E be a uniformly convex and smooth Banach space and let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as positively homogeneous nonexpansive mappings of E into itself. Then, there exists a unique sunny generalized nonexpansive retraction R of E onto $F(\mathcal{S})$. Furthermore, for any $x \in E$, $\lim_{s\to\infty} RT_s x$ exists in $F(\mathcal{S})$, where $\lim_{s\to\infty} RT_s x = q$ means $\lim_{s\to\infty} \|RT_s x - q\| = 0$.*

Proof. We have from Lemma 2.2 that $F(\mathcal{S})$ is closed and convex. Furthermore, we have from Lemma 2.9 that $JF(\mathcal{S})$ are closed and convex. Then from Lemmas 2.5, 2.7 and 2.10 , there exists a unique sunny generalized nonexpansive retraction R of E onto $F(\mathcal{S})$. For an invariant mean μ on $C(S)$, there exists $q \in F(\mathcal{S})$ such that

$$\mu_t \langle RT_t x, x^* \rangle = \langle q, x^* \rangle, \quad \forall x^* \in E^*.$$

Then we have that

$$\mu_t \langle RT_{t+s} x, x^* \rangle = \mu_t \langle RT_t x, x^* \rangle = \langle q, x^* \rangle, \quad \forall x^* \in E^*.$$

Thus we have that

$$q \in \overline{co}\{RT_{t+s}x : t \in S\}, \quad \forall s \in S. \tag{4.1}$$

From Lemma 2.6, we know that

$$0 \leq \langle v - Rv, JRv - Ju \rangle, \quad \forall v \in E,\ u \in F(\mathcal{S}). \tag{4.2}$$

We have from (4.2) and (2.3) that

$$\begin{aligned} 0 &\leq 2\langle v - Rv, JRv - Ju \rangle \\ &= \phi(v,u) + \phi(Rv,Rv) - \phi(v,Rv) - \phi(Rv,u) \\ &= \phi(v,u) - \phi(v,Rv) - \phi(Rv,u). \end{aligned}$$

Hence we have that

$$\phi(Rv,u) \leq \phi(v,u) - \phi(v,Rv), \quad \forall v \in E,\ u \in F(\mathcal{S}). \tag{4.3}$$

Since $\phi(T_s z, u) \leq \phi(z,u)$ for all $s \in S$, $u \in F(\mathcal{S})$ and $z \in E$, it follows from Lemma 2.8 that

$$\phi(T_{t+s}x, RT_{t+s}x) \leq \phi(T_{t+s}x, RT_s x) \leq \phi(T_s x, RT_s x). \tag{4.4}$$

Hence we have from (4.4) and Theorem 3.4 that

$$\phi(T_s x, RT_s x) \to \inf_{w\in S} \phi(T_w x, RT_w x), \quad \text{as } s \to \infty. \tag{4.5}$$

Putting $u = RT_s x$ and $v = T_{t+s}x$ in (4.3), we have that

$$\begin{aligned} \phi(RT_{t+s}x, RT_s x) &\leq \phi(T_{t+s}x, RT_s x) - \phi(T_{t+s}x, RT_{t+s}x) \\ &\leq \phi(T_s x, RT_s x) - \phi(T_{t+s}x, RT_{t+s}x) \\ &\leq \phi(T_s x, RT_s x) - \inf_{w\in S} \phi(T_w x, RT_w x). \end{aligned}$$

Using (4.1), we have that

$$\phi(q, RT_s x) \leq \phi(T_s x, RT_s x) - \inf_{w\in S} \phi(T_w x, RT_w x), \quad \forall s \in S.$$

Thus we have from (4.5) and Lemma 2.1 that

$$\|RT_s x - q\| \to 0, \quad \text{as } s \to \infty.$$

Therefore $\{RT_s x\}$ converges strongly to $q \in F(\mathcal{S})$. This completes the proof. $\square$

Let S be a semitopological semigroup and let $\{\mu_\alpha\}$ be a net of means on $C(S)$. Then $\{\mu_\alpha\}$ is said to be *asymptotically invariant* if for each $f \in C(S)$ and $s \in S$,

$$\mu_\alpha(f) - \mu_\alpha(l_s f) \to 0 \quad \text{and} \quad \mu_\alpha(f) - \mu_\alpha(r_s f) \to 0.$$

Theorem 4.3. *Let S be a commutative semitopological semigroup. Let E be a uniformly convex and smooth Banach space and let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as positively homogeneous nonexpansive mappings of E into itself. If a net $\{\mu_\alpha\}$ of means on $C(S)$ is asymptotically invariant. Then for any $x \in C$, $\{T_{\mu_\alpha} x\}$ converges weakly to $z_0 \in F(\mathcal{S})$. Additionally, if the norm of E is a Fréchet differentiable, then $\lim_s R_{F(\mathcal{S})} T_s x = z_0$, where $R_{F(\mathcal{S})}$ is the sunny generalized nonexpansive retraction of E onto $F(\mathcal{S})$.*

Proof. Since $\{\mu_\alpha\}$ is a net of means on $C(S)$, it has a cluster point μ in the weak* topology. We show that μ is an invariant mean on $C(S)$. In fact, since the set

$$\{\lambda \in C(S)^* : \lambda(e) = \|\lambda\| = 1\}$$

is closed in the weak* topology, it follows that μ is a mean on $C(S)$. Furthermore, any $\varepsilon > 0$, $f \in C(S)$ and $s \in S$, there exists α_0 such that

$$|\mu_\alpha(f) - \mu_\alpha(l_s f)| \le \frac{\varepsilon}{3}, \quad \forall \alpha \ge \alpha_0.$$

Since μ is a cluster point of $\{\mu_\alpha\}$, we can choose $\beta \ge \alpha_0$ such that

$$|\mu_\beta(f) - \mu(f)| \le \frac{\varepsilon}{3} \quad \text{and} \quad |\mu_\beta(l_s f) - \mu(l_s f)| \le \frac{\varepsilon}{3}.$$

Hence we have

$$\begin{aligned} |\mu(f) - \mu(l_s f)| &\le |\mu(f) - \mu_\beta(f)| \\ &\quad + |\mu_\beta(f) - \mu_\beta(l_s f)| + |\mu_\beta(l_s f) - \mu(l_s f)| \\ &\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\mu(f) = \mu(l_s f), \quad \forall f \in C(S),\ s \in S.$$

Let $x \in E$ and define $D = \{z \in E : \|z\| \le \|x\|\}$. Then D is nonempty, bounded, closed and convex. Furthermore, since T_s is nonexpansive for each $s \in S$ and $0 \in F(\mathcal{S})$, D is invariant under T_s and hence $\{T_s x : s \in S\}$ and $\{T_{\mu_\alpha} x\}$ are in D. We show that $\{T_{\mu_\alpha} x\}$ converges weakly to $z_0 \in F(\mathcal{S})$. Since $\{T_{\mu_\alpha} x\}$ is a bounded net in D, there exists a subnet $\{T_{\mu_{\alpha_\beta}} x\}$ of $\{T_{\mu_\alpha} x\}$ converges weakly to some $z \in D$. If λ is a cluster point of $\{\mu_{\alpha_\beta}\}$ in the weak* topology, then λ is a cluster point of $\{\mu_\alpha\}$, too. Then λ is an invariant mean on $C(S)$. Without loss of generality, we have from $\{T_{\mu_{\alpha_\beta}} x\} \rightharpoonup z$ that

$$\lambda_s \langle T_s x, y^* \rangle = \lim_\beta\, (\mu_{\alpha_\beta})_s \langle T_s x, y^+ \rangle = \lim_\beta\, \langle T_{\mu_{\alpha_\beta}} x, y^+ \rangle = \langle z, y^* \rangle, \quad \forall y^* \in E^*.$$

Since λ is an invariant mean on $C(S)$, we have

$$z = T_\lambda x \in \cap_{s \in S} \overline{co} \{T_{t+s} x : t \in S\} \cap F(\mathcal{S}).$$

We know from Lemma 4.1 that the set

$$\cap_s \overline{co} \{T_{t+s} x : t \in S\} \cap F(\mathcal{S})$$

consists of one point z_0. Therefore $\{T_{\mu_\alpha} x\}$ converges weakly to $z_0 \in F(\mathcal{S})$.

Additionally, assume that the norm of E is a Fréchet differentiable. We have from Lemma 4.2 that there exists the sunny generalized nonexpansive retraction $R = R_{F(\mathcal{S})}$ of E onto $F(\mathcal{S})$ and $\{RT_s x\}$ converges strongly to a point $q \in F(\mathcal{S})$. Rewriting the characterization of the retraction R, we have that

$$0 \le \langle T_t x - RT_t x, JRT_t x - Ju \rangle, \quad \forall u \in F(\mathcal{S})$$

and hence

$$\begin{aligned}\langle T_t x - RT_t x, Ju - Jq \rangle &\le \langle T_t x - RT_t x, JRT_t x - Jq \rangle \\ &\le \|T_t x - RT_t x\| \cdot \|JRT_t x - Jq\| \\ &\le K \|JRT_t x - Jq\|,\end{aligned}$$

where K is an upper bound for $\|T_t x - RT_t x\|$. Remembering that J is continuous because the norm of E is a Fréchet differentiable, we apply an invariant mean μ to both sides of this inequality. Then we have that

$$\langle z_0 - q, Ju - Jq \rangle \le 0.$$

This holds for any $u \in F(\mathcal{S})$. Putting $u = z_0$, we have $\langle z_0 - q, Jz_0 - Jq \rangle \le 0$. Since J is monotone, we have $\langle z_0 - q, Jz_0 - Jq \rangle = 0$. Since E is strictly convex, we have $z_0 = q$. Thus $z_0 = \lim_{s\to\infty} R_{F(\mathcal{S})} T_s x$. $\square$

Compare Theorem 4.3 with that of [10]. Though the assumption of a mapping in Theorem 4.3 is stronger than that of [10], the assumption of a Banach space is weaker. Furthermore, the limit points are characterized by sunny generalized nonexpansive retractions.

5. Applications

In this section, we apply Theorem 4.3 to get some nonlinear ergodic theorems in Banach spaces which are related to positively homogeneous nonexpansive mappings. The following theorem was proved by Takahashi, Wong and Yao [27].

Theorem 5.1 (Takahashi, Wong and Yao [27]). *Let E be a uniformly convex and smooth Banach space. Let $T : E \to E$ be a positively homogeneous nonexpansive mapping. Then for any $x \in E$,*

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to $z_0 \in F(T)$. Additionally, if the norm of E is a Fréchet differentiable, then $z_0 = \lim_{n\to\infty} R_{F(T)} T^n x$, where $R_{F(T)}$ is the sunny generalized nonexpansive retraction of E onto $F(T)$.

Proof. Let $S = \{0\} \cup \mathbb{N}$. For any $f = (x_0, x_1, x_2, \dots) \in B(S)$, define

$$\mu_n(f) = \frac{1}{n} \sum_{k=0}^{n-1} x_k, \quad \forall n \in \mathbb{N}.$$

Then $\{\mu_n : n \in \mathbb{N}\}$ is an asymptotically invariant sequence of means on $B(S)$; see [24, p.78]. Furthermore, we have that for any $x \in E$ and $n \in \mathbb{N}$,

$$T_{\mu_n} x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x.$$

Therefore, we have the desired result from Theorem 4.3. □

Theorem 5.2. *Let E be a uniformly convex and smooth Banach space and let $T : E \to E$ be a positively homogeneous nonexpansive mapping. Then for any $x \in E$,*

$$S_r x = (1-r)\sum_{k=0}^{\infty} r^k T^k x$$

converges weakly to $z_0 \in F(T)$ as $r \to 1$ with $0 < r < 1$. Additionally, if the norm of E is a Fréchet differentiable, then $z_0 = \lim_{n\to\infty} R_{F(T)}T^n x$, where $R_{F(T)}$ is the sunny generalized nonexpansive retraction of E onto $F(T)$.

Proof. Let $S = \{0\} \cup \mathbb{N}$. For any $f = (x_0, x_1, x_2, \dots) \in B(S)$ and $r \in \mathbb{R}$ with $0 < r < 1$, define

$$\mu_r(f) = (1-r)\sum_{k=0}^{\infty} r^k x_k, \quad \forall r \in (0,1).$$

Then $\{\mu_r : r \in (0,1)\}$ is an asymptotically invariant net of means on $B(S)$; see [24, p.79]. Furthermore, we have that for any $x \in E$ and $r \in (0,1)$,

$$T_{\mu_r} x = (1-r)\sum_{k=0}^{\infty} r^k T^k x.$$

Therefore, we have the desired result from Theorem 4.3. □

Let $S = \mathbb{R}^+ = \{t \in \mathbb{R} : 0 \le t < \infty\}$. Then a family $\mathcal{S} = \{S(t) : t \in \mathbb{R}^+\}$ of mappings of E into itself is called a *positively homogeneous nonexpansive semigroup* on E if $\mathcal{S}$ satisfies the following:

(1) $S(t+s)x = S(t)S(s)x, \quad \forall x \in E,\ t, s \in \mathbb{R}^+$;
(2) $S(0)x = x, \quad \forall x \in E$;
(3) for each $x \in E$, the mapping $t \mapsto S(t)x$ from $\mathbb{R}^+$ into E is continuous;
(2) for each $t \in \mathbb{R}^+$, $S(t)$ is positively homogeneous and nonexpansive.

Theorem 5.3. *Let E be a uniformly convex and smooth Banach space and let $\mathcal{S} = \{S(t) : t \in \mathbb{R}^+\}$ be a positively homogeneous nonexpansive semigroup on E. Then for any $x \in E$,*

$$S_\lambda x = \frac{1}{\lambda}\int_0^\lambda S(t)x\,dt$$

converges weakly to $z_0 \in F(\mathcal{S})$ as $\lambda \to \infty$. Additionally, if the norm of E is a Fréchet differentiable, then $z_0 = \lim_{t\to\infty} R_{F(\mathcal{S})}S(t)x$, where $R_{F(\mathcal{S})}$ is the sunny generalized nonexpansive retraction of E onto $F(\mathcal{S})$.

Proof. Let $S = \mathbb{R}^+$. For any $f \in C(\mathbb{R}^+)$, define

$$\mu_\lambda(f) = \frac{1}{\lambda}\int_0^\lambda f(t)\,dt, \quad \forall \lambda \in (0,\infty).$$

Then $\{\mu_\lambda : \lambda \in (0,\infty)\}$ is an asymptotically invariant net of means on $C(\mathbb{R}^+)$; see [24, p.80]. Furthermore, we have that for any $x \in E$ and $\lambda \in (0,\infty)$,

$$T_{\mu_\lambda} x = \frac{1}{\lambda}\int_0^\lambda S(t)x\,dt.$$

Therefore, we have the desired result from Theorem 4.3. □

Theorem 5.4. *Let E be a uniformly convex and smooth Banach space and let $\mathcal{S} = \{S(t) : t \in \mathbb{R}^+\}$ be a positively homogeneous nonexpansive semigroup on E. Then for any $x \in E$,*

$$r \int_0^\infty e^{-rt} S(t)x dt$$

converges weakly to $z_0 \in F(\mathcal{S})$ as $r \to 0$ with $0 < r < \infty$. Additionally, if the norm of E is a Fréchet differentiable, then $z_0 = \lim_{t\to\infty} R_{F(\mathcal{S})} S(t)x$, where $R_{F(\mathcal{S})}$ is the sunny generalized nonexpansive retraction of E onto $F(\mathcal{S})$.

Proof. Let $S = \mathbb{R}^+$. For any $f \in C(\mathbb{R}^+)$, define

$$\mu_r(f) = r \int_0^\infty e^{-rt} f(t)dt, \quad \forall r \in (0, \infty).$$

Then $\{\mu_r : r \in (0, \infty)\}$ is an asymptotically invariant net of means on $C(\mathbb{R}^+)$; see [24, p.82]. Furthermore, we have that for any $x \in E$ and $r \in (0, \infty)$,

$$T_{\mu_r} x = r \int_0^\infty e^{-rt} S(t)x dt.$$

Therefore, we have the desired result from Theorem 4.3. □

Theorem 5.5. *Let E be a uniformly convex and smooth Banach space and let $S, T : E \to E$ be positively homogeneous nonexpansive mappings with $ST = TS$. Then for any $x \in E$,*

$$\frac{1}{n^2} \sum_{i,j=0}^{n-1} S^i T^j x$$

converges weakly to $z_0 \in F(S) \cap F(T)$ as $n \to \infty$. Additionally, if the norm of E is a Fréchet differentiable, then $z_0 = \lim_{i,j\to\infty} R_{F(S)\cap F(T)} S^i T^j x$, where $R_{F(S)\cap F(T)}$ is the sunny generalized nonexpansive retraction of E onto $F(S) \cap F(T)$.

Proof. Let $S = \{0, 1, 2, \dots\} \times \{0, 1, 2, \dots\}$ and let $\mathcal{S} = \{S^i T^j : (i, j) \in S\}$. For any $f \in B(S)$, define

$$\mu_n(f) = \frac{1}{n^2} \sum_{i,j=0}^{n-1} f(i, j), \quad \forall n \in \mathbb{N}.$$

Then $\{\mu_n : n \in \mathbb{N}\}$ is an asymptotically invariant sequence of means on $B(S)$; see [24, p.83]. Furthermore, we have that for any $x \in E$ and $n \in \mathbb{N}$,

$$T_{\mu_n} x = \frac{1}{n^2} \sum_{i,j=0}^{n-1} S^i T^j x.$$

Therefore, we have the desired result from Theorem 4.3. □

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References

[1] Ya. I. Alber, *Metric and generalized projection operators in Banach spaces: properties and applications*, Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, Dekker, New York, 1996, 15–50.

[2] J.-B. Baillon. *Un theoreme de type ergodique pour les contractions non lineaires dans un espace de Hilbert*, C. R. Acad. Sci. Paris Ser. A-B **280** (1975), 1511-1514.

[3] F. E. Browder, *Convergence theorems for sequences of nonlinear operators in Banach spaces*, Math. Z. **100** (1967), 201–225.

[4] F. E. Browder, *Nonlinear operators and nonlinear equations of evolution in Banach spaces*, Nonlinear functional analysis (Proc. Sympos. Pure Math., Vol XVIII, Part2, Chicago, III., 1968), Amer. Math. Soc., Providence, R.I., 1973, pp. 251–262.

[5] R. E. Bruck, *A simple proof of the mean ergodic theorems for nonlinear contractions in Banach spaces*, Israel J. Math. **32** (1974), 107–116.

[6] R. E. Bruck, *On the convex approximation property and the asymptotic behavior of nonlinear contractions in Banach spaces*, Israel J. Math. **38** (1981), 304–314.

[7] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge University Press, Cambridge, 1990.

[8] N. Hirano, K. Kido and W. Takahashi, *The existence of nonexpansive retractions in Banach space*, J. Math. Soc. Japan **38** (1986), 1-7.

[9] N. Hirano, K. Kido and W. Takahashi, *Asymptotic behavior of commutative semigroups of nonexpansive mappings in Banach spaces*, Nonlinear Anal. **10** (1986), 229-249.

[10] N. Hirano, K. Kido and W. Takahashi, *Nonexpansive retractions and nonlinear ergodic theorems in Banach spaces*, Nonlinear Anal. **12** (1988), 1269-1281.

[11] T. Ibaraki and W. Takahashi, *Convergence theorems for new projections in Banace spaces*, RIM Kokyuroku **1484** (2006), 150–160.

[12] T. Ibaraki and W. Takahashi, *A new projection and convergence theorems for the projections in Banach spaces*, J. Approx. Theory **149** (2007), 1–14.

[13] T. Ibaraki and W. Takahashi, *Mosco convergence of sequences of retracts of four nonlinear projections in Banach spaces*, in Nonlinear Analysis and Convex Analtsis (W. Takahashi and T. Tanaka Eds.), Yokohama Publishers, Yokohama, 2007, pp. 139–147.

[14] T. Ibaraki and W. Takahashi, *Fixed point theorems for nonlinear mappings of nonexpansive type in Banach spaces*, J. Nonlinear Convex Anal. **71** (2009), 21–32.

[15] T. Ibaraki and W. Takahashi, *Generalized nonexpansive mappings and a proximal-type algorithm in Banach spaces*, Contemp. Math. **513**, Amer. Math. Soc., Providence, RI, 2010, pp. 169–180.

[16] W. Inthakon, S. Dhompongsa and W. Takahashi, *Strong convergence theorems for maximal monotone operators and generalized nonexpansive mappings in Banach spaces*, J. Nonlinear Convex Anal. **11** (2010), 45–63.

[17] S. Itoh and W. Takahashi, *The common fixed points theory of singlevalued mappings and multi-valued mappings*, Pacific J. Math. **79** (1978), 493–508.

[18] S. Kamimura and W. Takahashi, *Strong convergence of proximal-type algorithm in a Banach space*, SIAM J. Optim. **13** (2002), 938-945.

[19] F. Kohsaka and W. Takahashi, *Generalized nonexpansive retractions and a proximal-type algorithm in Banach spaces*, J. Nonlinear Convex Anal. **8** (2007), 197-209.

[20] L.-J. Lin and W. Takahashi, *Attractive point theorem and ergodic theorems for nonlinear mappings in Hilbert space*, Taiwanese J. Math. **13** (2012), to appear.

[21] L.-J. Lin, W. Takahashi and Z.-T. Yu, *AttractivepPoint theorems and ergodic theorems for 2-generalized nonspreading mappings in Banach spaces*, J. Nonlinear Convex Anal. **14** (2013), to appear.

[22] W. Takahashi, *A nonlinear ergodic theorem for an amenable semigroup of nonexpansive mappings in a Hilbert space*, Proc. Amer. Math. Soc. **81** (1981), 253–256.

[23] W. Takahashi, *A nonlinear ergodic theorem for a reversible semigroup of nonexpansive mappings in a Hilbert space*, Proc. Amer. Math. Soc. **97** (1986), 55–58.

[24] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, 2000.

[25] W. Takahashi, *Convex Analysis and Approximation of Fixed Points*, Yokohama Publishers, Yokohama, 2000.

[26] W. Takahashi, *Introduction to Nonlinear and Convex Analysis*, Yokohama Publishers, Yokohama, 2009.

[27] W. Takahashi, N.-C. Wong and J.-C. Yao, *Nonlinear ergodic theorem for positively homogeneous nonexpansive mappings in Banach spaces*, to appear.

[28] W. Takahashi and J.-C. Yao, *Fixed point theorems and ergodic theorems for nonlinear mappings in Hilbert spaces*, Taiwanese J. Math. **15** (2011), 457–472.

[29] W. Takahashi and J.-C. Yao, *Weak and strong convergence theorems for positively homogenuous nonexpansive mappings in Banach spaces*, Taiwanese J. Math. **15** (2011), 961–980.

[30] H. K. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Anal. **16** (1991), 1127-1138.

[31] K. Yosida, *Mean ergodic theorem in Banach spaces*, Proc. Imp. Acad. Tokyo **14** (1938), 292–294.

(Wataru Takahashi) Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, Tokyo 152-8552, Japan

E-mail address: `wataru@is.titech.ac.jp`

(Ngai-Ching Wong) Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung 80424, Taiwan

E-mail address: `wong@math.nsysu.edu.tw`

(Jen-Chih Yao) Center for General Education, Kaohsiung Medical University, Kaohsiung 80702, Taiwan

E-mail address: `yaojc@kmu.edu.tw`