

On the decomposition into Discrete, Type II and Type III C^* -algebras

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Abstract

We obtained a “decomposition scheme” of C^* -algebras. We show that the classes of discrete C^* -algebras (as defined by Peligard and Zsidó), type II C^* -algebras and type III C^* -algebras (both defined by Cuntz and Pedersen) form a good framework to “classify” C^* -algebras. In particular, we found that these classes are closed under strong Morita equivalence, hereditary C^* -subalgebras as well as taking “essential extension” and “normal quotient”. Furthermore, there exist the largest discrete finite ideal $A_{d,1}$, the largest discrete essentially infinite ideal $A_{d,\infty}$, the largest type II finite ideal $A_{II,1}$, the largest type II essentially infinite ideal $A_{II,\infty}$, and the largest type III ideal A_{III} of any C^* -algebra A such that $A_{d,1} + A_{d,\infty} + A_{II,1} + A_{II,\infty} + A_{III}$ is an essential ideal of A . This “decomposition” extends the corresponding one for W^* -algebras.

We also give a closer look at C^* -algebras with Hausdorff primitive ideal spaces, AW^* -algebras as well as local multiplier algebras of C^* -algebras. We find that these algebras can be decomposed into continuous fields of prime C^* -algebras over a locally compact Hausdorff space, with each fiber being non-zero and of one of the five types mentioned above.

1. Introduction

Murray and von Neumann defined in [21] (see also [14, 20]) three types of W^* -algebras according to the abelianness and finiteness properties of their projections. Since a C^* -algebra needs not have any non-zero projection, a similar classification for C^* -algebras cannot go verbatim. Cuntz and Pedersen defined (in [10]) type II and type III C^* -algebras according to certain abelianness and finiteness properties of their positive elements. They used these, together with type I C^* -algebras, to obtain a classification scheme that captures some features of the W^* -algebra counterpart.

In [25], we use open projections in A^{**} to obtain another classification scheme for C^* -algebras parallel to the one of Murray and von Neumann. Meanwhile, we also observe that discrete C^* -algebras (as defined by Peligard and Zsidó), type II C^* -algebras and type III C^* -algebras also form a good classification scheme, and some of the results in

[25] have their counterparts in this scheme. We develop a more comprehensive theory in the current paper. Note that the overlap materials between the current paper and [25] is not significant. Actually, only the arguments of Lemma 3.2 and Theorem 5.3 have overlap with the corresponding results in [25], and some of the statements that have correspondences in [25] have different proofs here. Moreover, most of the results in this paper are completely new, e.g., all the results in Subsections 3.3 and 3.5, as well as Sections 4 and 6 have no correspondences in [25] at all. Conversely, more than half of the results in [25] has no correspondence in the current paper neither.

Let us first recall the notion of discrete, type II and type III C^* -algebras. A positive element $x \in A_+$ is said to be *abelian* (in A) if \overline{xAx} is an abelian algebra (see [26, p.191]). As in the literature, A is said to be *anti-liminary* if there is no non-zero abelian element in A_+ . Following Cuntz and Pedersen ([10]), for $x, y \in A_+$, we write $x \sim y$ if there is a sequence $\{z_k\}_{k \in \mathbb{N}}$ in A such that $x = \sum_{k=1}^{\infty} z_k^* z_k$ and $y = \sum_{k=1}^{\infty} z_k z_k^*$ (in norm). A positive element x is said to be *finite* in A if one has $y = x$ whenever $y \in A_+$ satisfying $0 \leq y \leq x$ and $y \sim x$. It can be shown that a projection in a W^* -algebra is finite as a projection in the sense of Murray-von Neumann if and only if it is finite as a positive element in the sense of Cuntz-Pedersen (see e.g. the proof of Proposition 3.9(d) in Section 3).

DEFINITION 1.1. *A C^* -algebra A is said to be*

- (a) *discrete if every non-zero element in A_+ dominates a non-zero abelian element in A ([27, Definition 2.1]).*
- (b) *finite (respectively, semi-finite) if every non-zero element in A_+ is finite in A (respectively, dominates a non-zero positive finite element in A) ([10, p.140]).*
- (c) *of type II if it has no non-zero abelian element and it is semi-finite ([10, p.149]).*
- (d) *of type III if it has no non-zero finite element ([10, p.149]).*

It follows from the definition that any C^* -subalgebra of a finite C^* -algebra is finite. A W^* -algebra M is a type I, type II or type III W^* -algebra if and only if M is a discrete, type II or type III C^* -algebra (see Proposition 3.9(d)). We also recall that a type I C^* -algebra is discrete but the converse is not true (e.g. $\mathcal{B}(\ell^2)$). In fact, we obtained in [25, Proposition 4.3(a)] the following relation:

A C^ -algebra A is of type I if and only if A as well as all the primitive quotient C^* -algebras of A are discrete.*

It is not hard to see that A is discrete if and only if every non-zero hereditary C^* -subalgebra of A contains a non-zero abelian hereditary C^* -algebra, or equivalently, a non-zero abelian element (see [27, Theorem 2.3]). We obtain alternative looks of discrete, type II and type III C^* -algebras in the following theorem. Below, by a *normal ideal*, we mean the set J^\perp of annihilators of another ideal J . Observe that an ideal in a W^* -algebra is a normal ideal if and only if it is weak- $*$ -closed. This theorem summarizes Theorem 3.3, Remark 3.4 and Corollary 3.7.

THEOREM 1.2. *Let A be a C^* -algebra.*

- (a) *A is discrete if and only if every non-zero closed ideal (or equivalently, every non-zero “normal ideal”) of A contains a non-zero abelian hereditary C^* -subalgebra.*
- (b) *A is of type II if and only if there is no non-zero abelian hereditary C^* -subalgebra of A and every non-zero closed ideal (or equivalently, every non-zero “normal ideal”) of A contains a non-zero finite hereditary C^* -subalgebra.*
- (c) *A is of type III if and only if it contains no non-zero finite hereditary C^* -subalgebra.*

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We will show that the above three types of C^* -algebras are invariance under strong Morita equivalence and under essential extension. Here, a C^* -algebra B is an *essential extension* of a C^* -algebra A if A is an essential ideal of B (in the sense that $I \cap A \neq (0)$ for every non-zero closed ideal I of B).

THEOREM 1.3. *Let A and B be C^* -algebras.*

- (a) *Assume that B is either strongly Morita equivalent to A or is an essential extension of A . Then A is discrete (respectively, of type II, of type III or semi-finite) if and only if B is so (see Theorem 3.3(c) and Proposition 3.5(b)).*
- (b) *If A is discrete (respectively, of type II, of type III or semi-finite), B is a hereditary C^* -subalgebra of A , and J is a “normal ideal” of A , then both B and A/J are discrete (respectively, of type II, of type III or semi-finite) (see Proposition 3.5(b)ℰ(c)).*
- (c) *The class of discrete (respectively, semi-finite or type II) C^* -algebras is the smallest class that contains all abelian (respectively, finite or anti-liminary finite) C^* -algebras and is closed under strong Morita equivalence and essential extension (see Theorem 3.8).*
- (d) *A is of type III if and only if for every hereditary C^* -subalgebra $B \subseteq A$, there is an essential closed ideal of B with the bidual being a properly infinite W^* -algebra (see Corollary 4.7).*

We say that a C^* -algebra is *essentially infinite* if it does not contain any non-zero finite ideal. By Proposition 4.6(a), a C^* -algebra is essentially infinite if there is an essential closed ideal with its bidual being a properly infinite W^* -algebra. Thus, part (d) above tells us that a C^* -algebra is of type III if and only if all of its hereditary C^* -subalgebras are essentially infinite.

We obtain the following decomposition scheme of C^* -algebras. In this theorem, a *normal quotient* is a quotient by a normal ideal, and the term “universal” is used to indicate the fact that it is the “biggest such quotient”.

THEOREM 1.4. *Let A, B be C^* -algebras.*

- (a) *There exists the largest discrete (respectively, semi-finite, type II and type III) hereditary C^* -subalgebra A_d (respectively, A_{sf} , A_{II} and A_{III}) of A (see Theorem 5.3(a)).*
- (b) *A_d , A_{sf} , A_{II} and A_{III} are ideals of A such that A_d , A_{II} and A_{III} are disjoint and $A_{III} \cap A_{sf} = (0)$ (see Theorem 5.3(a)).*
- (c) *$A_d + A_{II} + A_{III}$ is an essential ideal of A , and $A_d + A_{II}$ is an essential ideal of A_{sf} (see Theorem 5.3(b)).*
- (d) *A/A_d (respectively, $A/(A_{II} + A_{III})^{\perp\perp}$, A/A_{III} and A/A_{sf}) is the universal anti-liminary (respectively, discrete, semi-finite and type III) “normal quotient” of A (see Corollary 5.5(a) and Theorem 5.3(c)ℰ(d)).*
- (e) *If A is semi-finite, then A/A_{II} (respectively, A/A_d) is the universal discrete (respectively, type II) “normal quotient” of A (see Corollary 5.5(b)).*
- (f) *If B is a hereditary C^* -subalgebra of A , then $B_d = A_d \cap B$, $B_{sf} = A_{sf} \cap B$, $B_{II} = A_{II} \cap B$ and $B_{III} = A_{III} \cap B$ (see Proposition 5.6(a)).*
- (g) *If J is an essential closed ideal of A , then $A_{\#} = \{x \in A : xJ \subseteq J_{\#}\}$, for $\# = d, sf, II, III$ (see Proposition 5.6(b)).*
- (h) *If A and B are strongly Morita equivalent, then $A_{\#}$ and $B_{\#}$ (for $\# = d, sf, II, III$) corresponds to each other, under the canonical bijection between ideals of A and ideals of B given by the imprimitivity bimodule (see Corollary 5.4).*
- (i) *There exist the largest finite ideal A_1 and the largest essentially infinite ideal A_{∞} of*

A. One has $A_1 \cap A_\infty = (0)$ and $A_1 + A_\infty$ is an essential ideal of A (see Remark 4.3, Lemma 4.4 and Theorem 4.5(b)).

(j) A/A_1 is the universal essentially infinite “normal quotient” of A , and A/A_∞ is the universal finite “normal quotient” of A (see Theorem 4.5(a)&(b)).

(k) If A is discrete (respectively, semi-finite or of type II), then so are A/A_1 and A/A_∞ (see Theorem 4.5(c)).

(l) Let $J \subseteq A$ be a closed ideal, and $q : A \rightarrow A/J^\perp$ be the quotient map. Then $J_1 = J \cap A_1$, $J_\infty = J \cap A_\infty$, $q(A)_1 = q(A_1)^{\perp\perp}$ and $q(A)_\infty = q(A_\infty)^{\perp\perp}$ (see Theorem 4.5(d)&(e)).

(m) $A_{d,1} := A_d \cap A_1$ (respectively, $A_{II,1} := A_{II} \cap A_1$) is the largest discrete finite (respectively, type II finite) ideal of A . On the other hand, $A_{d,\infty} := A_d \cap A_\infty$ (respectively, $A_{II,\infty} := A_{II} \cap A_\infty$) is the largest discrete essentially infinite (respectively, type II essentially infinite) ideal of A . One has $A_{d,1} + A_{d,\infty} + A_{II,1} + A_{II,\infty} + A_{III}$ being an essential ideal of A (see Corollary 5.7).

(n) If A is a W^* -algebra, then A_d , A_{II} and A_{III} are respectively, the type I, the type II and the type III W^* -summands of A . Furthermore, A_1 and A_∞ are respectively, the finite part and the infinite part of A (see Theorem 5.3(e)).

From these, for any C^* -algebra A , one has

$$A_{d,1} \oplus A_{d,\infty} \oplus A_{II,1} \oplus A_{II,\infty} \oplus A_{III} \subseteq A \subseteq M(A_{d,1}) \oplus M(A_{d,\infty}) \oplus M(A_{II,1}) \oplus M(A_{II,\infty}) \oplus M(A_{III}),$$

and the C^* -algebras $M(A_{d,1})$, $M(A_{d,\infty})$, $M(A_{II,1})$, $M(A_{II,\infty})$ and $M(A_{III})$ are also discrete finite, discrete essentially infinite, type II finite, type II essentially infinite and type III, respectively.

As seen in Theorems 1.3(b) and 1.4(c), normal ideals play an important role in the structure theory of C^* -algebras. Hence, prime C^* -algebras (i.e., C^* -algebra containing no non-zero normal ideal) can be considered as the counterpart of factors in the C^* -world. The following tells us that every prime C^* -algebra is of one of the five types as in the above “decomposition”.

COROLLARY 1.5. *Any prime C^* -algebra is of one of the five types: discrete finite, discrete essentially infinite, type II finite, type II essentially infinite, or type III (see Proposition 6.1).*

EXAMPLE 1.6. *(a) Discrete finite prime C^* -algebras are exactly matrix algebras (see Proposition 6.2(b)).*

(b) Discrete essentially infinite prime C^ -algebras are those algebras that contain $\mathcal{K}(H)$ as an essential ideal for some infinite dimensional Hilbert space H (see Proposition 6.2(a)). A concrete example is the unitalization of $\mathcal{K}(\ell^2)$.*

(c) For a countable ICC group Γ , its reduced group C^ -algebra $C_r^*(\Gamma)$ is a type II finite prime C^* -algebra (see Example 6.3(b)).*

(d) A simple non-type I AF-algebra that admits no tracial state is a type II essentially infinite prime C^ -algebra (see Example 6.3(c)).*

(e) The Calkin algebra is a type III prime C^ -algebra (see Example 6.3(d)).*

One consequence of the above is the following corollary.

COROLLARY 1.7. *(a) If the primitive ideal space $\text{Prim}(A)$ of A is Hausdorff, or A is an AW^* -algebra, or A is the local multiplier algebra of a C^* -algebra, then A can be represented as the algebra of C_0 -sections of an (F) -Banach bundle over an open dense*

subset Ω of $\text{Prim}(ZM(A))$ with each fiber being a non-zero prime C^* -algebra of one of the five types. If, in addition, A is discrete (respectively, essentially infinite), there is a dense subset of Ω on which each fiber is discrete (respectively, essentially infinite) (see Corollary 6.11).

(b) If the primitive ideal space of A is extremally disconnected (in particular, when A is AW^* -algebra), then $A = A_{\text{d},1} \oplus A_{\text{d},\infty} \oplus A_{\text{II},1} \oplus A_{\text{II},\infty} \oplus A_{\text{III}}$ (see Proposition 6.6(a)).

2. Notations and preliminaries

Throughout this article, A and B are C^* -algebras, $Z(A)$ is the center of A , A_+ is the positive cone of A , $\text{Proj}^1(A)$ is the set of non-zero projections in A , A^{**} is the bidual of A (equipped with the canonical W^* -algebra structure), and $M(A) \subseteq A^{**}$ is the multiplier algebra of A . As usual, we write $ZM(A)$ for $Z(M(A))$. For any subsets $X, Y, Z \subseteq A$, we write XY (respectively, XYZ) for the linear span of $\{xy : x \in X, y \in Y\}$ (respectively, $\{xyz : x \in X, y \in Y, z \in Z\}$). We also write

$$X^\perp := \{a \in A : aX = \{0\} = Xa\}.$$

Notice that if I is a closed ideal of A , then I^\perp is a closed ideal and $I^\perp = \{a \in A : aI = (0)\}$ (since any closed ideal is $*$ -invariant).

A C^* -subalgebra B of A is said to be *hereditary* if B_+ is a *hereditary subcone* of A_+ in the following sense:

$$B_+ = \{a \in A_+ : a \leq x, \text{ for some } x \in B_+\}. \quad (2.1)$$

Clearly, the intersection of two hereditary C^* -subalgebras is hereditary (we use the convention that the zero subalgebra is hereditary). It is well-known that (see e.g. [19, Theorem 3.2.2]) a $*$ -invariant closed subspace $B \subseteq A$ is a hereditary C^* -subalgebra if and only if $BAB \subseteq B$. Hence, any closed ideal is a hereditary C^* -subalgebra. Furthermore, if $D \subseteq A$ is any C^* -subalgebra, then D^\perp is a hereditary C^* -subalgebra. Moreover, if D is a hereditary C^* -subalgebra of B and B is a hereditary C^* -subalgebra of A , then D is a hereditary C^* -subalgebra of A (in fact, if $a \in A_+$ satisfying $a \leq x \in D_+$, then $a \in B_+$ because D_+ is a subset of the hereditary subcone B_+ of A_+ , and hence $a \in D_+$ because D_+ is a hereditary subcone of B_+).

Let us also give a brief account on open projections, which was introduced by Akemann in [1] (see also [2, 18, 26, 27, 30] for more information). A projection $p \in A^{**}$ is called an *open projection of A* if there is an increasing net $\{a_i\}_{i \in \mathcal{I}}$ of positive elements in A_+ with $\lim_i a_i = p$ in the $\sigma(A^{**}, A^*)$ -topology. A projection $q \in A^{**}$ is said to be *closed* if $1 - q$ is open. We use $\text{OP}^1(A)$ to denote the collection of non-zero open projections of A . In the case when A is commutative, open projections of A are exactly the images (in A^{**}) of characteristic functions of open subsets of the spectrum of A .

By [26, Proposition 3.11.9], we know that a projection $p \in A^{**}$ is open if and only if it lies in the norm-closure of the set $(\tilde{A}_{\text{sa}})^m$ of $\sigma(A^{**}, A^*)$ -limits of increasing nets in the self-adjoint part \tilde{A}_{sa} of the unitalization \tilde{A} of A . Furthermore, it was shown in [26, Theorem 3.12.9] that

$$M(A)_{\text{sa}} = (\tilde{A}_{\text{sa}})^m \cap (\tilde{A}_{\text{sa}})_m,$$

where $M(A)_{\text{sa}}$ is the self-adjoint part of the multiplier algebra $M(A)$ of A and $(\tilde{A}_{\text{sa}})_m$ is the set of $\sigma(A^{**}, A^*)$ -limits of decreasing nets in \tilde{A}_{sa} . Thus, every element in $\text{Proj}^1(M(A)) \cup \{0\}$ is both an open projection and a closed projection of A . Conversely, if a projection

$p \in A^{**}$ is both open and closed, then it is the $\sigma(A^{**}, A^*)$ -limit of an increasing in A_+ as well as a $\sigma(A^{**}, A^*)$ -limit of a decreasing in \tilde{A}_{sa} , which means that $p \in M(A)_{\text{sa}}$.

For every projection (respectively, central projection) e in A^{**} the C^* -subalgebra,

$$\text{her}_A(e) := eA^{**}e \cap A$$

is a hereditary C^* -subalgebra (respectively, a closed ideal). Let us list the following two facts from [26] (which may be used implicitly throughout this article):

- O1). The assignment: $e \mapsto \text{her}_A(e)$ is a bijection from the set of open (respectively, central open) projections on A onto the set of hereditary C^* -subalgebras (respectively, closed ideals) of A (see [26, Remark 3.11.10]).
- O2). A projection $e \in \text{Proj}^1(A^{**})$ is an open projection of A if and only if e belongs to the $\sigma(A^{**}, A^*)$ -closure of $\text{her}_A(e)$ (see [26, Proposition 3.11.9]). In this case, any approximate unit in $\text{her}_A(e)$ will $\sigma(A^{**}, A^*)$ -converge to e (see the proof of [26, Proposition 3.11.9]).

The following proposition contains some well-known facts about open projections and hereditary C^* -subalgebras.

PROPOSITION 2.1. *Suppose that A is a C^* -algebra and $e, p, q \in \text{OP}^1(A)$ with $p, q \in Z(A^{**})$.*

- (a) *If $z(e)$ is the central cover of e in A^{**} , then $z(e) \in \text{OP}^1(A)$ and $\text{her}_A(z(e))$ is the closed ideal, $A \text{her}_A(e)A$, generated by $\text{her}_A(e)$.*
- (b) *$ep \in \text{OP}^1(A)$ and $\text{her}_A(ep) = \text{her}_A(e) \cap \text{her}_A(p) = \text{her}_A(e) \text{her}_A(p) \text{her}_A(e)$.*
- (c) *If $\text{her}_A(e) \subseteq \text{her}_A(p) + \text{her}_A(q)$, then one has $\text{her}_A(e) \cap \text{her}_A(p) \neq (0)$ or $\text{her}_A(e) \cap \text{her}_A(q) \neq (0)$.*

Proof. (a) Set $B := \text{her}_A(e)$ and $I := \overline{ABA}$. Denote by $z_0 \in \text{OP}^1(A) \cap Z(A^{**})$ the projection with $I = \text{her}_A(z_0) = z_0A^{**} \cap A$. As $B \subseteq I$, we know that $e \leq z_0$. If $z \in \text{Proj}^1(A^{**}) \cap Z(A^{**})$ satisfying $e \leq z$, then $B \subseteq zA^{**}$ which implies $I \subseteq zA^{**}$ and hence $z_0 \leq z$ (see (O2)). These show that $z_0 = z(e)$.

(b) Suppose that $\{x_i\}_{i \in \mathfrak{J}}$ and $\{y_j\}_{j \in \mathfrak{J}}$ are increasing nets in $\text{her}_A(e)_+$ and $\text{her}_A(p)_+$ that $\sigma(A^{**}, A^*)$ -converge to e and p , respectively. For a fixed $i \in \mathfrak{J}$, it follows from

$$epx_i^{1/2}y_jx_i^{1/2} = x_i^{1/2}y_jx_i^{1/2} = x_i^{1/2}y_jx_i^{1/2}ep$$

(because $p \in Z(A^{**})$) that the increasing net $\{x_i^{1/2}y_jx_i^{1/2}\}_{j \in \mathfrak{J}}$ lies in $\text{her}_A(ep)$, and it will $\sigma(A^{**}, A^*)$ -converge to $x_i p = x_i^{1/2} p x_i^{1/2}$. Consequently, ep , being the $\sigma(A^{**}, A^*)$ -limit of $\{x_i p\}_{i \in \mathfrak{J}}$, lies inside the $\sigma(A^{**}, A^*)$ -closure of $\text{her}_A(ep)$, and Statement (O2) implies that ep is an open projection of A .

Let D be the hereditary C^* -subalgebra $\text{her}_A(e) \cap \text{her}_A(p)$ of A and $f \in \text{OP}^1(A)$ such that $D = \text{her}_A(f)$. Since p is central, $ep = e \wedge p$, and we know that $\text{her}_A(ep) \subseteq D$ (and so, $ep \leq f$). Conversely, as $f \leq e \wedge p$, we know that $D \subseteq \text{her}_A(ep)$.

On the other hand, we denote $D_0 := \text{her}_A(e) \text{her}_A(p) \text{her}_A(e)$. As every element in D is a product of three elements in D_0 , we know that $D \subseteq D_0$. Conversely, as $\text{her}_A(p)$ is an ideal and $\text{her}_A(e)$ is hereditary, we know that $D_0 \subseteq D$.

(c) Since $\text{her}_A(p) + \text{her}_A(q) \subseteq \text{her}_A(p + q - pq)$, we know that $e \leq p + q - pq$. Suppose on the contrary that

$$\text{her}_A(e) \cap \text{her}_A(p) = (0) = \text{her}_A(e) \cap \text{her}_A(q).$$

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By part (b), we know that $ep = eq = 0$. Hence, we will arrive at the contradiction that $e = e(p + q - pq) = 0$. \square

REMARK 2.2. Suppose that $e, f \in \text{OP}^1(A)$ with $e \leq f$ and $B, D \subseteq A$ are hereditary C^* -subalgebras.

(a) By [25, Remark 2.2(b)], one may identify

$$\text{OP}^1(\text{her}_A(f)) = \{r \in \text{OP}^1(A) : r \leq f\}.$$

Let \bar{e}^f be the closure of e in $\text{her}_A(f)$ (see [1, Definition II.11]); i.e., \bar{e}^f is the smallest closed projection of $\text{her}_A(f)$ that dominates e . Consequently, $f - \bar{e}^f$ is the largest element in

$$\{r \in \text{OP}^1(A) \cup \{0\} : r \leq f \text{ and } re = 0\}$$

(notice that f is the identity of $\text{her}_A(f)$). From which, one can obtain

$$\text{her}_A(e)^\perp = \text{her}_A(1 - \bar{e}^1). \quad (2.2)$$

Indeed, we see from Statement (O2) that $\text{her}_A(e)^\perp = \{x \in A : ex = 0 = xe\}$. So, if $p \in \text{OP}^1(A)$, then $\text{her}_A(p) \subseteq \text{her}_A(e)^\perp$ if and only if $pe = 0$, which is equivalent to $p \leq 1 - \bar{e}^1$. Thus, Relation (2.2) follows from the description of $f - \bar{e}^f$ in the above and the fact that $\text{her}_A(e)^\perp$ is a hereditary C^* -subalgebra of A .

(b) D is said to be disjoint from B if $BD = (0)$. Clearly, D is disjoint from B if and only if $D \subseteq B^\perp$. As in [30], we say that B is essential in A if there is no non-zero hereditary C^* -subalgebra of A that is disjoint from B (or equivalently, $B^\perp = (0)$).

We say that e is dense in f if $\bar{e}^f = f$. Relation (2.2) (applying to the case when A is replaced by $\text{her}_A(f)$) tells us that e is dense in f if and only if $\text{her}_A(e)$ is essential in $\text{her}_A(f)$.

(c) The ideal $I := \overline{ABA}$ is essential in A as a hereditary C^* -subalgebra if and only if for every non-zero closed ideal J of A , one has $BJ \neq (0)$ (or equivalently, B^\perp does not contain a non-zero ideal of A). In fact, if I is essential in A but there is a non-zero closed ideal $J \subseteq A$ with $BJ = (0)$, then

$$IJ = \overline{ABA}J \subseteq \overline{ABJ} = (0),$$

which is a contradiction. Conversely, suppose that B satisfies the said condition, and $D \subseteq A$ is a non-zero hereditary C^* -subalgebra. If $J := \overline{ADA}$, then $BJ \neq (0)$, and hence

$$(0) \neq ABADA \subseteq IDA$$

(by considering an approximate identity in A), which implies that $ID \neq (0)$ as required.

(d) Let E be another C^* -algebra. Then A contains (a $*$ -isomorphic copy of) E as an essential ideal if and only if there is an injective $*$ -homomorphism $\varphi : A \rightarrow M(E)$ such that $E \subseteq \varphi(A)$.

In fact, if A contains E as an essential ideal, then by [19, Theorem 3.1.8], there is an injective $*$ -homomorphism from A to $M(E)$ extending the inclusion map $E \subseteq M(E)$. Conversely, suppose that such a map φ exists. Then E is clearly an ideal of $\varphi(A)$. Moreover, as E is an essential ideal of $M(E)$ (see e.g. [19, p.82]), one has

$$\{a \in \varphi(A) : aE = \{0\} = Ea\} \subseteq \{x \in M(E) : xE = \{0\} = Ex\} = \{0\}.$$

This shows that E is an essential ideal of $\varphi(A)$ (see part (b) above).

Next, we give a brief account for the notion of strong Morita equivalence. The readers may consult some standard literature on this subject (e.g., [16]) for more information. Let X be a right A -module equipped with an “ A -valued inner product”, i.e. a map $\langle \cdot, \cdot \rangle_A : X \times X \rightarrow A$ such that it is A -linear in the second variable and it satisfies

$$\langle x, y \rangle_A^* = \langle y, x \rangle_A, \quad \langle x, x \rangle_A \geq 0 \quad (x, y \in X)$$

as well as $\{x \in X : \langle x, x \rangle_A = 0\} = \{0\}$. If X is complete under the norm defined by

$$\|z\| := \|\langle z, z \rangle_A\|^{1/2} \quad (z \in X),$$

then it is called a *Hilbert A -module*. A Hilbert A -module is said to be *full* if the linear span of $\{\langle x, y \rangle_A : x, y \in X\}$ is dense in A . For any $x, y \in X$, we define $\theta_{x,y} : X \rightarrow X$ by

$$\theta_{x,y}(z) := x\langle y, z \rangle_A \quad (z \in X).$$

The closure, $\mathcal{K}_A(X)$, of the linear span of $\{\theta_{x,y} : x, y \in X\}$ in the Banach space of bounded linear operators on X is naturally a C^* -algebra with the involution satisfying $\theta_{x,y}^* := \theta_{y,x}$. If B is another C^* -algebra such that there exists a full Hilbert A -module X with $B \cong \mathcal{K}_A(X)$, then we say that A and B are *strongly Morita equivalent*.

REMARK 2.3. *Suppose that X is a full Hilbert A -module and $B = \mathcal{K}_A(X)$.*

(a) *Let \tilde{X} be the conjugate vector space of X with the canonical conjugate linear bijection being denoted by $x \mapsto x^\sim$. One may equip \tilde{X} with a Hilbert B -module structure as follows:*

$$y^\sim b = (b^* y)^\sim \quad \text{and} \quad \langle x^\sim, y^\sim \rangle_B := \theta_{x,y} \quad (x, y \in X; b \in B).$$

For every $a \in A$, the map defined by $\Psi(a)(x^\sim) := (xa^)^\sim$ belongs to $\mathcal{K}_B(\tilde{X})$, and Ψ is a $*$ -isomorphism from A onto $\mathcal{K}_B(\tilde{X})$. In fact, Ψ is clearly a linear homomorphism from A to the algebra of bounded linear operators on \tilde{X} . It is easy to see that $\Psi(\langle x, y \rangle_A) = \theta_{x^\sim, y^\sim}$ ($x, y \in X$). This implies that $\Psi(A) \subseteq \mathcal{K}_B(\tilde{X})$ (because X is full) and that $\Psi(A)$ is dense in $\mathcal{K}_B(\tilde{X})$. Furthermore, Ψ preserves the adjoint since*

$$\langle \Psi(a^*)(x^\sim), y^\sim \rangle = \langle (xa^*)^\sim, y^\sim \rangle = \theta_{xa^*, y^\sim} = \theta_{x, ya^*} = \langle x^\sim, \Psi(a)y^\sim \rangle \quad (x, y \in X; a \in A).$$

Hence, $\Psi : A \rightarrow \mathcal{K}_B(\tilde{X})$ is a $$ -homomorphism, which implies that its range is closed and thus it is also surjective. Finally, if $\Psi(a) = 0$, then $a\langle x, y \rangle_A = \langle xa^*, ya^* \rangle = 0$ for any $x, y \in X$ (because $(xa^*)^\sim = (ya^*)^\sim = 0$) and the fullness of X implies that $aAa^* = \{0\}$, and hence $a = 0$.*

(b) *One may equip $D := \begin{pmatrix} A & \tilde{X} \\ X & B \end{pmatrix}$ with a canonical C^* -algebra structure (which is*

called the linking algebra). Consider $e := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M(D)$. Then $A \cong eDe = \text{her}_D(e)$

and $B \cong (1-e)D(1-e) = \text{her}_D(1-e)$. Moreover, as the closed ideal of D generated by A is the whole algebra D and the same is true for B , we know from Proposition 2.1(a) that $z(e) = 1 = z(1-e)$. As a more explicit reference, the readers may consult, e.g. [5, Theorem II.7.6.9].

(c) *For every closed ideal I of A , the subset $X_I := \{xa : x \in X; a \in I\} \subseteq X$ coincides with its linear span XI and is a full Hilbert I -module. Moreover,*

$$\mathcal{K}_{A,I}(X) := \overline{\text{span}\{\theta_{x,y} : x, y \in XI\}}$$

is an ideal of $B = \mathcal{K}_A(X)$ that can be identified with $\mathcal{K}_I(XI)$.

In fact, if F is the closure of XI , then clearly F is a Hilbert I -module containing X_I ,

and it follows from the Cohen factorization theorem (see e.g. Theorem 1.11.10 in [7, p.61]) that elements in F are of the form ya for some $y \in F \subseteq X$ and $a \in I$, which gives the required inclusion $F \subseteq XI$. On the other hand, since any element in I is a product of three elements in I , we know from fullness of X that XI is a full Hilbert I -module.

It is clear that $\mathcal{K}_{A,I}(X)$ is an ideal of B . Moreover, one has

$$T(X) \subseteq XI \quad (T \in \mathcal{K}_{A,I}(X)),$$

and it is easy to check that $T \mapsto T|_{XI}$ is a surjective $*$ -homomorphism from $\mathcal{K}_{A,I}(X)$ onto $\mathcal{K}_I(XI)$. Suppose that $T|_{XI} = 0$. If $x \in X$ and $\{u_i\}_{i \in \mathbb{N}}$ is an approximate unit of I , then

$$0 = T(xu_i) = T(x)u_i \rightarrow T(x) \in XI,$$

which implies that $T = 0$.

Furthermore, part (a) ensures that the map $I \mapsto \mathcal{K}_I(XI)$ is a bijection from the collection of closed ideals of A onto that of B .

(d) Suppose that C is a hereditary C^* -subalgebra of A such that $\overline{ACA} = A$. Then C is strongly Morita equivalent to A . In fact, if one takes $X := \overline{CA}$ and equips it with the canonical Hilbert A -module structure, then $\theta_{x_1 a_1, x_2 a_2} \mapsto x_1 a_1 a_2^* x_2$ induces a $*$ -isomorphism from $\mathcal{K}_A(X)$ onto C , because $\theta_{x_1 a_1, x_2 a_2}(yb) = x_1 a_1 a_2^* x_2 y b$ ($x_1, x_2, y \in C; a_1, a_2, b \in A$). In particular, if A is a separable simple C^* -algebra, then for any non-zero hereditary C^* -subalgebra $D \subseteq A$, one has $D \otimes \mathcal{K}(\ell^2) \cong A \otimes \mathcal{K}(\ell^2)$.

3. Two new looks of discreteness, type II and type III

Let us start the main content of this paper with the following simple lemma, which is implicitly included in [10]. Since some of them are not explicitly stated there, we will give a full account here.

LEMMA 3.1. Let $B \subseteq A$ be a non-zero hereditary C^* -subalgebra and \mathcal{F}^A be the set of all non-zero positive elements that are finite in A .

- (a) If $b \in A$ satisfying $bb^*, b^*b \in B$, then $b \in B$.
- (b) $\mathcal{F}^B = \mathcal{F}^A \cap B$.
- (c) If $x \in \mathcal{F}^A$ and $\epsilon \in (0, \|x\|)$, then $\overline{(x - \epsilon)_+ A (x - \epsilon)_+}$ is a finite C^* -algebra.
- (d) If $x \in \mathcal{F}^A$ such that 0 is an isolated point of $\sigma(x) \cup \{0\}$ (in particular, if x is a projection), then \overline{xAx} is a finite C^* -algebra.

Proof. (a) Let $e \in \text{OP}^1(A)$ with $B := \text{her}_A(e)$. As $b^*b \in \text{her}_A(e)$, by considering the polar decomposition of b , we see that $be = b$. Similarly, we have $eb = b$. Hence, $b = ebe \in \text{her}_A(e)$.

(b) It is obvious that $\mathcal{F}^A \cap B \subseteq \mathcal{F}^B$. Conversely, suppose that $x \in \mathcal{F}^B$. Consider $y \in A_+$ and a sequence $\{z_k\}_{k \in \mathbb{N}}$ in A such that $y \leq x$, $y = \sum_{k=1}^{\infty} z_k z_k^*$ and $x = \sum_{k=1}^{\infty} z_k^* z_k$. Since B_+ is a hereditary subcone of A_+ (see (2.1)), we have $y \in B_+$ and $z_k^* z_k, z_k z_k^* \in B_+$ ($k \in \mathbb{N}$). By part (a), we know that $z_k \in B$. Now, $x \in \mathcal{F}^B$ gives $y = x$.

(c) Let $D := \overline{(x - \epsilon)_+ A (x - \epsilon)_+}$ and set

$$\mathcal{F}_0 := \{a \in A_+ \setminus \{0\} : a = ay \text{ for some } y \in \mathcal{F}^A\}.$$

By [10, Lemma 4.1], one has $\mathcal{F}_0 \subseteq \mathcal{F}^A$. Consider $f(t) := \max\{t - \epsilon, 0\}$ ($t \in \sigma(x)$). There exists $g \in C(\sigma(x))_+$ as well as $\lambda > 0$ satisfying $g(t)f(t) = f(t)$ and $g(t) \leq \lambda t$ ($t \in \sigma(x)$).

Hence, $g(x) \leq \lambda x$, and this gives $g(x) \in \mathcal{F}_A$ (because of [10, Lemma 4.1]). Moreover, we know from $f(x) = f(x)g(x)$ that for any $z \in D_+$, one has $zg(x) = z$, which implies

$$z \in \mathcal{F}_0 \cap D \subseteq \mathcal{F}^A \cap D \subseteq \mathcal{F}^D.$$

Consequently, D is a finite C^* -algebra.

(d) As 0 is an isolated point of $\sigma(x) \cup \{0\}$, one can find $g \in C(\sigma(x) \cup \{0\})_+$ and $\lambda > 0$ such that $g(t)t = t$ and $g(t) \leq \lambda t$ ($t \in \sigma(x) \cup \{0\}$). The same argument as in part (c) will give the required conclusion. \square

The corresponding statement of part (b) above for abelian elements follows from the fact that if $x \in B_+ \setminus \{0\}$ is abelian, then $xAx \subseteq x^{1/2}(x^{1/2}Ax^{1/2})x^{1/2} \subseteq \overline{x^{1/2}Bx^{1/2}} = \overline{xBx}$ and hence, is an abelian algebra.

Our next lemma is a crucial step toward the new looks and the decomposition scheme. A different proof of the statement concerning abelian elements can be found in [25, Proposition 3.8(a)].

LEMMA 3.2. *Let A and B be two strongly Morita equivalent C^* -algebras. Then A has a non-zero abelian element (respectively, finite elements) if and only if B does.*

Proof. We will only give the proof for the statement concerning finite elements since the other statement follows from a similar argument. Let D and $e \in \text{Proj}^1(M(D)) \subseteq \text{OP}^1(D)$ be as in Remark 2.3(b). In particular, $A = \text{her}_D(e)$ and $z(e) = 1$ in $\text{OP}^1(D)$.

Suppose that $\mathcal{F}^B \neq \emptyset$. Lemma 3.1(b) implies that $\mathcal{F}^D \neq \emptyset$. By Lemma 3.1(c), we know that D contains a non-zero finite hereditary C^* -subalgebra D_0 . Let $p \in \text{OP}^1(D)$ with $D_0 = \text{her}_D(p)$. Then [27, Theorem 1.9] gives $e_0, e_1, p_0, p_1 \in \text{OP}^1(D) \cup \{0\}$ such that $e_0, e_1 \leq e$, $p_0, p_1 \leq p$, $e_0e_1 = 0$, $p_0p_1 = 0$,

$$\overline{e_0 + e_1}^e = e, \quad \overline{p_0 + p_1}^p = p, \quad z(e_0)z(p_0) = 0 \quad \text{and} \quad \text{her}_D(e_1) \cong \text{her}_D(p_1)$$

(notice that the equivalence relation in [27] produces a $*$ -isomorphism between $\text{her}_D(e_1)$ and $\text{her}_D(p_1)$).

Let us first show that $p_1 \neq 0$. Suppose on the contrary that $p_1 = 0$. The relation $\text{her}_D(e_1) \cong \text{her}_D(p_1)$ implies $e_1 = 0$ (see Statement (O1)). Thus, $\overline{e_0}^e = e$ tells us that $z(e_0)$ is dense in $z(e) = 1$ (by [27, Lemma 1.8]). This implies that $z(p_0) = 0$ (because $z(e_0)z(p_0) = 0$ and we have Proposition 2.1(a) as well as Remark 2.2(a)). Hence, $p_0 = 0$, which is absurd since $\overline{p_0}^p = p \neq 0$.

Now, $\text{her}_D(p_1) \subseteq D_0$ is a non-zero finite hereditary C^* -subalgebra of D . As $\text{her}_D(e_1)$ is $*$ -isomorphic to $\text{her}_D(p_1)$, we conclude that $\text{her}_D(e_1)$ is a non-zero finite hereditary C^* -subalgebra of $\text{her}_D(e)$. Consequently, $\mathcal{F}^A \neq \emptyset$ (by Lemma 3.1(b)) as required. \square

When both A and B are separable, there is a simpler proof of the above, using the fact that A and B are stably isomorphic whenever they are strongly Morita equivalent. In fact, if $\mathcal{F}^A \neq \emptyset$ and $p \in \mathcal{K}(\ell^2)$ is a rank one projection, we know from Lemma 3.1(c) that there is a non-zero finite hereditary C^* -subalgebra D of $A \otimes p \subseteq A \otimes \mathcal{K}(\ell^2)$. Let E be the corresponding non-zero finite hereditary C^* -subalgebra of $B \otimes \mathcal{K}(\ell^2)$ under the stably isomorphism. Then $(1 \otimes q)E(1 \otimes q) \neq \{0\}$ for some rank one projection $q \in \mathcal{K}(\ell^2)$, and we know that $\mathcal{F}^B \neq \emptyset$.

3.1. The first set of new looks

The following theorem is our first set of new looks. Note that there is no ambiguity in part (b) of this theorem because of Lemma 3.1(b).

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THEOREM 3.3. (a) *A is discrete if and only if every non-zero closed ideal of A contains a non-zero abelian element.*

(b) *A is semi-finite if and only if every non-zero closed ideal of A contains a non-zero finite element.*

(c) *Let A and B be two strongly Morita equivalent C^* -algebras. If A is discrete (respectively, anti-liminary, semi-finite, of type II or of type III), then so is B.*

Proof. (a) It is clear that if A is discrete, the required condition holds. Conversely, suppose that A satisfies the condition as stated, and $D \subseteq A$ is a non-zero hereditary C^* -algebra. If $I := \overline{ADA}$, then Remark 2.3(d) tells us that I is strongly Morita equivalent to D. Now, we know from Lemma 3.2 that D contains a non-zero abelian element and hence A is discrete (by [27, Theorem 2.3]).

(b) Since the “only if part” is clear (recall that the cone of a closed ideal of A is a hereditary subcone of A_+), we only need to consider the “if part”. Let $a \in A_+ \setminus \{0\}$ and set $D := \overline{aAa}$. By Remark 2.3(d), D is strongly Morita equivalent to \overline{ADA} . The hypothesis and Lemma 3.2 produces $x \in \mathcal{F}^D = \mathcal{F}^A \cap D$ with $\|x\| = 1$ (see Lemma 3.1(b)). Notice that if $a^{1/2}x^{1/2} = 0$, then we have a contradiction that $Dx^{1/2} = 0$. Moreover, since $x^{1/2}ax^{1/2} \leq \|a\|x$ and $a^{1/2}xa^{1/2} \sim x^{1/2}ax^{1/2}$, we know from [10, Lemma 4.1] that $a^{1/2}xa^{1/2} \in \mathcal{F}^A$. As $a^{1/2}xa^{1/2} \leq a$, we see that A is semi-finite.

(c) It follows from Lemma 3.2 that if A is anti-liminary (respectively, type III), then so is B. Let J be a non-zero closed ideal of B. By Remark 2.3(c), there is a closed ideal $I \subseteq A$ that is strongly Morita equivalent to J. If A is discrete (respectively, semi-finite), then by part (a) (respectively, part (b)) as well as Lemma 3.2, we know that J contains a non-zero abelian (respectively, finite) element, and hence B is discrete (respectively, semi-finite). Since a C^* -algebra is of type II if and only if it is anti-liminary and semi-finite, we know that the property of being type II is also preserved under strong Morita equivalence. \square

REMARK 3.4. (a) *It is clear that one can replace “abelian element” in Theorem 3.3(a) by “abelian hereditary C^* -subalgebra”. Moreover, by Lemma 3.1(c), one may also replace “finite element” in Theorem 3.3(b) by “finite hereditary C^* -subalgebra”. The same reason also tells us that:*

a C^* -algebra is of type III if and only if it has no non-zero finite hereditary C^* -subalgebra.

(b) *Remark 2.3(d) and Theorem 3.3(c) implies that every discrete (respectively, anti-liminary, semi-finite, type II or type III) hereditary C^* -subalgebra of A is contained in a discrete (respectively, anti-liminary, semi-finite, type II or type III) closed ideal of A.*

The following statement is a direct consequence of Theorem 3.3(b) and [22, Theorem 2.7]:

Let A be a C^ -algebra and α be an action of a compact group G on A. Suppose that the linear span of functions of the form $t \mapsto \alpha_t(a)b$, where $a, b \in A$, is dense in $C(G; A)$. Then the fixed point algebra A^α is discrete (respectively, anti-liminary, semi-finite, of type II or of type III) if and only if the crossed product $A \rtimes_\alpha G$ has the same property.*

3.2. Some permanence properties

With the help of Theorem 3.3, we will obtain in Proposition 3.5 below some permanence properties of discreteness, type II, type III and semi-finiteness. Apart from part (a), these properties seem to be new. Notice that, by Remark 2.2(c), the hypothesis in part (b) means that B^\perp does not contain any non-zero ideal of A, but the presentation in the statement here seems more informative.

PROPOSITION 3.5. *Let B be a discrete (respectively, type II, type III, semi-finite or anti-liminary) C^* -algebra.*

(a) *If A is a hereditary C^* -subalgebra of B , then A is discrete (respectively, of type II, of type III, semi-finite or anti-liminary).*

(b) *If B is a hereditary C^* -subalgebra of a C^* -algebra A such that $I := \overline{ABA}$ is an essential ideal of A , then A is discrete (respectively, of type II, of type III, semi-finite or anti-liminary). In particular, $M(B)$ is discrete (respectively, of type II, of type III, semi-finite or anti-liminary).*

(c) *If B is a closed ideal of a C^* -algebra A , then both $B^{\perp\perp}$ and A/B^\perp are discrete (respectively, of type II, of type III, semi-finite or anti-liminary).*

(d) *If I is an ideal of B , then B/I^\perp is discrete (respectively, of type II, of type III, semi-finite or anti-liminary).*

Proof. (a) This part follows from the definitions, Lemma 3.1(b) as well as the paragraph following Lemma 3.1.

(b) We will only consider the case when B is of type II (as the argument for the other cases are similar). By Remark 2.3(d) and Theorem 3.3(c), we know that I is of type II. Suppose that $(0) \neq D \subseteq A$ is a hereditary C^* -subalgebra. Then ID is non-zero, because I is essential. Hence, $D \cap I = DID \neq (0)$ and is a hereditary C^* -subalgebra of I . This ensures that D contains a non-zero finite element. On the other hand, if $D = \overline{aAa}$ for an abelian element $a \in A_+$, then $D \cap I$ is a non-zero abelian hereditary C^* -subalgebra of I , which is impossible. These show that A is of type II.

(c) Let $\varphi : A \rightarrow M(B)$ be the canonical $*$ -homomorphism. As $\ker \varphi = B^\perp$, we know that $\varphi|_{B^{\perp\perp}}$ is injective and so is the induced $*$ -homomorphism $\hat{\varphi} : A/B^\perp \rightarrow M(B)$. Since both $\varphi(B^{\perp\perp})$ and $\hat{\varphi}(A/B^\perp)$ contains the image of B as an essential ideal (see Remark 2.2(d)), the conclusion follows from part (b).

(d) By part (a), we know that I is discrete (respectively, of type II, of type III, semi-finite or anti-liminary). Hence, by part (c), we know that B/I^\perp is discrete (respectively, of type II, of type III, semi-finite or anti-liminary). \square

DEFINITION 3.6. (a) *We say that A is an essential extension of B if it contains B as an essential ideal.*

(b) *An ideal $I \subseteq A$ is called a normal ideal if $I = J^\perp$ for some ideal $J \subseteq A$.*

(c) *If $I \subseteq A$ is a normal ideal, then A/I is called a normal quotient of A .*

Notice that the name ‘‘normal ideal’’ comes from the fact that an ideal of a W^* -algebra M is a normal ideal if and only if it is $\sigma(M, M_*)$ -closed.

Proposition 3.5(b) implies that the properties of being discrete, type II, type III, semi-finite and anti-liminary are stable under essential extensions. Furthermore, Proposition 3.5(d) states that these types are preserved under normal quotients. In contrast, the quotient of a discrete C^* -algebra by an arbitrary closed ideal needs not be discrete (e.g. $\mathcal{B}(\ell^2)$ is a discrete C^* -algebra but $\mathcal{B}(\ell^2)/\mathcal{K}(\ell^2)$ is of type III).

COROLLARY 3.7. *A is discrete (respectively, semi-finite) if and only if every non-zero normal ideal of A contains a non-zero abelian (respectively, finite) hereditary C^* -subalgebra.*

Proof. If A is discrete (respectively, semi-finite), then A will satisfy the said condition because of Theorem 3.3(a) (respectively, Theorem 3.3(b)). Conversely, assume that the

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said condition is satisfied. Let I be a non-zero closed ideal of A . The assumption tells us that $I^{\perp\perp}$ contains a non-zero abelian (respectively, finite) element. It follows from the definition (respectively, from Lemma 3.1(c)) that $I^{\perp\perp}$ contains a non-zero abelian (respectively, finite) hereditary C^* -subalgebra D . As I is an essential ideal of $I^{\perp\perp}$, we know that $I \cap D = DID \neq (0)$. Now, Remark 3.4(a) implies that A is discrete (respectively, semi-finite). \square

3.3. The second set of new looks

The following can be regarded as another new looks of discreteness, semi-finiteness and type II.

THEOREM 3.8. *Let \mathcal{A} , \mathcal{F} and \mathcal{F}_{al} denote the class of abelian C^* -algebras, the class of finite C^* -algebras and the class of finite and anti-liminary finite C^* -algebras, respectively. Let \mathcal{D} (respectively, \mathcal{S} and \mathcal{T}_2) be the smallest class containing \mathcal{A} (respectively, \mathcal{F} and \mathcal{F}_{al}) which is closed under strong Morita equivalences and essential extensions. Then \mathcal{D} (respectively, \mathcal{S} and \mathcal{T}_2) is the class of discrete (respectively, semi-finite and type II) C^* -algebras.*

Proof. By Theorem 3.3(c) and Proposition 3.5(b), we know that every member in \mathcal{D} (respectively, \mathcal{S} and \mathcal{T}_2) is discrete (respectively, semi-finite and type II).

Suppose that A is a non-zero discrete C^* -algebra. Then [27, Theorem 2.3(iii)] gives an abelian hereditary C^* -subalgebra A_1 of A such that $\overline{AA_1A}$ is an essential ideal of A . Thus, by Remark 2.3(d), we know that $A \in \mathcal{D}$.

Secondly, assume that A is a semi-finite C^* -algebra. In order to show that A is a member of \mathcal{S} , it suffices to show that there is a finite hereditary C^* -subalgebra $A_2 \subseteq A$ with $\overline{AA_2A}$ being an essential ideal of A (because of Remark 2.3(d)). Indeed, by Zorn's Lemma, there exists a maximal family $\{B_i\}_{i \in \mathcal{J}}$ of finite hereditary C^* -subalgebras of A such that $\overline{AB_iA}$ and $\overline{AB_jA}$ are disjoint for all $i \neq j$, and we set

$$A_2 := \overline{\sum_{i \in \mathcal{J}} B_i}.$$

Because A_2 is the c_0 -direct sum of finite C^* -algebras, it is finite. Moreover, the ideal $\overline{AA_2A}$ is essential, by the maximality of $\{B_i\}_{i \in \mathcal{J}}$ and Remark 3.4(a).

Finally, let A be a type II C^* -algebra. As in above, we consider a maximal family $\{D_i\}_{i \in \mathcal{J}}$ of anti-liminary finite hereditary C^* -subalgebras with $\overline{AD_iA}$ and $\overline{AD_jA}$ being disjoint when $i \neq j$. If $A_3 := \overline{\sum_{i \in \mathcal{J}} D_i}$, then A_3 is an anti-liminary finite hereditary C^* -subalgebra of A . Again by Remark 3.4(a) and the maximality, the ideal $\overline{AA_3A}$ is essential in A , and we know that A is a member of \mathcal{T}_2 . \square

From the proof above, a C^* -algebra is type II (respectively, semi-finite or discrete) if and only if it contains an essential ideal that is strongly Morita equivalent to an anti-liminary finite (respectively, a finite or an abelian) C^* -algebra.

3.4. C^* -algebras of real rank zero and W^* -algebras

We want to compare discreteness, type II and type III of C^* -algebras with the corresponding properties of W^* -algebras. These comparisons could be known, but since we do not find them in the literature, we present them in Proposition 3.9(d) below for later reference. On our way, we also give alternative descriptions of these types in the case of real rank zero C^* -algebras.

PROPOSITION 3.9. *Let A be a C^* -algebra of real rank zero.*

(a) *A is discrete (respectively, semi-finite) if and only if every non-zero projection in A dominates a non-zero projection that is an abelian element of A (respectively, is a finite element in A).*

(b) *A is of type II if and only if every non-zero projection in A is not abelian but dominates a non-zero projection that is finite in A .*

(c) *A is of type III if and only if it has no non-zero projection which is finite in A .*

(d) *If, in addition, A is a W^* -algebra, then A is a type I, type II, type III or semi-finite W^* -algebra if and only if it is, respectively, a discrete, type II, type III or semi-finite C^* -algebra.*

Proof. (a) Suppose that A is discrete (respectively, semi-finite). Any projection $p \in A$ will dominate a non-zero abelian (respectively, finite) element $x \in A_+$. As $(pAp)_+$ is a hereditary subcone of A , we know that $x \in pAp$. Hence, pAp will contain the non-zero abelian hereditary C^* -subalgebra \overline{xAx} (respectively, the non-zero finite hereditary C^* -subalgebra \overline{yAy} for $y := (x - p\|x\|/2)_+$; see Lemma 3.1(c)), and any non-zero projection in this subalgebra will be dominated by p .

Conversely, assume that the said condition holds. If $I \subseteq A$ is a non-zero closed ideal and $p \in \text{Proj}^1(I)$, then p will dominate a non-zero abelian (respectively, finite) element and this element will belong to I (as I_+ is a hereditary subcone of A_+). Thus, A is discrete (respectively, semi-finite) by Theorem 3.3.

(b) This part follows directly from part (a).

(c) The “only if part” is clear, and the converse follows from Lemma 3.1(c).

(d) By parts (a), (b) and (c) above, it suffices to show that a projection is finite in A if and only if it is finite in the sense of Murry and von Neumann. In fact, one of the implications is clear because the equivalence relation “ \sim ” (about positive elements) in the beginning of Section 1, when restricted to the set of projections, is weaker than the Murry-von Neumann equivalence. Conversely, suppose that a projection $p \in \text{Proj}^1(A)$ is finite in the sense of Murry and von Neumann. Then pAp is finite as a W^* -algebra, and the set of normal tracial states will separate 0 from other positive elements in pAp (i.e., for any $x \in (pAp)_+ \setminus \{0\}$, there exists a normal tracial state τ with $\tau(x) \neq 0$). Hence, we know from [10, Theorem 3.4] that pAp is a finite C^* -algebra and $p \in pAp$ is finite in A . \square

3.5. Purely infinite C^* -algebras and tracially infinite C^* -algebras

Another application of Proposition 3.5 is Proposition 3.10 below, which shows that type III is weaker than the notion of purely infiniteness as defined by Cuntz in [9] (in the case of simple C^* -algebras) as well as by Kirchberg and Rørdam [15] (in the general case). Note that Proposition 3.10 implies, in particular, [15, Proposition 4.4].

In order to obtain this comparison, we will investigate another property that looks very similar to type III (compare Proposition 3.10(a) as well as Corollary 4.7 in the next section), and lies between type III and pure infiniteness. This property will also be considered in Section 6 below.

Let us first recall the notion of traces on C^* -algebras. As in [26, §5.2], by a *trace*, we mean an additive and positively homogeneous map (i.e., a weight) $\tau : A_+ \rightarrow [0, \infty]$ satisfying

$$\tau(u^*xu) = \tau(x)$$

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for any $x \in A_+$ and unitary u in the unitalization of A . A trace τ is said to be *lower semi-continuous* if $\{x \in A_+ : \tau(x) \leq \lambda\}$ is norm closed in A , for every $\lambda \in \mathbb{R}_+$. Moreover, as in [11, Definition 6.1.1], τ is called *semi-finite* if

$$\tau(x) = \sup\{\tau(y) : 0 \leq y \leq x \text{ and } \tau(y) < +\infty\} \quad (x \in A_+).$$

Note that this semi-finiteness is slightly stronger than the one in [10]. In the following, we denote by $\mathcal{T}(A)$ the set of all lower semi-continuous semi-finite traces on A , and by $T_s(A)$ the set of tracial states (i.e. states that are also traces) on A .

PROPOSITION AND DEFINITION 3.10. *A C^* -algebra A is said to be tracially infinite if it contains an essential closed ideal J with $\mathcal{T}(J) = \{0\}$.*

(a) *A is tracially infinite if and only if A contains an essential ideal J such that for every hereditary C^* -subalgebra $B \subseteq J$, its bidual B^{**} is a properly infinite W^* -algebra.*

(b) *If A is tracially infinite, then A is of type III. In particular, every purely infinite C^* -algebra is of type III.*

Proof. (a) It suffices to show that for any C^* -algebra D , one has $\mathcal{T}(D) = \{0\}$ if and only if $T_s(B) = \emptyset$ for every non-zero hereditary C^* -subalgebra $B \subseteq D$ (we may then apply this statement to $D = J$; note that B^{**} is a properly infinite W^* -algebra if and only if $T_s(B) = \emptyset$).

Suppose that there exist a non-zero hereditary C^* -subalgebra $B \subseteq D$ and an element $\varphi \in T_s(B)$. Let τ be the lower semi-continuous trace extension of φ as given in the proof of [10, Lemma 4.6]; namely,

$$\tau(x) := \sup\{\varphi(y) : y \in B_+, y \sim z \leq x\} \quad (x \in D_+).$$

It is not hard to check that τ is semi-finite, and $\mathcal{T}(D) \neq \emptyset$.

Conversely, if $\tau \in \mathcal{T}(D) \setminus \{0\}$, then the semi-finiteness of τ will produce an element $x \in D_+$ with $0 < \tau(x) < +\infty$. Without loss of generality, assume that $\|x\| = 1$. It is clear that $\|x - (x - \frac{1}{n})_+\| \rightarrow 0$ and the lower semi-continuity of τ tells us that $\tau((x - \frac{1}{n})_+) \rightarrow \tau(x)$. Set $y := (x - \frac{1}{n_0})_+$, where n_0 is big enough so that $\tau((x - \frac{1}{n_0})_+) > 0$. Using a similar argument as that of Lemma 3.1(c), one can find $g \in C(\sigma(x))_+$ and $\lambda > 0$ such that $y = yg^{1/2}(x)$ and $g(x) \leq \lambda x$. Thus,

$$\tau(z) = \tau(g^{1/2}(x)zg^{1/2}(x)) \leq \lambda\|z\|\tau(x) < +\infty \quad (z \in (\overline{yDy})_+),$$

and hence $T_s(\overline{yDy}) \neq \emptyset$.

(b) Let J be an essential closed ideal of A with $\mathcal{T}(J) = \{0\}$. By Proposition 3.5(b), it suffices to show that J is of type III. Suppose on the contrary that J contains a non-zero finite hereditary C^* -subalgebra B (see Remark 3.4(a)). Then [10, Theorem 3.4] implies that $T_s(B) \neq \emptyset$, and the argument of part (a) produces a non-zero element in $\mathcal{T}(J)$, which is a contradiction.

For the second statement, if A is purely infinite, it follows from [15, Proposition 5.1] that $\mathcal{T}(A) = \{0\}$ and A is tracially infinite. \square

Recall that pure infiniteness passes to quotients (see [15, Theorem 4.19]) but the quotient of a type III C^* -algebra can be semi-finite (see e.g. [10, Remark 3.14]). Consequently, these two properties are different. On the other hand, we do not know whether tracial infiniteness coincides with type III. One can find in [24] some equivalences of the statement: “every type III C^* -algebra is tracially infinite”.

The following is a direct application of Proposition 3.10, Theorem 3.3(c) and Proposition 3.5(b).

COROLLARY 3.11. *Let A be a purely infinite C^* -algebra.*

- (a) *If A is strongly Morita equivalent to a C^* -algebra B , then B is of type III.*
- (b) *An essential extension of A (in particular, $M(A)$) is of type III.*

Notice that if A and B are strongly Morita equivalent separable C^* -algebras with A being purely infinite, then [15, Theorem 4.23] implies that B is also purely infinite. We do not know if the same is true for non-separable C^* -algebras.

EXAMPLE 3.12. *For every non-unital C^* -algebra B , the C^* -algebras $M(B \otimes \mathcal{O}_\infty)$ and $M(B \otimes \mathcal{O}_2)$ are of type III (by Corollary 3.11(b) and [15, Proposition 4.5]).*

4. Essentially infinite C^* -algebras and another equivalent form of type III C^* -algebras

As seen in Theorem 3.8, stability under both strong Morita equivalence as well as essential extension plays an important role in understanding discrete and type II C^* -algebras. Clearly, finiteness is not preserved under strong Morita equivalence (e.g. \mathbb{C} is strongly Morita equivalent to C^* -algebra $\mathcal{K}(\ell^2)$ that is not finite). However, it is stable under essential extension, as stated in part (a) of the following lemma. This lemma could be known, but we present it here for completeness.

- LEMMA 4.1.** (a) *If B is a finite C^* -algebra, then every essential extension of B is finite.*
 (b) *If I is a non-zero closed ideal of A , every $\tau \in T_s(I)$ extends to an element of $T_s(A)$.*

Proof. (a) Since a C^* -subalgebra of a finite C^* -algebra is finite, we need only to consider the essential extension $M(B)$ of B (see Remark 2.2(d)). By [10, Theorem 3.4], $T_s(B)$ separates 0 from other positive elements in B ; in other words,

$$\{x \in B_+ : \tau(x) = 0, \text{ for all } \tau \in T_s(B)\} = \{0\}.$$

For any element $\tau \in T_s(B)$, let $(\pi_\tau, \mathfrak{H}_\tau, \xi_\tau)$ be its GNS construction. If $\pi_0 := \bigoplus_{\tau \in T_s(B)} \pi_\tau$, then the above tells us that π_0 is faithful and hence extends to a faithful $*$ -representation $\tilde{\pi}_0$ of $M(B)$. On the other hand, each $\sigma \in T_s(B)$ extends to a tracial state $\bar{\sigma}$ on $D := \bigoplus_{\tau \in T_s(B)}^{\ell^\infty} \pi_\tau(B)''$ such that

$$\bar{\sigma}((y_\tau)_{\tau \in T_s(B)}) = \langle y_\sigma \xi_\sigma, \xi_\sigma \rangle \quad ((y_\tau)_{\tau \in T_s(B)} \in D).$$

As $\tilde{\pi}_0(M(B)) \subseteq D$, we can define $\tilde{\sigma} := \bar{\sigma} \circ \tilde{\pi}_0$. It is easy to see that $(\pi_{\tilde{\sigma}}, \mathfrak{H}_{\tilde{\sigma}})$ is the canonical extension of $(\pi_\sigma, \mathfrak{H}_\sigma)$ on $M(B)$. Moreover, we have $\ker \pi_{\tilde{\sigma}} = \{x \in M(B) : \tilde{\sigma}(x^*x) = 0\}$, since $\tilde{\sigma}$ is a tracial state. Now, if $a \in M(B)_+$ satisfying $\tilde{\sigma}(a) = 0$ for every $\sigma \in T_s(B)$, then $\pi_{\tilde{\sigma}}(a^{1/2}) = 0$ ($\sigma \in T_s(B)$), which gives $\tilde{\pi}_0(a^{1/2}) = 0$ and hence $a^{1/2} = 0$. Consequently, $M(B)$ is finite (by [10, Theorem 3.4]).

(b) As I is an essential closed ideal of $I^{\perp\perp}$, the argument of part (a) tells us that τ can be extended to a tracial state τ_0 on $I^{\perp\perp}$. Furthermore, as $I^{\perp\perp} + I^\perp$ is an essential ideal of A , the trivial extension of τ_0 on $I^{\perp\perp} + I^\perp$ (i.e. it vanishes on I^\perp) will again extends to a tracial state on A . \square

Motivated by the classification theory of W^* -algebras, we make the following definition of essentially infiniteness. Note that a W^* -algebra is essentially infinite if and only if it is

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properly infinite, but the name “properly infinite C^* -algebras” is already in used in the literature. The term “essentially infinite” also comes from Proposition 4.6(a) below.

DEFINITION 4.2. *A C^* -algebra A is said to be essentially infinite if it contains no non-zero finite closed ideal.*

REMARK 4.3. (a) *Let us set*

$$A_\infty := \bigcap \{J^\perp : J \text{ is a finite closed ideal of } A\} \quad \text{and} \quad A_1 := A_\infty^\perp.$$

It is easy to see that A_∞ is essentially infinite. On the other hand, if I is an arbitrary essentially infinite ideal of A , then for any finite ideal $J \subseteq A$, one has $I \cap J = \{0\}$, which implies that $I \subseteq A_\infty$.

(b) *If $I \subseteq A$ is a non-zero closed ideal such that $I \cap A_\infty = \{0\}$, then by the definition, $I \subseteq A_1$. This shows that $A_1 + A_\infty$ is an essential ideal of A .*

(c) *It is clear that any non-zero closed ideal of an essentially infinite C^* -algebra is essentially infinite.*

(d) *If A is simple, then either $A = A_\infty$ or $A = A_1$.*

We will show that A_1 and A_∞ can be viewed as the “finite part” and the “infinite part” of A , respectively. Let us first give the following lemma.

LEMMA 4.4. *Let \mathcal{J} be a maximal collection of pairwise disjoint non-zero finite ideals of A . Then $J_0 := \overline{\sum_{J \in \mathcal{J}} J}$ is a finite ideal of A such that $J_0 + A_\infty$ is essential in A . Moreover, $A_\infty = J_0^\perp$.*

Proof. J_0 is finite because it is a c_0 -direct sum of finite C^* -algebras. Suppose that I is a non-zero closed ideal of A . If I contains a non-zero finite ideal, then $I \cap J_0 \neq \{0\}$ (otherwise, \mathcal{J} cannot be a maximal family). If I does not contain a non-zero finite ideal, then $I \subseteq A_\infty$ (see Remark 4.3(a)). This shows that $J_0 + A_\infty$ is essential.

Furthermore, it is clear that $A_\infty \subseteq J_0^\perp$. Conversely, as J_0^\perp cannot contain any non-zero finite ideal (because \mathcal{J} is maximal), J_0^\perp is essentially infinite and is contained in A_∞ . \square

Observe that the construction in Lemma 4.4 needs not give a maximal finite ideal of A . For example, if $A = \ell^\infty$ and \mathcal{J} is the collection of all one-dimensional ideals of A , then $J_0 = c_0$, while the abelian C^* -algebra ℓ^∞ is itself finite.

THEOREM 4.5. *Let A be a C^* -algebra, $I \subseteq A$ be a closed ideal of A and $q : A \rightarrow A/I^\perp$ be the quotient map.*

(a) *A_1^\perp coincides with the largest essentially infinite ideal A_∞ of A , and A/A_1 is the universal essentially infinite normal quotient of A ; i.e., if a normal quotient of A (see Definition 3.6(c)) is essentially infinite, then the corresponding quotient map will factor through A/A_1 .*

(b) *A_1 is the largest finite ideal of A , and A/A_∞ is the universal finite normal quotient of A ; i.e., the quotient map of any finite normal quotient of A factors through A/A_∞ .*

(c) *If A is discrete (respectively, of type II, semi-finite or anti-liminary), then so are A/A_1 and A/A_∞ .*

(d) *$I_1 = I \cap A_1$ and $I_\infty = I \cap A_\infty$.*

(e) *$q(A_1) \cong A_1/(I^\perp)_1$ (respectively, $q(A_\infty) \cong A_\infty/(I^\perp)_\infty$) and is an essential ideal of $(A/I^\perp)_1$ (respectively, $(A/I^\perp)_\infty$).*

(f) *If A is a W^* -algebra, then A_1 and A_∞ are, respectively, the finite part and the infinite part of A .*

Proof. (a) We learn from Remark 4.3(a) that A_∞ is the largest essentially infinite ideal of A . Suppose that $\theta : A \rightarrow M(A_\infty)$ is the canonical embedding. Then $\ker \theta = A_\infty^\perp = A_1$ and θ induces a *-monomorphism $\hat{\theta} : A/A_1 \rightarrow M(A_\infty)$. Since $\hat{\theta}(A/A_1)$ contains $\theta(A_\infty)$ as an essential ideal (see Remark 2.2(d)), if there exists a non-zero finite ideal J of A/A_1 , then $\hat{\theta}(J) \cap \theta(A_\infty)$ will be a non-zero finite ideal of $\theta(A_\infty)$ which is a contradiction. Thus, A/A_1 is essentially infinite.

Assume that A/I^\perp is essentially infinite. Then I is *-isomorphic to the ideal $q(I)$ of A/I^\perp and hence is essentially infinite (see Remark 4.3(c)). Therefore, $I \subseteq A_\infty$ and $A_1 \subseteq I^\perp$, which means that q factors through A/A_1 .

On the other hand, θ restricts to an injection from $A_1^\perp = A_\infty^{\perp\perp}$ to $M(A_\infty)$. It follows from Remark 2.2(d) that $\theta(A_\infty)$ is an essential ideal of $\theta(A_1^\perp)$, and the same argument as the essential infiniteness of A/A_1 implies that A_1^\perp is essentially infinite. Consequently, we obtain the converse inclusion $A_1^\perp \subseteq A_\infty$.

(b) Let \mathcal{J} and J_0 be as in Lemma 4.4. Suppose that $\varphi : A \rightarrow M(J_0)$ is the canonical *-homomorphism. Since $\varphi|_{A_1}$ is a *-monomorphism (because $\ker \varphi = J_0^\perp = A_\infty$), we know that A_1 is finite (see Lemma 4.1(a) and Remark 2.2(d)). Moreover, if $J \subseteq A$ is a finite ideal, then $J \cap A_\infty = \{0\}$ (by the definition of A_∞), and $J \subseteq A_\infty^\perp = A_1$.

To show the finiteness of A/A_∞ , we note that the canonical map from A to $M(A_1)$ induces an injection on A/A_∞ (as $A_\infty = A_1^\perp$ by part (a)). Hence, we know from the finiteness of A_1 and Lemma 4.1(a) that A/A_∞ is finite.

Assume that A/I^\perp is finite. Since q restricts to an injection on I , one may identify I with an ideal of A/I^\perp and hence is finite. Thus, $I \subseteq A_1$ and $A_\infty = A_1^\perp \subseteq I^\perp$. Consequently, q factors through A/A_∞ .

(c) By part (a) and the definition of A_1 , both A/A_1 and A/A_∞ are normal quotient of A . Thus, this part follows from Proposition 3.5(d).

(d) If J is any closed ideal of A , then, by considering an approximate unit of I , one has

$$J^\perp \cap I = (J \cap I)^\perp \cap I. \quad (4.1)$$

This gives

$$A_\infty \cap I = \bigcap \{(J \cap I)^\perp \cap I : J \text{ is a finite ideal of } A\} = I_\infty,$$

and $A_1 \cap I = A_\infty^\perp \cap I = (A_\infty \cap I)^\perp \cap I = I_\infty^\perp \cap I = I_1$.

(e) By part (d) and Relation (4.1), one has

$$(I^\perp)_1 = I^\perp \cap A_1 = (I_1)^\perp \cap A_1$$

and the canonical *-homomorphism from A_1 to $M(I_1)$ induces a *-monomorphism from

$$q(A_1) \cong A_1/(I^\perp \cap A_1) = A_1/(I^\perp)_1 = A_1/((I_1)^\perp \cap A_1)$$

to $M(I_1)$. It follows from Lemma 4.1(a) and Remark 2.2(d) that $A_1/((I_1)^\perp \cap A_1)$ (whose faithful image in $M(I_1)$ contains I_1) is a finite C^* -algebra. Consequently, $q(A_1) \subseteq q(A)_1$ (because of part (b)).

Suppose that J is a non-zero closed ideal of the largest finite ideal $q(A)_1$ of $q(A)$. Since q restricts to an injection from $q^{-1}(J) \cap I^{\perp\perp}$ to the finite C^* -algebra J , we know that $q^{-1}(J) \cap I^{\perp\perp}$ is an ideal of A_1 and is non-zero (because $q^{-1}(J) \not\subseteq I^\perp$). Thus, $J \cap q(A)_1 \neq (0)$, and $q(A)_1$ is an essential ideal of $q(A)_1$.

In a similar way, the canonical map from A_∞ to $M(I_\infty)$ induces a *-monomorphism

from

$$q(A_\infty) \cong A_\infty / (I^\perp \cap A_\infty) = A_\infty / (I^\perp)_\infty = A_\infty / ((I_\infty)^\perp \cap A_\infty)$$

(see (4.1)) to $M(I_\infty)$. Remark 2.2(d) tells us that the image of $q(A_\infty)$ in $M(I_\infty)$ contains I_∞ as an essential ideal. Since I_∞ is essentially infinite, so is $q(A_\infty)$ (by the definition) and hence $q(A_\infty) \subseteq q(A)_\infty$ (because of part (a)).

Consider J' to be a non-zero closed ideal of the largest essentially infinite ideal $q(A)_\infty$ of $q(A)$. Since q restricts to an injection from $q^{-1}(J') \cap I^{\perp\perp}$ to an ideal of the essentially infinite C^* -algebra J' (see Remark 4.3(c)), we know that $q^{-1}(J') \cap I^{\perp\perp}$ is a non-zero (because $q^{-1}(J') \not\subseteq I^\perp$) closed ideal of A_∞ . Thus, $J' \cap q(A_\infty) \neq (0)$, and $q(A_\infty)$ is an essential ideal of $q(A)_\infty$.

(f) Recall that a W^* -algebra is finite in the sense of Murray-von Neumann if and only if it is finite in the sense of Cuntz-Pedersen (see the proof of Proposition 3.9(d)). As J^\perp is $\sigma(A, A_*)$ -closed for every ideal $J \subseteq A$, we see that A_1 is the largest $\sigma(A, A_*)$ -closed finite ideal of A (as $A_1 = A_\infty^\perp$). Moreover, A_∞ is the largest $\sigma(A, A_*)$ -closed ideal of A that contains no $\sigma(A, A_*)$ -closed finite ideal (see part (a)). \square

We learn from [10, Theorem 3.4] that A is finite if and only if it has enough tracial states; in the sense that $\bigcap_{\tau \in T_s(A)} \ker \pi_\tau = (0)$ (we use the convention that this intersection is A when $T_s(A) = \emptyset$). In the following, we will establish that A is essentially infinite if and only if $\bigcap_{\tau \in T_s(A)} \ker \pi_\tau$ is an essential ideal of A . Notice that it is possible for an essentially infinite C^* -algebra to have a tracial state. For example, if A is the unitalization of $\mathcal{K}(\ell^2)$, then A is essentially infinite (by Proposition 4.6(a) below), but the canonical non-degenerate one-dimensional representation τ of A is a tracial state.

PROPOSITION 4.6. *Let A be a C^* -algebra.*

(a) *The following statements are equivalent.*

1) *A is essentially infinite.*

2) $\bigcap_{\tau \in T_s(A)} \ker \pi_\tau$ *is an essential ideal of A .*

3) *There is an essential ideal J of A with J^{**} being a properly infinite W^* -algebra.*

(b) *If A is separable, then A is essentially infinite if and only if $\ker \pi_\tau$ is an essential ideal of A for every $\tau \in T_s(A)$*

Proof. (a) Set $J_0 := \bigcap_{\tau \in T_s(A)} \ker \pi_\tau$.

1) \Rightarrow 2). Suppose on the contrary that $J_0^\perp \neq (0)$. As the representation $\bigoplus_{\tau \in T_s(A)} \pi_\tau$ is injective on J_0^\perp , one knows that

$$\{x \in (J_0^\perp)_+ : \tau(x) = 0, \text{ for all } \tau \in T_s(A)\} = \{0\}.$$

Thus, we arrive at the contradiction that J_0^\perp is a finite closed ideal of A (using [10, Theorem 3.4]).

2) \Rightarrow 3). We will verify that J_0^{**} is a properly infinite W^* -algebra. Assume on the contrary that one can find $\tau_0 \in T_s(J_0)$. By Lemma 4.1(b), τ_0 extends to an element $\tau_1 \in T_s(A)$. Since $\pi_{\tau_1}(J_0) = \{0\}$, one has $\tau_0((J_0)_+) = \tau_1((J_0)_+) = \{0\}$, which gives the contradiction that $\tau_0 = 0$.

3) \Rightarrow 1). Suppose on the contrary that A contains a non-zero finite closed ideal I . Then $I_0 := I \cap J \neq (0)$. As I_0 is finite, it has a tracial state τ and τ extends to a tracial state on J (because of Lemma 4.1(b)), which contradicts the proper infiniteness of J^{**} .

(b) Assume that there is a non-zero finite closed ideal $I \subseteq A$. By [10, Corollary 3.6], there

is a faithful tracial state τ on I , and it extends to $\tau_0 \in T_s(A)$ (see Lemma 4.1(b)). For any $x \in (I \cap \ker \pi_{\tau_0})_+$, we have $\tau_0(x) = 0$, and the faithfulness of τ gives $x = 0$. Hence, $I \cap \ker \pi_{\tau_0} = \{0\}$ and $\ker \pi_{\tau_0}$ is not an essential ideal. The converse follows directly from part (a). \square

The above and Remark 3.4(a) give the following. The reader may note the similarity and difference between Statement (3) below and Proposition 3.10(a).

COROLLARY 4.7. *The following statements are equivalent.*

- 1) *A is of type III.*
- 2) *All hereditary C^* -subalgebras of A are essentially infinite.*
- 3) *For every hereditary C^* -subalgebra $B \subseteq A$, there is an essential ideal I of B with I^{**} being a properly infinite W^* -algebra.*

In particular, all type III C^ -algebra are essentially infinite.*

EXAMPLE 4.8. (a) *If G is a separable connected non-amenable locally compact group, then it follows from Proposition 4.6(a) and the main result of [23] that the reduced group C^* -algebra $C_r^*(G)$ is essentially infinite.*

(b) *Let Γ be a countably infinite amenable group and $c_0(\Gamma)^1$ be the unitalization of $c_0(\Gamma)$. Consider β to be the unique extension of the left translation action of Γ on $c_0(\Gamma)$ to $c_0(\Gamma)^1$, and $B := c_0(\Gamma)^1 \rtimes_{\beta} \Gamma$. The primitive ideal space $\text{Prim}(c_0(\Gamma)^1)$ equals the one-point compactification, $\Gamma \cup \{\infty\}$, of Γ . For any $t \in \Gamma \setminus \{e\}$, one has*

$$\text{Prim}(c_0(\Gamma)^1)^t := \{x \in \Gamma \cup \{\infty\} : \tilde{\beta}_t(x) = x\} = \{\infty\},$$

where $\tilde{\beta}$ is the induced action of Γ on $\text{Prim}(c_0(\Gamma)^1)$. Hence, the action β is almost free in the sense of [17, Definition 4]. Suppose that J is a non-zero closed ideal of B . It follows from [17, Theorem 9] that there is a non-zero β -invariant ideal $I \subseteq c_0(\Gamma)^1$ with $I \rtimes_{\beta} \Gamma \subseteq J$. Since $c_0(\Gamma)$ is essential in $c_0(\Gamma)^1$, we know that $I \cap c_0(\Gamma) \neq \{0\}$, and this gives

$$I \rtimes_{\beta} \Gamma \cap c_0(\Gamma) \rtimes_{\beta} \Gamma \neq \{0\}.$$

Consequently, $\mathcal{K}(\ell^2(\Gamma)) = c_0(\Gamma) \rtimes_{\beta} \Gamma$ is an essential ideal of B , and Proposition 4.6(a) tells us that B is essentially infinite.

On the other hand, the short exact sequence $0 \rightarrow c_0(\Gamma) \rightarrow c_0(\Gamma)^1 \rightarrow \mathbb{C} \rightarrow 0$ induces the exact sequence

$$0 \rightarrow c_0(\Gamma) \rtimes_{\beta} \Gamma \rightarrow B \rightarrow C^*(\Gamma) \rightarrow 0.$$

From which, we know that there exists a tracial state on B .

(c) *The simple C^* -algebras obtained by Rørdam in Sections 5 and 6 of [28] that contain a finite projection and an infinite projection (both in the sense of Murray and von Neumann) are of type III.*

More generally, every stably infinite simple C^* -algebra A (in the sense that $A \otimes \mathcal{K}(\ell^2)$ contains an infinite projection) is of type III. In fact, suppose that A is a simple C^* -algebra such that $A \otimes \mathcal{K}(\ell^2)$ contains an infinite projection p . Then $p(A \otimes \mathcal{K}(\ell^2))p$ is a simple C^* -algebra (see Remark 2.3(d)) with its identity p being infinite. Thus, $T_s(p(A \otimes \mathcal{K}(\ell^2))p) = \emptyset$ (because any tracial state on a simple C^* -algebra is faithful). Since $p(A \otimes \mathcal{K}(\ell^2))p$ is simple and unital, the above implies $\mathcal{F}(p(A \otimes \mathcal{K}(\ell^2))p) = \{0\}$. This means that $p(A \otimes \mathcal{K}(\ell^2))p$ is of type III (see Proposition 3.10(b)). Now, it follows from Remark 2.3(d) and Theorem 3.3(c) that A is also of type III.

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As a partial converse to part (c) above, one knows from [6, Corollary II.4.11] and the main result of [13] that if A is a simple exact stably finite unital C^* -algebra, then A has a tracial state, and hence is finite (or equivalently, semi-finite as A is simple and unital). However, we do not know if the same is true when the exactness assumption or the unital assumption is removed.

5. *The decomposition scheme in terms of discreteness, type II and type III*

In this section, we will consider the remaining statements in Theorem 1.4. Let us begin with the following two simple lemmas.

LEMMA 5.1. *If A is not discrete, then A contains a non-zero closed ideal of either type II or type III.*

Proof. By Theorem 3.3(a), there is a non-zero anti-liminary closed ideal $I \subseteq A$. If I does not contain a non-zero finite element as well, then it is of type III. Otherwise, I will contain a non-zero finite hereditary C^* -subalgebra B (see Lemma 3.1(c)). The non-zero ideal $J := \overline{BI}$ of I is semi-finite, because of Remark 2.3(d) and Theorem 3.3(c). Hence, J is of type II. \square

LEMMA 5.2. *Let I and J be two closed ideals of A such that $I \cap J = (0)$ and $I + J$ is essential. If $\varphi : A \rightarrow M(I)$ is the canonical $*$ -homomorphism, then φ restricts to an injection on J^\perp .*

Proof. Observe that $(I + J)^\perp = J^\perp \cap I^\perp$. If $J^\perp \cap I^\perp$ is non-zero, then the assumption on $I + J$ gives the contradiction that $(I + J)^\perp \cap (I + J) \neq (0)$. Consequently, $J^\perp \cap I^\perp = (0)$ and $\varphi|_{J^\perp}$ is injective (as $\ker \varphi = I^\perp$). \square

THEOREM 5.3. *Let A be a C^* -algebra.*

- (a) *There exists the largest discrete (respectively, semi-finite, type II and type III) hereditary C^* -subalgebra A_d (respectively, A_{sf} , A_{II} and A_{III}) of A . Moreover, A_d , A_{sf} , A_{II} and A_{III} are ideals of A such that A_d , A_{II} and A_{III} are disjoint and $A_{III} \cap A_{sf} = (0)$.*
- (b) *$A_d + A_{II} + A_{III}$ is an essential ideal of A and $A_d + A_{II}$ is an essential ideal of A_{sf} .*
- (c) *$A_{al} := A_d^\perp$ is the largest anti-liminary hereditary C^* -subalgebra of A .*
- (d) *$A_d = A_{al}^\perp = (A_{II} + A_{III})^\perp$, $A_{II} = A_{sf} \cap A_d^\perp$, $A_d = A_{sf} \cap A_{II}^\perp$, $A_{III} = A_{sf}^\perp = (A_d + A_{II})^\perp$ and $A_{sf} = A_{III}^\perp$.*
- (e) *If A is a W^* -algebra, then A_d , A_{II} and A_{III} are respectively, the type I, the type II and the type III W^* -algebra summands of A .*

Proof. (a) Let us first construct the largest type II hereditary C^* -subalgebra of A and verify that it is an ideal. In order to do so, we will show that the collection \mathcal{J}_{II} of all type II closed ideals of A is a directed set, and that

$$A_{II} := \overline{\sum_{J \in \mathcal{J}_{II}} J},$$

is the largest type II hereditary C^* -subalgebra of A , which is clearly an ideal.

For the first claim, we will verify that $J_1 + J_2 \in \mathcal{J}_{II}$ for any arbitrary elements $J_1, J_2 \in \mathcal{J}_{II}$. In fact, if B is a non-zero hereditary C^* -subalgebra of $J_1 + J_2$, then by Proposition 2.1(c), we have $B \cap J_1 \neq (0)$ or $B \cap J_2 \neq (0)$. If $B \cap J_1 \neq (0)$, then one obtains a non-zero finite element $x \in \mathcal{F}^{B \cap J_1} \subseteq \mathcal{F}^B$. Similarly, if $B \cap J_2 \neq (0)$, then B will also contains a non-zero finite element. On the other hand, suppose that there exists a non-zero abelian

element $a \in (J_1 + J_2)_+$. Set $D := \overline{a(J_1 + J_2)a}$. Again, it follows from Proposition 2.1(c) that $D \cap J_1 \neq (0)$ or $D \cap J_2 \neq (0)$. However, this contradicts with $J_1, J_2 \in \mathcal{J}_{\text{II}}$, because $D \cap J_1 \subseteq J_1$ and $D \cap J_2 \subseteq J_2$ are abelian hereditary C^* -subalgebras. Thus, $J_1 + J_2 \in \mathcal{J}_{\text{II}}$.

Now, for any non-zero closed ideal J of A , let us denote by p_J the element in $\text{OP}^1(A) \cap Z(A^{**})$ with $J = \text{her}_A(p_J)$. It is easy to see that $p_{A_{\text{II}}}$ is the $\sigma(A^{**}, A^*)$ -limit of the increasing net $\{p_J\}_{J \in \mathcal{J}_{\text{II}}}$. Suppose that $e \in \text{OP}^1(A)$ satisfying $\text{her}_A(e) \subseteq A_{\text{II}}$. Then

$$e = ep_{A_{\text{II}}}e = w^*\text{-}\lim_{J \in \mathcal{J}_{\text{II}}} ep_Je,$$

and there is $J \in \mathcal{J}_{\text{II}}$ with $ep_J \neq 0$. This means that $\text{her}_A(e) \cap J$ is non-zero (see Proposition 2.1(b)), and is a hereditary C^* -subalgebra of J . Hence, $\text{her}_A(e) \cap J$ contains a non-zero finite element. On the other hand, assume that there is a non-zero abelian positive element a in A_{II} . Consider $f \in \text{OP}^1(A)$ such that $\text{her}_A(f) = \overline{aAa} \subseteq A_{\text{II}}$. As in the above, one can find $J \in \mathcal{J}_{\text{II}}$ with $fp_J \neq 0$. Thus, $\text{her}_A(f) \cap J$ is a non-zero abelian hereditary C^* -subalgebra of J , which contradicts J being of type II. Consequently, one has $A_{\text{II}} \in \mathcal{J}_{\text{II}}$. Finally, if $B \subseteq A$ is a hereditary C^* -subalgebra of type II, then, by Remark 2.3(d) and Theorem 3.3(c), one knows that $B \subseteq \overline{ABA} \subseteq A_{\text{II}}$.

The existences of A_{sf} , A_{II} and A_{III} as well as the fact that they are ideals follow from similar arguments. The disjointness statements are obvious.

(b) This part follows directly from Lemma 5.1 (namely, every non-zero non-discrete closed ideal interests A_{II} or A_{III}).

(c) If A_{al} contains a non-zero abelian positive element x , then $\overline{xAx} \subseteq A_{\text{d}}$, and we have a contradiction that $x \in A_{\text{al}} \cap A_{\text{d}}$. Thus, A_{al} is anti-liminary. Furthermore, if $B \subseteq A$ is a non-zero anti-liminary hereditary C^* -subalgebra, then $BA_{\text{d}}B = A_{\text{d}} \cap B = (0)$, which means that $B \subseteq A_{\text{d}}^\perp = A_{\text{al}}$.

(d) By parts (a) and (c), one has $A_{\text{d}} \subseteq A_{\text{al}}^\perp \subseteq (A_{\text{II}} + A_{\text{III}})^\perp$. Since $A_{\text{d}} \cap (A_{\text{II}} + A_{\text{III}}) = (0)$ and $A_{\text{d}} + A_{\text{II}} + A_{\text{III}}$ is an essential ideal of A , we obtain from Lemma 5.2 a $*$ -monomorphism from $(A_{\text{II}} + A_{\text{III}})^\perp$ to $M(A_{\text{d}})$ whose image clearly contains A_{d} . Therefore, Remark 2.2(d) and Proposition 3.5(b) tells us that $(A_{\text{II}} + A_{\text{III}})^\perp$ is discrete; i.e., $(A_{\text{II}} + A_{\text{III}})^\perp \subseteq A_{\text{d}}$. These give the first equality.

Secondly, since $A_{\text{II}}A_{\text{d}} = (0)$ and $A_{\text{d}} + A_{\text{II}}$ is essential in A_{sf} (see part (b)), we have $A_{\text{II}} \subseteq A_{\text{sf}} \cap A_{\text{d}}^\perp$ and Lemma 5.2 produces a $*$ -monomorphism from $A_{\text{sf}} \cap A_{\text{d}}^\perp$ to $M(A_{\text{II}})$. As in the above, this gives the opposite inclusion $A_{\text{sf}} \cap A_{\text{d}}^\perp \subseteq A_{\text{II}}$. The equality $A_{\text{d}} = A_{\text{sf}} \cap A_{\text{II}}^\perp$ follows from the same argument.

To establish the fourth equality, let us set $J_3 := (A_{\text{d}} + A_{\text{II}})^\perp$. As $A_{\text{III}}A_{\text{sf}} = (0)$ and $A_{\text{d}} + A_{\text{II}} \subseteq A_{\text{sf}}$, we see that $A_{\text{III}} \subseteq A_{\text{sf}}^\perp \subseteq J_3$. It remains to show that J_3 is of type III. In fact, suppose on the contrary that J_3 contains a non-zero finite hereditary C^* -subalgebra B . Then $B \cap A_{\text{d}} = (0)$ (because $J_3 \subseteq A_{\text{d}}^\perp$). This implies that B is of type II, which contradicts $J_3 \subseteq A_{\text{II}}^\perp$.

Finally, it is clear that $A_{\text{sf}} \subseteq A_{\text{III}}^\perp$. Conversely, since $A_{\text{sf}} \cap A_{\text{III}} = (0)$ and $A_{\text{sf}} + A_{\text{III}}$ contains the essential ideal $A_I + A_{\text{II}} + A_{\text{III}}$ of A , one learns from Lemma 5.2 that there is a $*$ -monomorphism from A_{III}^\perp to $M(A_{\text{sf}})$ whose image clearly contains A_{sf} . It now follows from Remark 2.2(d) and Proposition 3.5(b) that A_{III}^\perp is semi-finite, and thus $A_{\text{III}}^\perp \subseteq A_{\text{sf}}$.

(e) This part follows from Proposition 3.9(d) and the fact that A_{d} , A_{II} and A_{III} are $\sigma(A, A_*)$ -closed, because of part (d) (and the fact that $(I + J)^\perp = J^\perp \cap I^\perp$ for any two ideals I and J). \square

This theorem produces the following result concerning the stability of the above decomposition under strong Morita equivalence (see Remark 2.3(c) for the notation).

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COROLLARY 5.4. *If X is a full Hilbert A -module and $B := \mathcal{K}_A(X)$, then $B_{\text{d}} = \mathcal{K}_A(XA_{\text{d}})$, $B_{\text{II}} = \mathcal{K}_A(XA_{\text{II}})$ and $B_{\text{III}} = \mathcal{K}_A(XA_{\text{III}})$.*

Proof. It follows from Remark 2.3(c) that the assignment $I \mapsto \mathcal{K}_A(XI)$ is a bijection from the set of closed ideals of A to that of B , and that the ideal I of A is strongly Morita equivalent to $\mathcal{K}_A(XI)$. Since A_{d} (respectively, A_{II} and A_{III}) is the largest discrete (respectively, type II and type III) ideal of A , we know from the above and Theorem 3.3(c) that B_{d} (respectively, B_{II} and B_{III}) is the largest discrete (respectively, type II and type III) ideal of B . \square

On the other hand, we have the following result. The meaning of universal “#” normal quotient in its statement is understood in similar ways as those of parts (a) and (b) of Theorem 4.5.

COROLLARY 5.5. (a) *A/A_{al} (respectively, A/A_{d} , A/A_{III} and A/A_{sf}) is the universal discrete (respectively, anti-liminary, semi-finite and type III) normal quotient of A .*
 (b) *$A_{\text{sf}}/A_{\text{II}}$ (respectively, $A_{\text{sf}}/A_{\text{d}}$) is the universal discrete (respectively, type II) normal quotient of A_{sf} .*

Proof. (a) It follows from Theorem 5.3(c) (respectively, Theorem 5.3(d)) that $A/A_{\text{al}} = A/A_{\text{d}}^{\perp}$ (respectively, $A/A_{\text{d}} = A/A_{\text{al}}^{\perp}$, $A/A_{\text{III}} = A/A_{\text{sf}}^{\perp}$ and $A/A_{\text{sf}} = A/A_{\text{III}}^{\perp}$), and hence it is a normal quotient. Moreover, we know from Proposition 3.5(c) that A/A_{al} is discrete (respectively, A/A_{d} is anti-liminary, A/A_{III} is semi-finite and A/A_{sf} is of type III).

Let I be an ideal of A and $q : A \rightarrow A/I^{\perp}$ be the quotient map. Suppose that A/I^{\perp} is discrete (respectively, anti-liminary, semi-finite and type III). Then q restricts to an injection on I with $q(I)$ being an ideal of A/I^{\perp} . Proposition 3.5(a) tells us that I is discrete (respectively, anti-liminary, semi-finite and type III), and so $I \subseteq A_{\text{d}}$ (respectively, $I \subseteq A_{\text{al}}$, $I \subseteq A_{\text{sf}}$ and $I \subseteq A_{\text{III}}$). Consequently, $A_{\text{al}} \subseteq I^{\perp}$ (respectively, $A_{\text{d}} \subseteq I^{\perp}$, $A_{\text{III}} \subseteq I^{\perp}$ and $A_{\text{II}} \subseteq I^{\perp}$), because of parts (c) and (d) of Theorem 5.3, and we conclude that q factors through A/A_{al} (respectively, A/A_{d} , A/A_{III} and A/A_{sf}).

(b) This part follows from the argument of part (a). Notice that $A_{\text{II}} = A_{\text{sf}} \cap A_{\text{d}}^{\perp}$ and $A_{\text{d}} = A_{\text{sf}} \cap A_{\text{II}}^{\perp}$ (see Theorem 5.3(d)). \square

Furthermore, we want to see how type decompositions pass to hereditary C^* -subalgebras, essential extensions and normal quotients.

PROPOSITION 5.6. *Let $B \subseteq A$ be a hereditary C^* -subalgebra and $I, J \subseteq A$ be closed ideals with J being essential. Let $q : A \rightarrow A/I^{\perp}$ be the quotient map.*

- (a) *$B_{\text{d}} = A_{\text{d}} \cap B$, $B_{\text{sf}} = A_{\text{sf}} \cap B$, $B_{\text{al}} = A_{\text{al}} \cap B$, $B_{\text{II}} = A_{\text{II}} \cap B$ and $B_{\text{III}} = A_{\text{III}} \cap B$.*
 (b) *Consider $\# = \text{d, sf, al, II or III}$. Then $q(A_{\#})$ is an essential ideal of $(A/I^{\perp})_{\#}$ and*

$$(A/I^{\perp})_{\#} = q(\{a \in A : aI \subseteq I_{\#}\}). \quad (5.1)$$

In particular, $A_{\#} = \{a \in A : aJ \subseteq J_{\#}\}$.

- (c) *$A_{\text{II}} = \{x \in A : xJ_{\text{d}} = (0) \text{ and } xJ \subseteq J_{\text{sf}}\}$*
 (d) *$A_{\text{III}} = \{x \in A : xJ_{\text{sf}} = (0)\} = \{x \in A : xJ_{\text{d}} = (0) \text{ and } xJ_{\text{II}} = (0)\}$.*

Proof. (a) By Proposition 3.5(a), $A_{\text{d}} \cap B$, $A_{\text{sf}} \cap B$, $A_{\text{al}} \cap B$, $A_{\text{II}} \cap B$ and $A_{\text{III}} \cap B$ are respectively, discrete, semi-finite, anti-liminary, type II and type III hereditary C^* -subalgebras of B . On the other hand, if $D \subseteq B$ is a hereditary C^* -subalgebra which is

discrete (respectively, semi-finite, anti-liminary, type \mathbb{II} and type \mathbb{III}), then D is a hereditary C^* -subalgebra of A and hence is contained in A_d (respectively, A_{sf} , A_{al} , $A_{\mathbb{II}}$ and $A_{\mathbb{III}}$). These give the required statement.

(b) Let us first establish the third claim; i.e., $A_{\#} = \check{A}_{\#}$, where

$$\check{A}_{\#} := \{x \in A : xJ \subseteq J_{\#}\} \quad (\# = d, sf, al, \mathbb{II}, \mathbb{III}).$$

We will consider only the case when $\# = \mathbb{II}$ (because the other cases follow from similar arguments). In fact, it follows from $A_{\mathbb{II}}J = J_{\mathbb{II}}$ (see part (a)) that $A_{\mathbb{II}} \subseteq \check{A}_{\mathbb{II}}$. Consider $B \subseteq \check{A}_{\mathbb{II}}$ to be an arbitrary non-zero hereditary C^* -subalgebra. Then $B \cap J = BJB$ is contained in $J_{\mathbb{II}}$ and $BJB \neq (0)$, as J is essential. Thus, we obtain a non-zero finite element in $(B \cap J)_+$. On the other hand, suppose that $\check{A}_{\mathbb{II}}$ contains a non-zero abelian positive element a . If we set $D := \overline{aAa}$, then $D \cap J = DJD$ is a non-zero abelian hereditary C^* -subalgebra of $J_{\mathbb{II}}$, which is absurd. Consequently, $\check{A}_{\mathbb{II}}$ is of type \mathbb{II} and is contained in $A_{\mathbb{II}}$.

Secondly, we will verify Relation (5.1). Notice that, by considering the canonical $*$ -monomorphism $\varphi : A/I^{\perp} \rightarrow M(I)$, one sees that $\varphi(A/I^{\perp})$ as an essential extension of I (see Remark 2.2(d)). Thus, the last statement in this part that we have just established (i.e. the statement that concerns with $J_{\#}$) implies $(A/I^{\perp})_{\#} = \{x \in A/I^{\perp} : \varphi(x)I \subseteq I_{\#}\}$. This gives Relation (5.1), because we have

$$\varphi((A/I^{\perp})_{\#}) = \{\varphi(x) : x \in A/I^{\perp}; \varphi(x)I \subseteq I_{\#}\} = \{\varphi(q(a)) : a \in A; \varphi(q(a))I \subseteq I_{\#}\}$$

as well as $\varphi(q(a))x = ax$ for all $a \in A$ and $x \in I$.

Thirdly, we will show that $q(A_{\#})$ is an essential ideal of $(A/I^{\perp})_{\#}$. Indeed, by Relation (4.1) and part (a) above, one has

$$I^{\perp} \cap A_d = (I_d)^{\perp} \cap A_d,$$

which produces a $*$ -monomorphism from $q(A_d) \cong A_d/(I^{\perp} \cap A_d)$ to $M(I_d)$ with its image containing I_d . Therefore, Remark 2.2(d) and Proposition 3.5(b) imply that the ideal $q(A_d)$ of A/I^{\perp} lies inside $(A/I^{\perp})_d$. Moreover, suppose that $J \subseteq (A/I^{\perp})_d$ is a non-zero closed ideal. As q restricts to an injection from $q^{-1}(J) \cap I^{\perp\perp}$ to an ideal of the discrete C^* -algebra $(A/I^{\perp})_d$, we know that $q^{-1}(J) \cap I^{\perp\perp} \subseteq A_d$ (because of Proposition 3.5(a)) and that $q^{-1}(J) \cap I^{\perp\perp} \neq \{0\}$ (since $q^{-1}(J) \not\subseteq I^{\perp}$). These show that $J \cap q(A_d) \neq \{0\}$. The arguments for the statements concerning $q(A_{sf})$, $q(A_{al})$, $q(A_{\mathbb{II}})$ and $q(A_{\mathbb{III}})$ are the same.

(c) Obviously, for any $x \in A$, one has $xJ_d = (0)$ if and only if $xJJ_d = (0)$. Thus, this part follows from part (b) and the fact that $J_{\mathbb{II}} = J_{sf} \cap J_d^{\perp}$ (see Theorem 5.3(d)).

(d) Note that we have $xJ_{\#} = (0)$ if and only if $xJJ_{\#} = (0)$ for $\# = \mathbb{II}, sf$. Hence, this part follows from part (b) as well as the two equalities $J_{\mathbb{III}} = J_{sf}^{\perp} = (J_d + J_{\mathbb{II}})^{\perp}$ (as given in Theorem 5.3(d)). \square

COROLLARY 5.7. *Let $A_{d,1} := A_d \cap A_1$, $A_{d,\infty} := A_d \cap A_{\infty}$, $A_{\mathbb{II},1} := A_{\mathbb{II}} \cap A_1$ and $A_{\mathbb{II},\infty} := A_{\mathbb{II}} \cap A_{\infty}$. Then $A_{d,1}$, $A_{d,\infty}$, $A_{\mathbb{II},1}$ and $A_{\mathbb{II},\infty}$ are the largest discrete finite ideal, the largest discrete essentially infinite ideal, the largest type \mathbb{II} finite ideal and the largest type \mathbb{II} essentially infinite ideal respectively. These are disjoint normal ideals. Moreover, $A_{d,1} + A_{d,\infty} + A_{\mathbb{II},1} + A_{\mathbb{II},\infty} + A_{\mathbb{III}}$ is an essential ideal of A , or equivalently,*

$$A_{d,1} \oplus A_{d,\infty} \oplus A_{\mathbb{II},1} \oplus A_{\mathbb{II},\infty} \oplus A_{\mathbb{III}} \subseteq A \subseteq M(A_{d,1}) \oplus M(A_{d,\infty}) \oplus M(A_{\mathbb{II},1}) \oplus M(A_{\mathbb{II},\infty}) \oplus M(A_{\mathbb{III}}).$$

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Proof. We first note that by Theorem 4.5(d),

$$A_{d,1} = (A_d)_1, \quad A_{d,\infty} = (A_d)_\infty, \quad A_{\text{II},1} = (A_{\text{II}})_1 \quad \text{and} \quad A_{\text{II},\infty} = (A_{\text{II}})_\infty. \quad (5.2)$$

Thus, the first claim follows from Proposition 3.5(a), parts (a) and (b) of Theorem 4.5 as well as Theorem 5.3(a). These four ideals are clearly disjoint and they are normal because of Remark 4.3(a), Theorem 4.5(a), parts (c) and (d) of Theorem 5.3 as well as the fact that $I^\perp \cap J^\perp = (I + J)^\perp$ for any two closed ideals $I, J \subseteq A$. In order to establish the third statement, we consider a non-zero closed ideal $J \subseteq A$. By Theorem 5.3(b) and Proposition 2.1(c), we know that

$$J \cap A_d \neq \{0\}, \quad J \cap A_{\text{II}} \neq \{0\} \quad \text{or} \quad J \cap A_{\text{III}} \neq \{0\}.$$

If $J \cap A_d \neq \{0\}$, we know from Remark 4.3(b) as well as the first two equalities of (5.2) that $J \cap A_{d,1} \neq \{0\}$ or $J \cap A_{d,\infty} \neq \{0\}$. Similarly, if $J \cap A_{\text{II}} \neq \{0\}$, then $J \cap A_{\text{II},1} \neq \{0\}$ or $J \cap A_{\text{II},\infty} \neq \{0\}$. Thus, $A_{d,1} + A_{d,\infty} + A_{\text{II},1} + A_{\text{II},\infty} + A_{\text{III}}$ is an essential ideal. The displayed relations follow from the well-known fact that $M(B \oplus D) = M(B) \oplus M(D)$ for two C^* -algebras B and D . \square

Observe that the C^* -algebras $M(A_{d,1})$, $M(A_{d,\infty})$, $M(A_{\text{II},1})$, $M(A_{\text{II},\infty})$ and $M(A_{\text{III}})$ are also discrete finite, discrete essentially infinite, type II finite, type II essentially infinite and type III, respectively (by Proposition 3.5(b) and the argument of Theorem 4.5). Furthermore, one may obtain results similar to those in Corollary 5.5 for $A_{d,1}$, $A_{d,\infty}$, $A_{\text{II},1}$ and $A_{\text{II},\infty}$, but we leave them to the readers.

6. Some special cases

In this section, we will consider special classes of C^* -algebras for which we have a better decomposition in Corollary 5.7.

6.1. Prime C^* -algebras

The first special class is that of prime C^* -algebras. Recall that a C^* -algebra A is *prime* if $\{0\}$ is a prime ideal of A , or equivalently, A has no non-trivial normal ideal. Since the ideals $A_{d,1}$, $A_{d,\infty}$, $A_{\text{II},1}$, $A_{\text{II},\infty}$ and A_{III} are normal (see Corollary 5.7), we have the following.

PROPOSITION 6.1. *Any prime C^* -algebra is of one of the five types: discrete finite, discrete essentially infinite, type II finite, type II essentially infinite or type III.*

It is well-known that the kernel of a factor representation of a C^* -algebra is a prime ideal. If the C^* -algebra is separable, one has the strong converse that every prime ideal is primitive. These give the following well-known fact for a C^* -algebra A :

(P1). If there is an injective $*$ -homomorphism φ from A to a factor M with $\varphi(A)'' = M$, then A is prime.

(P2). If A is a separable prime C^* -algebra, there exists a faithful irreducible $*$ -representation of A .

Clearly, a W^* -algebra is a factor if and only if it is a prime C^* -algebra. Because of this, as well as Proposition 6.6(b) and Corollary 6.11 below, one may regard prime C^* -algebras as “ C^* -algebra factors”.

Let us present the following description of discrete prime C^* -algebras. This result is more or less a reformation of [27, Corollary 2.4].

PROPOSITION 6.2. *Let A be a C^* -algebra.*

(a) *The following statements are equivalent.*

- 1) *A is prime and discrete.*
- 2) *There exists a Hilbert space \mathfrak{H} such that $\mathcal{K}(\mathfrak{H}) \subseteq A \subseteq \mathcal{B}(\mathfrak{H})$.*
- 3) *A contains $\mathcal{K}(\mathfrak{H})$ as an essential ideal for some Hilbert space \mathfrak{H} .*

(b) *A is prime, discrete and finite if and only if $A \cong M_n(\mathbb{C})$ for some $n \in \mathbb{N}$.*

Proof. (a) 1) \Rightarrow 2). This follows directly from [27, Corollary 2.4].

2) \Rightarrow 3). This follows from Remark 2.2(d).

3) \Rightarrow 1). Since A contains $\mathcal{K}(\mathfrak{H})$ as an essential ideal, Proposition 3.5(b) implies that A is discrete. Moreover, if $J \subseteq A$ is a non-zero closed ideal, then $J \cap \mathcal{K}(\mathfrak{H}) = \mathcal{K}(\mathfrak{H})$ (as $\mathcal{K}(\mathfrak{H})$ is simple and essential) which means that $J^\perp \subseteq \mathcal{K}(\mathfrak{H})^\perp = \{0\}$. In other words, A has no non-zero normal ideal and hence it is prime.

(b) It is well-known that M_n is a discrete finite simple C^* -algebra. Conversely, if \mathfrak{H} is as in Statement (2) of part (a), then the ideal $\mathcal{K}(\mathfrak{H})$ of A is finite. Hence, $n := \dim \mathfrak{H} < \infty$ and $A \cong M_n(\mathbb{C})$. \square

For instance, the uniform Roe algebra of any uniformly locally finite metric space is an essentially infinite discrete prime C^* -algebra. In the following, we give some more examples of prime C^* -algebras of different types which are not W^* -algebra.

EXAMPLE 6.3. (a) *If Γ is a countably infinite amenable group, and β is as in Example 4.8(b), then $c_0(\Gamma)^1 \rtimes_\beta \Gamma$ contains $\mathcal{K}(\ell^2(\Gamma))$ as an essential ideal, and hence it is a discrete essentially infinite prime C^* -algebra.*

(b) *Let Γ be any countable ICC group. Since the group von Neumann algebra $L(\Gamma)$ is a factor, Statement (P1) above implies that the reduced group C^* -algebra $C_r^*(\Gamma)$ is prime. Moreover, since $C_r^*(\Gamma)$ has a faithful trace, it is finite. Consequently, by Propositions 6.1 and Proposition 6.2(b), $C_r^*(\Gamma)$ is type II finite.*

As in the case of W^ -algebra, if, in addition, Γ is amenable, then $C_r^*(\Gamma)$ is nuclear, and is non-isomorphic to the non-nuclear type II finite prime C^* -algebra $C_r^*(\mathbb{F}_2)$, where \mathbb{F}_2 is the free group on two generators.*

(c) *Suppose that $A \not\cong \mathcal{K}(\ell^2)$ is a simple AF-algebra that does not have a tracial state. Then A is a type II essentially infinite prime C^* -algebra. In fact, A is not finite as it has no tracial state. By [10, Proposition 4.11], A is semi-finite. However, since discrete simple C^* -algebras are of the form $\mathcal{K}(\ell^2)$, we know that A is not discrete.*

(d) *The Calkin algebra $\mathcal{B}(\ell^2)/\mathcal{K}(\ell^2)$ is a simple purely infinite C^* -algebra. Hence it is a type III prime C^* -algebra (by Proposition 3.10(b)).*

6.2. C^* -algebras with extremally disconnected primitive ideal spaces

The second special class are C^* -algebras of which all normal ideals are complementary. Before looking at this class, let us first give the following result concerning normal ideals of essential extensions.

LEMMA 6.4. *If A is an essential extension of a C^* -algebra B , the map $\Gamma : J \mapsto J \cap B$ is a bijection from the set of normal ideals of A onto that of B .*

Proof. Let $I \subseteq A$ be a closed ideal. We first show that

$$I^\perp = (I \cap B)^\perp. \quad (6.1)$$

Indeed, one clearly has $I^\perp \subseteq (I \cap B)^\perp$. Suppose on the contrary that there is $x \in$

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$(I \cap B)^\perp \setminus I^\perp$. Then $xI \neq (0)$ and hence \overline{xI} is a non-zero ideal of A . As B is an essential ideal of A , one can find $b \in \overline{xI} \cap B \setminus \{0\}$. Thus, b can be approximated in norm by elements of the form $\sum_{k=1}^n y_k x z_k$ where $y_1, \dots, y_n, z_1, \dots, z_n \in I$, and bb^* can be approximated by elements of the form $\sum_{k=1}^n y_k x z_k b^*$. Since $z_k b^* \in I \cap B$, we obtain the contradiction that $bb^* = 0$.

Now, if $J = I^\perp$, then by Relation (4.1), one has $J \cap B = (I \cap B)^\perp \cap B$ and hence $\Gamma(J)$ is a normal ideal of B . Furthermore, as

$$\Gamma(J)^{\perp\perp} = (J \cap B)^{\perp\perp} = J^{\perp\perp} = J \quad (6.2)$$

(by Relation (6.1)), we see that Γ is injective.

Finally, let J_0 be a normal ideal of B and $I_0 \subseteq B$ be an ideal satisfying $J_0 = I_0^\perp \cap B$. If $J := I_0^\perp$, then obviously, $\Gamma(J) = J_0$. \square

Recall that a topological space (not necessarily Hausdorff) is *extremally disconnected* if the closure of every open subset is again open. It happens that those C^* -algebras described in the beginning of this subsection are precisely those with extremally disconnected primitive spectra (equipped with the hull-kernel topology). We will give some other equivalent forms of these algebras in the following proposition (although not all of them are needed in this paper).

PROPOSITION 6.5. *The following statements are equivalent.*

- 1) *The primitive ideal space $\text{Prim}(A)$ is extremally disconnected.*
- 2) *$A = I^\perp + I^{\perp\perp}$, for every closed ideal $I \subseteq A$.*
- 3) *$\overline{p}^1 \in \text{OP}^1(A)$ for every $p \in \text{OP}^1(A) \cap Z(A^{**})$.*
- 4) *$\Lambda(q) := qA$ is a surjection (and hence a bijection) from $\text{Proj}^1(ZM(A)) \cup \{0\}$ onto the set of normal ideals of A .*
- 5) *$\text{Prim}(M(A))$ is extremally disconnected.*
- 6) *$ZM(A)$ is an AW^* -algebra and $\Delta : J \mapsto J \cap ZM(A)$ is an injection (equivalently, a bijection) from the set of normal ideals of $M(A)$ to the set of normal ideals of $ZM(A)$.*
- 7) *A is boundedly centrally closed (in the sense of [3, p.165]).*

Proof. 1) \Leftrightarrow 2). Let $I \subseteq A$ be an ideal. Since any element $P \in \text{Prim}(A)$ is prime, we know that $I \subseteq P$ or $I^\perp \subseteq P$. This ensures that $\text{Prim}(A) \setminus \text{hull}(I) \subseteq \{P \in \text{Prim}(A) : I^\perp \subseteq P\}$, which gives

$$I' := \bigcap \text{Prim}(A) \setminus \text{hull}(I) \supseteq \bigcap \{P \in \text{Prim}(A) : I^\perp \subseteq P\} = I^\perp.$$

Conversely, one has $I' \subseteq I^\perp$. In fact, suppose on the contrary that $I'I \neq \{0\}$. Then one can find $P_0 \in \text{Prim}(A)$ with $I'I \not\subseteq P_0$. Thus, $P_0 \notin \text{hull}(I)$, and we arrive at the contradiction that $I' \subseteq P_0$. In other words,

$$\overline{\text{Prim}(A) \setminus \text{hull}(I)} = \text{hull}(I^\perp). \quad (6.3)$$

Consequently, $\overline{\text{Prim}(A) \setminus \text{hull}(I)}$ is open if and only if $\text{hull}(I^\perp)$ is open, which, because of (6.3), is equivalent to

$$\text{Prim}(A) \setminus \text{hull}(I^\perp) = \overline{\text{Prim}(A) \setminus \text{hull}(I^\perp)} = \text{hull}(I^{\perp\perp}).$$

Therefore, $\overline{\text{Prim}(A) \setminus \text{hull}(I)}$ being open is, again because of (6.3), the same as

$$\text{hull}(I^\perp) \cap \text{hull}(I^{\perp\perp}) = \emptyset. \quad (6.4)$$

Finally, Relation (6.4) and $A = I^\perp + I^{\perp\perp}$ are equivalent, because a closed ideal $I_0 \subseteq A$ is proper if and only if there is $P \in \text{Prim}(A)$ satisfying $I_0 \subseteq P$.

2) \Rightarrow 3). Suppose that $p \in \text{OP}^1(A) \cap Z(A^{**})$. As the closed ideal $\text{her}_A(p)^\perp = \text{her}_A(1 - \bar{p}^1)$ (see Equality (2.2)), we know from Statement (2) that

$$\text{her}_A(1 - \bar{p}^1) + \text{her}_A(1 - \bar{p}^1)^\perp = A.$$

Hence, for any $a \in A$, there exist $x \in \text{her}_A(1 - \bar{p}^1)$ and $y \in \text{her}_A(1 - \bar{p}^1)^\perp$ with $a = x + y$. This shows that $(1 - \bar{p}^1)a = x \in A$. Consequently, $1 - \bar{p}^1$ belongs to $M(A)$, and hence is a central closed projection of A .

3) \Rightarrow 4). We first note that Λ is always injective, because $\text{Proj}^1(ZM(A)) \subseteq \text{OP}^1(A) \cap Z(A^{**})$ and we have Statement (O1). Moreover, it is clear that $\Lambda(q)$ equals $((1 - q)A)^\perp$ and hence is a normal ideal of A . Suppose that $I \subseteq A$ is a closed ideal and $J := I^\perp$. If $p, q \in \text{OP}^1(A) \cap Z(A^{**})$ with $I = \text{her}_A(p)$ and $J = \text{her}_A(q)$, then Relation (2.2) tells us that $q = 1 - \bar{p}^1$. As \bar{p}^1 is both open and closed (by Statement (3)), it belongs to $M(A)$ and so is q . Thus, $J = qA$ as required.

4) \Rightarrow 2). If Λ is surjective, then for any closed ideal $I \subseteq A$, there is $q \in \text{Proj}^1(ZM(A)) \cup \{0\}$ with $I^\perp = qA$, which gives $A = qA + (1 - q)A = I^\perp + I^{\perp\perp}$.

4) \Leftrightarrow 5). Notice that Statement (5) is exactly Statement (1) for the C^* -algebra $M(A)$. Thus, the equivalence of Statements (1) and (4) for the C^* -algebra $M(A)$ (which was established above) implies that Statement (5) is the same as the following statement:

4') For any normal ideal $J \subseteq M(A)$, there is $q \in \text{Proj}^1(ZM(A)) \cup \{0\}$ with $J = qM(A)$. Consequently, it suffices to show that Statement (4) for the C^* -algebra A is equivalent to Statement (4').

In fact, suppose that J is a normal ideal of $M(A)$. By Lemma 6.4, the ideal $J \cap A \subseteq A$ is normal. Hence, Statement (4) (for the C^* -algebra A) gives $q \in \text{Proj}^1(ZM(A)) \cup \{0\}$ with $J \cap A = qA$. Observe that if $x \in M(A)$ satisfying $xqA = (0)$, then $x = (1 - q)x$. This, together with Relation (6.2), tells us that

$$J = (J \cap A)^{\perp\perp} = (qA)^{\perp\perp} = ((1 - q)M(A))^\perp = qM(A)$$

(note that these \perp are taken in $M(A)$).

Conversely, if I is a normal ideal of A , then Lemma 6.4 and Statement (4') produce $q \in \text{Proj}^1(ZM(A)) \cup \{0\}$ satisfying $I = qM(A) \cap A = qA$, as is required.

1) \Rightarrow 6). Suppose that \mathcal{J} is a closed ideal of $ZM(A) \cong C_b(\text{Prim}(A))$ (by the Dauns-Hofmann theorem; see e.g. [26, Corollary 4.4.8]). Then

$$U_{\mathcal{J}} := \{J \in \text{Prim}(A) : f(J) \neq 0, \text{ for some } f \in \mathcal{J}\}$$

is open, and its closure $\overline{U_{\mathcal{J}}}$ is an open subset of $\text{Prim}(A)$ (by Statement (1)). One may regard $\mathcal{J}_0 := C_b(\overline{U_{\mathcal{J}}})$ and $\mathcal{J}_0 := C_b(\text{Prim}(A) \setminus \overline{U_{\mathcal{J}}})$ as disjoint ideals of $C_b(\text{Prim}(A))$ with their sum equals $C_b(\text{Prim}(A))$. Thus, $\mathcal{J}_0 = \mathcal{J}_0^\perp$ and $\mathcal{J}_0 = \mathcal{J}_0^\perp$ (here \perp are taken in $C_b(\text{Prim}(A))$). Since $\mathcal{J} \subseteq \mathcal{J}_0$ and $\mathcal{J}^\perp \subseteq \mathcal{J}_0$ (by the definition of $U_{\mathcal{J}}$, one has $g(U_{\mathcal{J}}) = \{0\}$ for any $g \in \mathcal{J}^\perp$), we see that $\mathcal{J}_0 = \mathcal{J}^\perp$ as well as $\mathcal{J}_0 = \mathcal{J}^{\perp\perp}$. Thus, $ZM(A) = \mathcal{J}^\perp + \mathcal{J}^{\perp\perp}$ and the primitive ideal space of $ZM(A)$ is extremally disconnected, because of the equivalence of Statements (1) and (2) for $ZM(A)$ (which was established above).

Let J be a normal ideal of $M(A)$. By Statement (4') (which was shown to be equivalent to Statement (1) in the above), there is $q \in \text{Proj}^1(ZM(A)) \cup \{0\}$ with $J = qM(A)$, and hence

$$J \cap ZM(A) = qZM(A),$$

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which is a normal ideal of $ZM(A)$. The injectivity of Δ follows from the fact that if $p, q \in \text{Proj}^1(ZM(A)) \cup \{0\}$ with $pZM(A) = qZM(A)$, then $p = q$.

On the other hand, we claim that Δ is surjective when $ZM(A)$ is an AW^* -algebra. In fact, since $\text{Prim}(ZM(A))$ is extremally disconnected, we know from Statement (4) for $ZM(A)$ (which was shown to be equivalent to Statement (1) for $ZM(A)$) that any normal ideal J_0 of $ZM(A)$ comes from a projection q_0 in $ZM(A)$, and we have $\Delta(q_0M(A)) = J_0$. 6) \Rightarrow 1). By the equivalence of Statements (1) and (4'), which was established above, it suffices to verify Statement (4'). Suppose that J is a non-zero normal ideal of $M(A)$. By the assumption, $J \cap ZM(A)$ is a normal ideal and there is $q \in \text{Proj}^1(ZM(A)) \cup \{0\}$ with $J \cap ZM(A) = qZM(A)$, because $ZM(A)$ is a commutative AW^* -algebra, and we can apply Statement (4) for $ZM(A)$ (which is equivalent to Statement (1) for $ZM(A)$, namely, $\text{Prim}(ZM(A))$ being extremally disconnected). Since $qM(A) \cap ZM(A) = qZM(A) = J \cap ZM(A)$, the injectivity of Δ tells us that $J = qM(A)$.

1) \Leftrightarrow 7). This is precisely [3, Proposition 2.9]. \square

If $\text{Prim}(A)$ is extremally disconnected, we know from the above that there are bijective correspondences between $\text{Proj}^1(ZM(A)) \cup \{0\}$, the set of normal ideals of A , the set of normal ideals of $M(A)$ as well as the set of normal ideals of $ZM(A)$.

In the case when A unital, the implication 1) \Rightarrow 6) also follows implicitly from [29, Theorem 2.1] and [4, Proposition 3.2]. However, it will be easier to prove this directly than to recall the definition of “quasi-standard” C^* -algebras and their properties.

Since the canonical map from the spectrum of a C^* -algebra A to $\text{Prim}(A)$ induces a bijection from the collection of open subsets (respectively, closed subsets) of the spectrum to the collection of open subsets (respectively, closed subsets) of $\text{Prim}(A)$, we know that $\text{Prim}(A)$ is extremally disconnected if and only if A has an extremally disconnected spectrum.

By [3, Example 2.10], all prime C^* -algebras, all AW^* -algebras and local multipliers algebras of all C^* -algebras will satisfy the equivalent conditions in Proposition 6.5. Moreover, Proposition 6.5 also tells us that if A has an extremally disconnected spectrum, then A is prime if and only if $\dim ZM(A) = 1$.

In the following, we denote by $\beta\mathfrak{J}$ the Stone-Cech compactification of a set \mathfrak{J} (as a discrete topological space).

PROPOSITION 6.6. *Let A be a C^* -algebra with an extremally disconnected spectrum.*

- (a) $A = A_{d,1} \oplus A_{d,\infty} \oplus A_{\text{II},1} \oplus A_{\text{II},\infty} \oplus A_{\text{III}}$.
- (b) Suppose, furthermore, that $ZM(A) \cong \ell^\infty(\mathfrak{J})$ for a set \mathfrak{J} . Then \mathfrak{J} is the disjoint union of subsets $\mathfrak{J}_{d,1}$, $\mathfrak{J}_{d,\infty}$, $\mathfrak{J}_{\text{II},1}$, $\mathfrak{J}_{\text{II},\infty}$ and $\mathfrak{J}_{\text{III}}$ such that for $\#$ equals $d, 1$ or d, ∞ or $\text{II}, 1$ or II, ∞ or III , one can find a continuous field $\{A_\omega\}_{\omega \in \beta\mathfrak{J}_\#}$ of C^* -algebras on $\beta\mathfrak{J}_\#$ with $A_\#$ being the algebra of continuous sections of $\{A_\omega\}_{\omega \in \beta\mathfrak{J}_\#}$. Moreover, for each $i \in \mathfrak{J}_\# \subseteq \beta\mathfrak{J}_\#$, the C^* -algebra A_i is prime and it is an ideal of $A_\#$. The canonical map $\Theta : A_\# \rightarrow \prod_{i \in \mathfrak{J}_\#}^\infty A_i$ is injective and $\overline{\sum_{i \in \mathfrak{J}_\#} A_i}$ can be regarded as an essential ideal of $A_\#$.

Proof. (a) By Relation 2.2, Proposition 6.5, as well as Theorems 4.5 and 5.3, the central open projections corresponding to the ideals $A_{d,1}$, $A_{d,\infty}$, $A_{\text{II},1}$, $A_{\text{II},\infty}$ and A_{III} belong to $M(A)$ (since these projections are both open and closed) with their sum being 1 (because this sum belongs to $M(A)$ and is dense in 1 by Corollary 5.7). This gives the conclusion. (b) Let $p_\#$ be the projection in $ZM(A)$ with $A_\# = p_\#A$ (see Statement (4) of Proposition 6.5). Then $p_\# \in \ell^\infty(\mathfrak{J})$ is the characteristic function for a subset $\mathfrak{J}_\#$, and \mathfrak{J} is the disjoint

union of $\mathfrak{J}_{d,1}$, $\mathfrak{J}_{d,\infty}$, $\mathfrak{J}_{\mathbb{I},1}$, $\mathfrak{J}_{\mathbb{I},\infty}$ and $\mathfrak{J}_{\mathbb{III}}$ (by part (a)). As $A = A_{d,1} \oplus A_{d,\infty} \oplus A_{\mathbb{I},1} \oplus A_{\mathbb{I},\infty} \oplus A_{\mathbb{III}}$, we know that

$$ZM(A) = ZM(A_{d,1}) \oplus ZM(A_{d,\infty}) \oplus ZM(A_{\mathbb{I},1}) \oplus ZM(A_{\mathbb{I},\infty}) \oplus ZM(A_{\mathbb{III}}),$$

and $ZM(A_{\#}) = \ell^\infty(\mathfrak{J}_{\#})$. Moreover, one may regard $A_{\#}$ as a Banach $\ell^\infty(\mathfrak{J}_{\#})$ -normed module, and hence consists of the continuous sections of a continuous field $\{A_\omega\}_{\omega \in \beta\mathfrak{J}_{\#}}$ of C^* -algebras (see e.g. [12]).

For every $i \in \mathfrak{J}_{\#}$, we consider p_i to be the projection in $ZM(A)$ corresponding to the point mass at i . Then we have

$$A_i \cong A/(1 - p_i)A \cong p_i A = \text{her}_A(p_i).$$

Suppose that J is a non-zero closed ideal of A_i and $p \in \text{OP}^1(A)$ with $J = \text{her}_A(p)$. As $p \leq p_i$ and $p_i \in M(A)$ (which implies that p_i is a closed projection of A), we know that $\bar{p}^1 \leq p_i$. Since $\bar{p}^1 \in M(A)$ (by Statement (3) of Proposition 6.5) and p_i is a minimal projection in $ZM(A)$, we know that $\bar{p}^1 = p_i$ and J is an essential ideal of A_i . This shows that A_i is prime.

Clearly, A_i is an ideal of $A_{\#}$ and $I_{\#} := \overline{\sum_{i \in \mathfrak{J}_{\#}} A_i}$ is a closed ideal of $A_{\#}$. Note that $A_{\#}$ also has an extremally disconnected spectrum. There exists, by Statement (4) of Proposition 6.5, a projection $q \in ZM(A_{\#})$ with $I_{\#}^\perp = qA_{\#}$. Hence, q is the characteristic function in $\ell^\infty(\mathfrak{J}_{\#})$ for a subset $S \subseteq \mathfrak{J}_{\#}$. If S is non-empty, then for each $i \in S$, we have the contradiction that $A_i \subseteq I_{\#} \cap I_{\#}^\perp$. Thus, $I_{\#}$ is an essential ideal of $A_{\#}$.

Finally, suppose that $a \in \ker \Theta$. If we regard a as a continuous section on $\{A_\omega\}_{\omega \in \beta\mathfrak{J}_{\#}}$, then its values at every $i \in \mathfrak{J}_{\#}$ is zero and hence a is the zero section (as $\mathfrak{J}_{\#}$ is dense in $\beta\mathfrak{J}_{\#}$). \square

Proposition 6.6(a) tells us that the decomposition in Corollary 5.7 extends the corresponding one for AW^* -algebras. Moreover, part (b) applies to all AW^* -algebras whose centers are of the form $\ell^\infty(\mathfrak{J})$.

6.3. Standard C^* -algebras and C^* -algebras with Hausdorff primitive ideal spaces

We end this paper by considering some situations under which the C^* -algebra can be “decomposed” as a continuous field of prime C^* -algebras. Let us begin with the following result.

PROPOSITION 6.7. *If A is discrete (respectively, essentially infinite or tracially infinite; see Definition 3.10), there is an open dense subset Ξ_d (respectively, Ξ_∞ or Ξ_{ti}) of $\text{Prim}(A)$ such that A/P is discrete (respectively, essentially infinite or tracially infinite) for every P in Ξ_d (respectively, Ξ_∞ or Ξ_{ti}).*

Proof. Suppose that A is discrete. By [27, Theorem 2.3(v)], the largest type I ideal A_{I} is essential. Thus,

$$\Xi_d := \text{Prim}(A) \setminus \text{hull}(A_{\text{I}})$$

is an open dense subset of $\text{Prim}(A)$. For every $P \in \Xi_d$, the image of A_{I} in A/P is a non-zero closed ideal of type I. Thus, by Proposition 6.1, we know that the prime C^* -algebra A/P is discrete.

Secondly, we consider the case when A is essentially infinite. By Proposition 4.6(a), there is an essential closed ideal J_0 of A with $T_s(J_0) = \emptyset$. Set

$$\Xi_\infty := \text{Prim}(A) \setminus \text{hull}(J_0).$$

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Then Ξ_∞ is an open dense subset of $\text{Prim}(A)$. If $P \in \Xi_\infty$ and $q : A \rightarrow A/P$ is the quotient map, then $q(J_0) \neq (0)$ and does not have a tracial state. Thus, $q(J_0)$ is essentially infinite and so is the prime C^* -algebra A/P (by Proposition 6.1).

In the case when A is tracially infinite, there is an essential closed ideal $J_1 \subseteq A$ with $\mathcal{T}(J_1) = \{0\}$. The set

$$\Xi_{\text{ti}} := \text{Prim}(A) \setminus \text{hull}(J_1).$$

is open and dense in $\text{Prim}(A)$. Let $P \in \Xi_{\text{ti}}$ and consider $q : A \rightarrow A/P$ to the quotient map. By the construction, $q(J_1) \neq (0)$.

Suppose that $\mathcal{T}(q(J_1)) \neq \{0\}$. By the argument of Proposition 3.10(a), one can find $y \in q(J_1)_+$ with $T_s(\overline{yq(J_1)y}) \neq \emptyset$. Thus, the hereditary C^* -subalgebra $q^{-1}(\overline{yq(J_1)y}) \cap J_1$ of J_1 (which is non-zero because J_1 is an essential ideal of A) has a tracial state τ . As in the proof of Proposition 3.10(a), the lower semi-continuous trace extension of τ to J_1 (as given in [10, Lemma 4.6]) is semi-finite. This gives a contradiction.

Now, because $\mathcal{T}(q(J_1)) = \{0\}$ and $q(J_1)$ is an essential ideal of A/P (as A/P is prime), we know that A/P is tracially infinite. \square

EXAMPLE 6.8. *Let $A = \mathcal{B}(\ell^2)$ and $P = \mathcal{K}(\ell^2)$. Then A is discrete and A/P is of type III. Therefore, one cannot expect $\Xi_{\text{d}} = \text{Prim}(A)$ in Proposition 6.7(b).*

By the Dauns-Hofmann theorem (see e.g. [26, Corollary 4.4.8]), one has a continuous map

$$\Phi : \text{Prim}(A) \rightarrow \text{Prim}(ZM(A))$$

with dense range. In fact, if we identify elements in $\text{Prim}(ZM(A))$ with one-dimensional unital $*$ -representations (i.e., characters) of $ZM(A)$, then $\Phi(P)(x) = \|x + \tilde{P}\|_{M(A)/\tilde{P}}$ for every $x \in ZM(A)_+$, where is \tilde{P} the unique element in $\text{Prim}(M(A))$ with $\tilde{P} \cap A = P$, i.e.,

$$\tilde{P} := \{m \in M(A) : mA, Am \subseteq P\}.$$

Thus, if ω is a character on $ZM(A)$, then $\Phi(P) = \omega$ if and only if $\ker(\omega) \subseteq \tilde{P}$.

Let I be a closed ideal of A . It is well-known that I is the algebra of bounded continuous sections of a continuous fields Υ^I of C^* -algebras over $\text{Prim}(ZM(A))$ with the fiber Υ_ω^I over $\omega \in \text{Prim}(ZM(A))$ being $I/\overline{I \ker(\omega)}$ (see e.g. [12]).

The following lemma gives another description for the fiber and it also gives a ‘‘visual description’’ of the image of an open subset of $\text{Prim}(A)$ under Φ . This lemma could be known, but since we do not find it in the literature, we give it argument here for completeness.

LEMMA 6.9. *Let $I \subseteq A$ be a non-zero closed ideal. Consider any $\omega_0 \in \text{Prim}(ZM(A))$ and $P_0 \in \text{Prim}(A) \setminus \text{hull}(I)$.*

- (a) $\Phi(P_0) = \omega_0$ if and only if $I \ker(\omega_0) \subseteq P_0$.
- (b) $\overline{I \ker(\omega_0)} = \bigcap \{P \in \text{Prim}(A) \setminus \text{hull}(I) : \Phi(P) = \omega_0\}$.
- (c) $\Phi(\text{Prim}(A) \setminus \text{hull}(I)) = \{\omega \in \text{Prim}(ZM(A)) : \Upsilon_\omega^I \neq (0)\}$.

Proof. (a) If $\omega_0 = \Phi(P_0)$, then $\ker(\omega_0) \subseteq \tilde{P}_0$, and we have $I \ker(\omega_0) \subseteq P_0$. Conversely, suppose that $I \ker(\omega_0) \subseteq P_0$. Let $Q_0 := P_0 \cap I \in \text{Prim}(I)$ and

$$\tilde{Q}_0 := \{m \in M(I) : mI, Im \subseteq Q_0\}.$$

Consider $\Psi : M(A) \rightarrow M(I)$ to be the canonical $*$ -homomorphism. Set $J := A\Psi^{-1}(\tilde{Q}_0)$.

As $I \ker(\omega_0) \subseteq P_0 \cap I$, one has $\Psi(\ker(\omega_0)) \subseteq \tilde{Q}_0$, and so

$$A \ker(\omega_0) \subseteq J.$$

Moreover, it follows from $\Psi(IJ) \subseteq I\tilde{Q}_0 \subseteq Q_0$ that $IJ \subseteq Q_0 \subseteq P_0$ (since Ψ restricts to the identity map on I). This implies that $J \subseteq P_0$ (as $I \not\subseteq P_0$ by the assumption). Thus, $A \ker(\omega_0) \subseteq P_0$, which implies that $\ker(\omega_0) \subseteq \tilde{P}_0$ and so, $\omega_0 = \Phi(P_0)$.

(b) Since $\overline{I \ker(\omega_0)}$ is a closed ideal of I , one has

$$\overline{I \ker(\omega_0)} = \bigcap \{P \in \text{Prim}(A) \setminus \text{hull}(I) : \overline{I \ker(\omega_0)} \subseteq P\},$$

and this part follows from part (a).

(c) It follows from part (a) that $\omega \in \Phi(\text{Prim}(A) \setminus \text{hull}(I))$ if and only if there exists $P \in \text{Prim}(A) \setminus \text{hull}(I) = \text{Prim}(I)$ such that $I \ker(\omega) \subseteq P$, which is equivalent to the closed ideal $\overline{I \ker(\omega)}$ of I being proper (in other words, $I/\overline{I \ker(\omega)} \neq (0)$). \square

Following [4, p.351] and [29, p.125], for each $\omega \in \text{Prim}(ZM(A))$, we call

$$G(\omega) := \bigcap \Phi^{-1}(\omega)$$

a *Glimm ideal* of A (here we use the usual convention that $\bigcap \emptyset = A$).

THEOREM 6.10. *Suppose that the map Φ is open and that every non-zero Glimm ideal of a C^* -algebra A is a prime ideal.*

(a) *A is the algebra of C_0 -sections of a continuous field of C^* -algebras over $\Omega_A := \Phi(\text{Prim}(A))$ with each fiber being a non-zero prime C^* -algebra (and hence of one of the five types).*

(b) *If A is discrete (respectively, essentially infinite or tracially infinite), there is an open dense subset Ω_d (respectively, Ω_∞ or Ω_{ti}) of Ω_A such that the fiber over elements in Ω_d (respectively, Ω_∞ or Ω_{ti}) are discrete (respectively, essentially infinite or tracially infinite).*

Proof. (a) Consider any $\omega \in \Omega_A$. By Lemma 6.9(b), one has

$$\overline{A \ker(\omega)} = G(\omega),$$

and hence is a prime ideal by the hypothesis. Moreover, it follows from Lemma 6.9(c) that $A/\overline{A \ker(\omega)} = \Upsilon_\omega^A \neq (0)$. Now, Proposition 6.1 tells us that they are of one of the five types.

(b) We will only consider the case when A is discrete since the other two cases follow from similar arguments. Suppose that A_I is the largest type I closed ideal of A and $\Xi_d := \text{Prim}(A) \setminus \text{hull}(A_I)$. By the proof of Proposition 6.7, Ξ_d is an open dense subset of $\text{Prim}(A)$. Set $\Omega_d := \Phi(\Xi_d)$. Then Ω_d is an open dense subset of Ω_A (by the hypothesis) satisfying

$$\Omega_d \subseteq \{\omega \in \Omega_A : G(\omega) \subseteq P, \text{ for some } P \in \Xi_d\}$$

(since $G(\omega) \subseteq P$ whenever $P \in \Phi^{-1}(\omega)$).

Pick any $\omega \in \Omega_d$. If A_I is contained in $G(\omega)$, then we know that $A_I \subseteq P$ for some $P \in \text{Prim}(A) \setminus \text{hull}(A_I)$, which is absurd. Consequently, for every $\omega \in \Omega_d$, the image of A_I in the prime C^* -algebra $\Upsilon_\omega^A = A/G(\omega)$ is a non-zero type I closed ideal, which implies that Υ_ω^A is discrete (because of Proposition 6.1). \square

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If $\text{Prim}(A)$ is Hausdorff (which is weaker than the assumption that the spectrum of A is Hausdorff), then Φ is injective and the hypothesis of Theorem 6.10 is satisfied. Moreover, if A is a “standard” C^* -algebras as in [29, p.127], then the two hypothesis in Theorem 6.10 are satisfied (see also [4, Theorem 3.4]). Thus, Theorem 6.10 gives the following corollary, thanks to [29, p.127] and [29, Theorem 2.4].

COROLLARY 6.11. *If either $\text{Prim}(A)$ is Hausdorff, or A is an AW^* -algebra, or A is the local multiplier algebra of another C^* -algebra, then A is the algebra of continuous C_0 -sections of a continuous fields of prime C^* -algebras over a locally compact Hausdorff space Ω_A with each fiber being non-zero and of one of the five types. If, in addition, A is discrete (respectively, essentially infinite or tracially infinite), there is an open dense subset of Ω_A on which the fiber are non-zero and discrete (respectively, essentially infinite or tracially infinite).*

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