NORMAL STATES ARE DETERMINED BY THEIR FACIAL DISTANCES

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Abstract. Let M be a semi-finite von Neumann algebra with normal state space $\mathfrak{S}(M)$. For any $\phi \in \mathfrak{S}(M)$, let $M_\phi := \{x \in M : x\phi = \phi x\}$ be the centralizer of ϕ with center $\mathcal{Z}(M_\phi)$. We show that for $\phi, \psi \in \mathfrak{S}(M)$, the following are equivalent.

- $\phi = \psi$.
- $\mathcal{Z}(M_\psi) \subseteq \mathcal{Z}(M_\phi)$ and $\phi|_{\mathcal{Z}(M_\phi)} = \psi|_{\mathcal{Z}(M_\phi)}$.
- ϕ, ψ have the same distances to all the closed faces of $\mathfrak{S}(M)$.

As an application, we give an alternative proof of the fact that metric preserving surjections between normal state spaces of semi-finite von Neumann algebras are induced by Jordan *-isomorphisms between the underlying algebras. We then use it to verify some facts concerning F-algebras and Fourier algebras of locally compact quantum groups.

1. Introduction

It is well-known that any point in an n-dimensional simplex Δ_n in the Euclidean space $\mathbb{R}^n$ is characterized by its vertex distances; namely, two points inside Δ_n have the same distances to all the vertices of Δ_n forces them to coincide. It is proved in an interesting paper of Geher ([5]) that, for an n-dimensional real Banach space X with $n \geq 3$, if for every n-simplex in X, the vertex distances do determine points in the n-simplex, then X is a Hilbert space. In other words, points in a compact convex set Δ of a non-Hilbert Banach space may not be determined by their distances from the extreme points of Δ.

It is natural to ask whether distances from closed faces will determine an element in a closed convex set. This paper concerns with such a question in the case of the normal state space $\mathfrak{S}(M)$ of a von Neumann algebra M. More precisely, we ask:

Question 1. Do the "facial distances" determine normal states in $\mathfrak{S}(M)$? More precisely, if ϕ, ψ are normal states of M, does the following hold:

$$\mathrm{dist}(\phi, F) = \mathrm{dist}(\psi, F) \text{ for every norm closed face } F \text{ of } \mathfrak{S}(M) \text{ implies } \phi = \psi?$$

We will give a positive answer to Question 1 when M is semi-finite. A first step to this answer is the following result in Section 2, which seems to be an interesting fact of its own. In fact, let ϕ be a normal state of a semi-finite von Neumann algebra M. If M_ϕ is the centralizer of ϕ with center $\mathcal{Z}(M_\phi)$, then as shown in Proposition 4 (see also Remark 7):

$$\phi \text{ is completely determined by } \mathcal{Z}(M_\phi) \text{ as well as the restriction of } \phi \text{ to } \mathcal{Z}(M_\phi).$$

With this tool, we establish our main result in Section 3, which partially answers Question 1.

Theorem 2. *Suppose that M is a semi-finite von Neumann algebra and $\phi, \psi \in \mathfrak{S}(M)$. If $\mathrm{dist}(\phi, F) = \mathrm{dist}(\psi, F)$ for every norm-closed face F of $\mathfrak{S}(M)$, then $\phi = \psi$.*

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As an application of Theorem 2, one can derive the following fact (see the Appendix).

Proposition 3. *Any metric preserving surjection $\Phi : \mathfrak{S}(M) \to \mathfrak{S}(N)$ between normal state spaces of semi-finite von Neumann algebras M and N is induced by a Jordan *-isomorphism $\Theta : N \to M$, in the sense that its predual map Θ_* extends Φ.*

Using this fact, one can generalize [17, Theorem 1.2] such that the type I assumptions on the dual von Neumann algebra of the F-algebra A_1 and the locally compact group G_1 can be relaxed to the semi-finiteness and the unimodularity, respectively.

After we obtained the proof of Proposition 3 (but before this paper was written down completely), we found that a better form of Proposition 3 was stated in Theorem 5.11(a) of the paper [21] by Mori (see Proposition 10). Note, however, that it does not seem possible to obtain our main result (i.e., Theorem 2) from results in [21].

Using Proposition 10, one can further generalize [17, Theorem 1.2] to any F-algebra and locally compact quantum group without any semi-finiteness restriction. We will present in Section 4 these further generalizations (see Proposition 12 and Corollary 13).

2. The restriction of a normal state to the center of its centralizer

Let M be a von Neumann algebra with normal state space $\mathfrak{S}(M)$ and center $\mathcal{Z}(M)$. We denote by $\mathcal{U}(M)$ and $\mathcal{P}(M)$ the set of unitaries and the set of projections, respectively, in M. For every $\phi \in \mathfrak{S}(M)$, we denote by $\mathbf{s}_\phi \in \mathcal{P}(M)$ the support projection of ϕ, and we also set

$$M_\phi := \{x \in M : x\phi = \phi x\}; \tag{2.1}$$

here, $(x\phi)(y) := \phi(yx)$ and $(\phi x)(y) := \phi(xy)$ $(y \in M)$. Following [7], we call M_ϕ the *centralizer* of ϕ. In the case when $\mathbf{s}_\phi = 1$, if $\{\sigma_t^\phi\}_{t\in\mathbb{R}}$ is the modular automorphism group of ϕ, then M_ϕ is precisely the fixed point algebra of the action σ^ϕ (see e.g., Definition 2.1 and Theorem 2.6 in Chapter VIII of [22]).

Suppose now that M is a semi-finite von Neumann algebra with a normal faithful semi-finite trace τ. We recall in the following the construction of the non-commutative L_1-space, $L_1(M,\tau)$, from [23]. Let $(\pi_\tau, \mathfrak{H}_\tau)$ be the GNS construction of τ. We identify M with $\pi_\tau(M) \subseteq \mathcal{L}(\mathfrak{H}_\tau)$. Consider $L_0(M,\tau)$ to be the collection of closed and densely defined operators T on $\mathfrak{H}_\tau$ affiliated with M satisfying $\tau\big(1-E_{|T|}([0,\lambda])\big) < +\infty$ for large enough λ, where $E_{|T|}$ is the spectral projection measure of the absolute value $|T|$ of T. It is well-known that $L_0(M,\tau)$ is closed under adjoints. Moreover, $L_0(M,\tau)$ is closed under closures of the additions and the multiplications (for densely defined closed operators). Hence, $L_0(M,\tau)$ is a *-algebra, and the von Neumann algebra M is a *-subalgebra of $L_0(M,\tau)$.

The trace τ extends to the cone $L_0^+(M,\tau)$ of positive self-adjoint elements in $L_0(M,\tau)$ as follows: $\tau(S) := \lim_{\epsilon\to 0} \tau\big(S(1+\epsilon S)^{-1}\big)$ for every $S \in L_0^+(M,\tau)$ (see, e.g., [22, p.174]), and it satisfies

$$\tau(uSu^*) = \tau(S) \qquad (S \in L_0^+(M,\tau), u \in \mathcal{U}(M)). \tag{2.2}$$

The subspace

$$L_1(M,\tau) := \{T \in L_0(M,\tau) : \tau(|T|) < \infty\}$$

is a Banach space under the norm given by $\|T\|_1 := \tau(|T|)$, and τ induces a linear functional, again denoted by τ, on $L_1(M,\tau)$. We denote by $L_1^+(M,\tau)$ the set of positive self-adjoint operators in $L_1(M,\tau)$. This set linearly spans $L_1(M,\tau)$. For any $S \in L_1(M,\tau)$ and $y \in M$, one has $Sy \in L_1(M,\tau)$ and $|\tau(Sy)| \leq \tau(|S|)\|y\|$. From this, one obtains an isometric order isomorphism from $L_1(M,\tau)$ onto M_* sending $S \in L_1(M,\tau)$ to the element $\tau_S \in M_*$ defined by

$$\tau_S(y) := \tau(Sy) \qquad (y \in M).$$

Proposition 4. *Suppose that M is a semi-finite von Neumann algebra. If $\phi, \psi \in \mathfrak{S}(M)$ satisfying $\mathcal{Z}(M_\psi) \subseteq \mathcal{Z}(M_\phi)$ and $\phi|_{\mathcal{Z}(M_\phi)} = \psi|_{\mathcal{Z}(M_\phi)}$, then $\phi = \psi$.*

Proof: Let τ be a normal faithful semi-finite trace on M. Fix any $S \in L_1^+(M,\tau)$. If $u \in \mathcal{U}(M_{\tau_S})$ and $x \in M$, it follows from $\tau_S(uxu^*) = \tau_S u(xu^*) = u\tau_S(xu^*) = \tau_S(x)$ and Relation (2.2) that $\tau_{u^*Su}(x) = \tau_S(x)$. The bijectivity of the assignment $S \mapsto \tau_S$ from $L_1(M,\tau)$ onto M_* tells us that $u^*Su = S$. Conversely, if $u \in \mathcal{U}(M)$ satisfying $u^*Su = S$, then Relation (2.2) implies that

$$\tau_S(uxu^*) = \tau(Suxu^*) = \tau(u^*Sux) = \tau(Sx) = \tau_S(x) \quad (x \in M).$$

In other words, $u\tau_S = \tau_S u$. Therefore, the following relation is established:

$$\mathcal{U}(M_{\tau_S}) = \{u \in \mathcal{U}(M) : u^*Su = S\} \tag{2.3}$$

Let $W^*(S)$ be the unital abelian von Neumann subalgebra of M generated by the spectral projections of S. Then Relation (2.3) tells us that $M_{\tau_S} = W^*(S)' \cap M$ which contains $W^*(S)$ and hence

$$W^*(S) = W^*(S)'' \cap M_{\tau_S} \subseteq \mathcal{Z}(M_{\tau_S}).$$

As S is affiliated with $W^*(S)$, it is affiliated with $\mathcal{Z}(M_{\tau_S})$.

Consider now $S, T \in L_1^+(M,\tau)$ such that $\phi = \tau_S$ and $\psi = \tau_T$. Then $\phi - \psi = \tau_{S-T}$. The hypothesis $\mathcal{Z}(M_{\tau_T}) \subseteq \mathcal{Z}(M_{\tau_S})$ implies that both S and T are affiliated with $\mathcal{Z}(M_{\tau_S})$, and so is the operator $R := S - T$. Moreover, it also follows from the hypothesis that

$$\tau(Rx) = 0 \qquad (x \in \mathcal{Z}(M_{\tau_S})). \tag{2.4}$$

We denote by E_R the spectral projection measure of R and set $e_n := E_R([0,n]) \in \mathcal{Z}(M_{\tau_S})$ $(n \in \mathbb{N})$. Then $Re_n \in \mathcal{Z}(M_{\tau_S})_+$ and the condition $\tau(Re_n) = 0$ (see (2.4)) implies that $Re_n = 0$ (since τ is faithful). Similarly, if $f_n := E_R([-n,0])$ $(n \in \mathbb{N})$, then $-Rf_n \in \mathcal{Z}(M_{\tau_S})_+$ and the condition $\tau(-Rf_n) = 0$ will give $Rf_n = 0$. This means that $R(e_n + f_n) = 0$. In other words,

$$R\xi = \Big(\int_{-n}^{n} \lambda \, dE_R(\lambda)\Big)\xi = R(e_n + f_n)\xi = 0 \qquad (\xi \in E_R([-n,n])\mathfrak{H}_\tau; n \in \mathbb{N}).$$

Since $\bigcup_{n\in\mathbb{N}} E_R([-n,n])\mathfrak{H}_\tau$ is a core for R, we conclude that $R = 0$, which means that $\phi = \psi$. □

The following example tells us that one cannot replace the condition $\phi|_{\mathcal{Z}(M_\phi)} = \psi|_{\mathcal{Z}(M_\phi)}$ with $\phi|_{\mathcal{Z}(M_\psi)} = \psi|_{\mathcal{Z}(M_\psi)}$ in Proposition 4.

Example 5. Consider M_4 to be the von Neumann algebra of 4×4 complex matrices. Let R and S be diagonal elements in M_4 with their diagonals being $(\frac{2}{5}, \frac{2}{5}, \frac{8}{5}, \frac{8}{5})$, and $(\frac{1}{5}, \frac{3}{5}, \frac{7}{5}, \frac{9}{5})$, respectively. Then

$$\{S\}' = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \subseteq M_2 \oplus M_2 = \{R\}'.$$

Thus, $\mathcal{Z}(\{R\}') = \mathbb{C}I_2 \oplus \mathbb{C}I_2 \subsetneq \mathcal{Z}(\{S\}')$, where I_2 is the identity of M_2. Moreover, if $\mathbf{tr}_4$ is the tracial state on M_4, then $\mathbf{tr}_4(RT) = \mathbf{tr}_4(ST)$, for every $T \in \mathcal{Z}(\{R\}')$, but certainly, $R \neq S$.

In the case of the von Neumann algebra $\mathcal{B}(\mathfrak{H})$ for a Hilbert space $\mathfrak{H}$, we can have a slightly better version of Proposition 4 as follows.

Example 6. Let R and S be two compact normal operators in $\mathcal{B}(\mathfrak{H})$ such that $\mathcal{Z}(\{S\}') \subseteq \mathcal{Z}(\{R\}')$.

(a) Let $\lambda_1, \lambda_2, \cdots$ be the set of all distinct eigenvalues of R (one of the λ_k could be zero) with eigenspaces $\mathfrak{K}_1, \mathfrak{K}_2, \cdots$ respectively. Consider $p_1, p_2, \cdots$ to be the projections (in $\mathcal{B}(\mathfrak{H})$) onto the subspaces $\mathfrak{K}_1, \mathfrak{K}_2, \cdots$, respectively. Then $R = \sum_{i=1}^N \lambda_i p_i$, where $N \in \mathbb{N} \cup \{\infty\}$ is the cardinality of the set of eigenvalues of R. Moreover,

$$\{R\}' = \bigoplus_{i=1}^N p_i \mathcal{B}(\mathfrak{H}) p_i \qquad \text{as well as} \qquad \mathcal{Z}(\{R\}') = \bigoplus_{i=1}^N \mathbb{C} p_i.$$

For each $T \in \mathcal{Z}(\{S\}') \subseteq \mathcal{Z}(\{R\}')$, we know that $T = \sum_{i=1}^N \nu_i p_i$ for some complex numbers $\nu_1, \nu_2, \cdots$.

(b) Suppose, in addition, that R and S are trace-class operators and that

$$\mathbf{tr}(RT) = \mathbf{tr}(ST) \quad \text{for any} \quad T \in \mathcal{Z}(\{R\}'), \tag{2.5}$$

where **tr** stands for the trace on trace-class operators. Then as $S \in \mathcal{Z}(\{S\}')$, part (a) tells us that $S = \sum_{i=1}^N \mu_i p_i$ for some complex numbers $\mu_1, \mu_2, \cdots$. Moreover, (2.5) implies that $\lambda_i = \mu_i$ for all i. Hence, $R = S$.

Because of the above example, we may consider $\big(\mathcal{Z}(M_\phi), \phi|_{\mathcal{Z}(M_\phi)}\big)$ as the "abstract spectral decomposition" of the normal state ϕ. Proposition 4 tells us that the abstract spectral decomposition of ϕ completely determines the normal state ϕ.

Remark 7. We would like to thank the referee for informing us that the assertion in Proposition 4 is no longer valid when M is not semi-finite. In fact, Herman and Takesaki gave in the Corollary on page 156 of [8] a normal faithful state ϕ in a type III-factor M such that $M_\phi = \mathbb{C}1$. Now, if $u \in M \setminus \mathbb{C}1$ is a non-trivial unitary, then $u\phi u^* \neq \phi$, but one has $M_{u\phi u^*} = uM_\phi u^* = \mathbb{C}1$ as well as $u\phi u^*|_\mathbb{C} = \phi|_\mathbb{C}$.

3. The facial distances of a normal state

For any projection $p \in M$, we put

$$F_0(p) := \{\psi \in \mathfrak{S}(M) : \psi(p) = 0\} \quad \text{and} \quad \tilde{F}_0(p) := \{f \in M_+^* : \|f\| = 1; f(p) = 0\}$$

We note that $F_0(p)$ is a norm-closed face of $\mathfrak{S}(M)$, and any norm-closed face of $\mathfrak{S}(M)$ is of the form $F_0(p)$ for a unique projection p (see e.g. [1, Theorem 3.35]). Moreover, $\tilde{F}_0(p)$ is the $\sigma(M^*, M)$-closure of $F_0(p)$. Notice that a normal state ϕ belongs to $F_0(p)$ if and only if $\mathbf{s}_\phi \leq 1 - p$. On the other hand, p belongs to M_ϕ, i.e., $p\phi = \phi p$, if and only if $p\phi$ is positive, or equivalently, $p\phi = (p\phi)^*$.

For any nonempty subset $S \subseteq M_+^*$ and $g \in M_+^*$, we set

$$\mathrm{dist}(g, S) := \inf\big\{\|g - f\| : f \in S\big\}.$$

Lemma 8. *Let M be a von Neumann algebra. For any $\phi \in \mathfrak{S}(M)$ and $p \in \mathcal{P}(M) \setminus \{0, 1\}$, the following statements are equivalent.*

- (C1) $p\phi = \phi p$.
- (C2) $\mathrm{dist}(\phi, F_0(1 - p)) = 2\phi(1 - p)$.
- (C3) *There exists $\psi_0 \in F_0(1 - p)$ with $\|\phi - \psi_0\| = 2\phi(1 - p)$.*
- (C4) *There exist $\psi_0 \in F_0(1 - p)$ and $\chi_0 \in F_0(p)$ such that $\|\phi - \psi_0\| + \|\phi - \chi_0\| = 2$.*

Proof. Assume that $\phi(p) = 0$, i.e., $\phi \in F_0(p)$. The Cauchy-Schwarz inequality gives $p\phi = \phi p = 0$, and Statement (C1) holds. Moreover, if we set $\chi_0 = \phi$ and take an arbitrary element $\psi_0 \in F_0(1 - p)$, then $\|\phi - \psi_0\| = 2$ (since $\mathbf{s}_\phi \mathbf{s}_{\psi_0} = 0$) and $\|\phi - \psi_0\| + \|\phi - \chi_0\| = 2$. This means that Statements (C2), (C3) and (C4) hold. Similarly, Statements (C1) - (C4) hold when $\phi(p) = 1$. In the following, we consider the case when $\phi(p) \in (0, 1)$.

(C1) $\Rightarrow$ (C4). If we set $\psi_0 := \frac{p\phi}{\phi(p)}$ and $\chi_0 := \frac{(1-p)\phi}{\phi(1-p)}$, then

$$\|\phi - \psi_0\| = \|p\phi - \psi_0\| + \|(1 - p)\phi\| = \big(1/\phi(p) - 1\big)\phi(p) + \phi(1 - p) = 2 - 2\phi(p)$$

and, similarly, $\|\phi - \chi_0\| = 2\phi(p)$.

(C4) $\Rightarrow$ (C3). Note that, in general,

$$\|\phi - \rho\| \geq |\phi(1 - 2p) - \rho(1 - 2p)| = |\phi(1 - 2p) - \rho(2 - 2p) + \rho(1)| = 2 - 2\phi(p) \quad (\rho \in F_0(1 - p)) \quad (3.1)$$

(because $1 - 2p$ has norm one). Similarly, for any $\psi \in F_0(p)$, one has $\|\phi - \psi\| \geq 2\phi(p)$. Hence, the condition $\|\phi - \psi_0\| + \|\phi - \chi_0\| = 2$ implies that

$$\|\phi - \psi_0\| = 2 - 2\phi(p) \quad \text{and} \quad \|\phi - \chi_0\| = 2\phi(p).$$

(C3) $\Rightarrow$ (C2). This part follows from (3.1).

(C2) $\Rightarrow$ (C1). For any $f \in \tilde{F}_0(1-p)$, the same argument of (3.1) tells us that $\|\phi - f\| \geq 2\phi(1-p)$. Hence, Statement (C2) implies that $\mathrm{dist}(\phi, \tilde{F}_0(1-p)) = 2\phi(1-p)$. As $\tilde{F}_0(1-p)$ is $\sigma(M^*, M)$-compact and the norm on M^* is $\sigma(M^*, M)$-lower semi-continuous (since it is the supremum of positive functions defined by norm one elements in M), we know that there exists $f_1 \in \tilde{F}_0(1-p)$ with $\|\phi - f_1\| = 2\phi(1-p)$.

It follows from $f_1 = pf_1p \in pM^*p$ that there exists $x \in pM^{**}p$ with

$$\|x\| = 1 \quad \text{and} \quad \|f_1 - p\phi p\| = (f_1 - p\phi p)(x).$$

Since $\|x - (1-p)\| = \max\{\|x\|, \|1-p\|\} = 1$, we have

$$\begin{aligned} 2 - 2\phi(p) &= \|f_1 - \phi\| \geq |(f_1 - \phi)(x - (1-p))| \\ &= |(f_1 - p\phi p)(x) + \phi(1-p)| = \|f_1 - p\phi p\| + 1 - \phi(p). \end{aligned}$$

Hence,

$$1 - \phi(p) = (f_1 - p\phi p)(p) \leq \|f_1 - p\phi p\| \leq 1 - \phi(p),$$

which means that

$$\|f_1 - p\phi p\| = 1 - \phi(p) = (f_1 - p\phi p)(p).$$

Therefore,

$$2 - 2\phi(p) = \|\phi - f_1\| \geq (\phi - f_1)(1 - 2p) = \phi(1-p) - (p\phi p)(p) + f_1(p) = 2 - 2\phi(p). \tag{3.2}$$

Suppose that $\phi - f_1 = g_+ - g_-$ is the Jordan decomposition. We learn from (3.2) that

$$\|\phi - f_1\| = (\phi - f_1)(1-p) - (\phi - f_1)(p) \leq g_+(1-p) + g_-(p) \leq \|g_+\| + \|g_-\| = \|\phi - f_1\|.$$

This forces $\|g_+\| = g_+(1-p)$ and $\|g_-\| = g_-(p)$. Consequently,

$$\phi - f_1 = (1-p)g_+(1-p) - pg_-p,$$

which gives $p(\phi - f_1) = (\phi - f_1)p$ and hence $p\phi = \phi p$ (as $pf_1 = f_1p$). □

Note that we also have $p \in \mathcal{P}(M_\phi) \setminus \{0, 1\}$ if and only if

(C5) $\mathrm{dist}(\phi, F_0(p)) + \mathrm{dist}(\phi, F_0(1-p)) = 2$.

In fact, it follows from Relation (3.1) that

$$\mathrm{dist}(\phi, F_0(p)) \geq 2\phi(p) \quad \text{and} \quad \mathrm{dist}(\phi, F_0(1-p)) \geq 2\phi(1-p).$$

Thus, the Relation (C5) is equivalent to $\mathrm{dist}(\phi, F_0(p)) = 2\phi(p)$ as well as $\mathrm{dist}(\phi, F_0(1-p)) = 2\phi(1-p)$.

We are now ready to prove our main result.

Proof of Theorem 2. Pick any $p \in \mathcal{P}(M) \setminus \{0, 1\}$. As said in the paragraph preceding this theorem, $p \in M_\phi$ if and only if $\mathrm{dist}(\phi, F_0(1-p)) + \mathrm{dist}(\phi, F_0(p)) = 2$. Thus, it follows from the hypothesis that $M_\phi = M_\psi$, and hence

$$\mathcal{Z}(M_\phi) = \mathcal{Z}(M_\psi).$$

On the other hand, for any $p \in \mathcal{P}(M_\phi) \setminus \{0, 1\}$, we know from Relation (C2) that

$$\phi(p) = \mathrm{dist}(\phi, F_0(p))/2.$$

From this, and the hypothesis, we know that $\phi(p) = \psi(p)$ for every $p \in \mathcal{P}(M_\phi)$. Now, the conclusion follows from Proposition 4. □

4. Applications and related results

Theorem 2 produces the following result.

Corollary 9. *If M is a semi-finite von Neumann algebra and $\Lambda : \mathfrak{S}(M) \to \mathfrak{S}(M)$ is a metric preserving bijection such that $\mathbf{s}_{\Lambda(\phi)} = \mathbf{s}_\phi$ for all $\phi \in \mathfrak{S}(M)$, then Λ is the identity map.*

Proof. For any closed face $F \subseteq \mathfrak{S}(M)$, there exists $p \in \mathcal{P}(M)$ such that $F = F_0(p)$, and the support preserving assumption implies that $\Lambda(F) = F$. Thus, for each $\phi \in \mathfrak{S}(M)$, one has

$$\operatorname{dist}(\phi, F) = \operatorname{dist}\big(\Lambda(\phi), \Lambda(F)\big) = \operatorname{dist}\big(\Lambda(\phi), F\big).$$

Now, Theorem 2 tells us that $\phi = \Lambda(\phi)$. □

Corollary 9 is an improvement of [17, Lemma 2.6] (which is itself a generalization of [16, Proposition 2.1]) in the sense that the type I assumption on M is relaxed to the semi-finiteness assumption. Using this, as well as the same proof as that of [17, Theorem 1.4], one can obtain the following result in the semi-finite case, namely, Proposition 3.

Proposition 10 (Mori [21, Theorem 5.11(a)])**.** *Suppose that M_1 and M_2 are two von Neumann algebras. If $\Phi : \mathfrak{S}(M_1) \to \mathfrak{S}(M_2)$ is a metric preserving bijection, there exists a Jordan *-isomorphism $\Theta : M_2 \to M_1$ whose predual map extends Φ.*

Note that the proof presented in [21, Theorem 5.11(a)] essentially referred to the discussion in Section 4 in that paper, and this makes the proof not easy to trace. We nevertheless got an alternative proof of this result in the case when M_1 is semi-finite using our Theorem 2, or more precisely Corollary 9 (see the Appendix).

Conversely, one can use Proposition 10 to obtain the following generalization of Corollary 9.

Corollary 11. *If M is a von Neumann algebra and $\Lambda : \mathfrak{S}(M) \to \mathfrak{S}(M)$ is a metric preserving bijection such that $\mathbf{s}_{\Lambda(\phi)} = \mathbf{s}_\phi$ for all $\phi \in \mathfrak{S}(M)$, then Λ is the identity map.*

Proof. By Proposition 10, there is a Jordan *-isomorphism $\Theta : M \to M$ such that $\Lambda = \Theta_*|_{\mathfrak{S}(M)}$. The support preserving assumption of Λ implies that $\Theta(\mathbf{s}_\phi) = \mathbf{s}_\phi$ for any $\phi \in \mathfrak{S}(M)$. Since Θ is weak-*-continuous and any element in $\mathcal{P}(M)$ is the supremum of an increasing net in $\{\mathbf{s}_\phi : \phi \in \mathfrak{S}(M)\}$, we see that Θ restricts to the identity map on $\mathcal{P}(M)$ and hence Θ is the identity. □

On the other hand, we can use Theorem 2 (in fact, Proposition 3) to give some applications to F-algebras as well as to Fourier algebras of locally compact quantum groups. More precisely, the type I assumption in [17, Theorem 1.2] can be relaxed to semi-finiteness and unimodularity, respectively. However, the same arguments also work when Proposition 3 is replaced by its more general form, namely, Proposition 10.

Let us recall that a Banach algebra A is an *F-algebra* if there is a von Neumann algebra structure on the dual space A^* such that the identity of the von Neumann algebra A^* is a homomorphism on A (see [6, 9, 14]). In this case, one has

$$\mathfrak{S}(A^*) = \big\{f \in A : f(A^*_+) \subseteq \mathbb{R}_+ \text{ and } f(1) = 1\big\},$$

which is closed under the multiplication of A. Moreover, $\mathfrak{S}(A^*)$ is a metric semigroup in the sense that

$$d(x_1y, x_2y) \leq d(x_1, x_2) \quad \text{and} \quad d(yx_1, yx_1) \leq d(x_1, x_2) \quad (x_1, x_2, y \in \mathfrak{S}(A^*)),$$

under the metric d induced by the norm on A^*.

The measure algebra $M(S)$ of a locally compact semigroup S is an F-algebra (see e.g., [2]). Other important examples of F-algebras include the Fourier algebra $A(G)$ and the Fourier-Stieltjes algebra

$B(G)$, when G is a locally compact group (see [4, 12]). We note that $B(G)$ is again an F-algebra when G is only a topological group (see [15, Corollary 4.7]). More generally, for a locally compact quantum group $\mathbb{G}$, the algebra $L^1(\widehat{\mathbb{G}})$ and the algebra $C_0^{\mathrm{u}}(\widehat{\mathbb{G}})^*$ are F-algebras (see e.g., [10]), where $C_0^{\mathrm{u}}(\widehat{\mathbb{G}})$ is the universal group C^*-algebra of $\mathbb{G}$ and $L^1(\widehat{\mathbb{G}})$ is the predual of the group von Neumann algebra, $L^\infty(\widehat{\mathbb{G}})$, of $\mathbb{G}$. We denote by $\Delta_{\widehat{\mathbb{G}}}$ and $\Delta^{\mathrm{u}}_{\widehat{\mathbb{G}}}$ the canonical comultiplications on $L^\infty(\widehat{\mathbb{G}})$ and $C_0^{\mathrm{u}}(\widehat{\mathbb{G}})$, respectively. We refer the readers to standard literature (e.g., [24]) for the notion of locally compact quantum groups.

Proposition 12. *Let A_1 and A_2 be F-algebras. Let $\mathbb{G}$ and $\mathbb{H}$ be locally compact quantum groups.*

(a) *Any metric semi-group isomorphism $\Phi : \mathfrak{S}(A_1^*) \to \mathfrak{S}(A_2^*)$ extends to an isometric algebra isomorphism from A_1 onto A_2.*

(b) *If $\Phi : \mathfrak{S}(L^\infty(\widehat{\mathbb{G}})) \to \mathfrak{S}(L^\infty(\widehat{\mathbb{H}}))$ is a metric semi-group isomorphism, then there is a map $\Theta : L^\infty(\widehat{\mathbb{H}}) \to L^\infty(\widehat{\mathbb{G}})$, which is either a *-isomorphism or an *-anti-isomorphism satisfying $\Delta_{\widehat{\mathbb{G}}} \circ \Theta = (\Theta \otimes \Theta) \circ \Delta_{\widehat{\mathbb{H}}}$ and $\Phi(\omega)(b) = \omega(\Theta(b))$ ($b \in L^\infty(\widehat{\mathbb{H}})$, $\omega \in \mathfrak{S}(L^\infty(\widehat{\mathbb{G}}))$).*

(c) *If $\Phi : \mathfrak{S}(C_0^{\mathrm{u}}(\widehat{\mathbb{G}})^{**}) \to \mathfrak{S}(C_0^{\mathrm{u}}(\widehat{\mathbb{H}})^{**})$ is a metric semi-group isomorphism, then there is a map $\Theta : C_0^{\mathrm{u}}(\widehat{\mathbb{H}}) \to C_0^{\mathrm{u}}(\widehat{\mathbb{G}})$, which is either a *-isomorphism or an *-anti-isomorphism satisfying $\Delta^{\mathrm{u}}_{\widehat{\mathbb{G}}} \circ \Theta = (\Theta \otimes \Theta) \circ \Delta^{\mathrm{u}}_{\widehat{\mathbb{H}}}$ and $\Phi(f)(y) = f(\Theta(y))$ ($y \in C_0^{\mathrm{u}}(\widehat{\mathbb{H}})$, $f \in \mathfrak{S}(C_0^{\mathrm{u}}(\widehat{\mathbb{G}})^{**})$).*

Proof. (a) By Proposition 10, there is a Jordan *-isomorphism $\Theta : A_2^* \to A_1^*$ (which is automatically weak-*-continuous) such that $\Phi = \Theta_*|_{\mathfrak{S}(A_1^*)}$. Thus, Φ extends to an isometric linear bijection $\bar{\Phi}$ from A_1 onto A_2. If $\phi, \psi \in A_1^+ \setminus \{0\}$, then $\phi/\phi(1), \psi/\psi(1) \in \mathfrak{S}(A_1^*)$, and we have

$$\Phi\big((\phi/\phi(1))(\psi/\psi(1))\big) = \Phi(\phi/\phi(1))\Phi(\psi/\psi(1))$$

(because Φ is a semi-group homomorphism), which gives $\bar{\Phi}(\phi\psi) = \bar{\Phi}(\phi)\bar{\Phi}(\psi)$. Now, as A_1 is a linear span of A_1^+, we know that $\bar{\Phi}$ is an algebra isomorphism.

(b) By part (a), the map Φ can be extended to a Banach algebra isomorphism from $L^1(\widehat{\mathbb{G}})$ to $L^1(\widehat{\mathbb{H}})$. The conclusions then follow from [3, Theorem 3.16]. Notice that the element $u \in L^\infty(\widehat{\mathbb{G}})$ as in the statement of [3, Theorem 3.16] is not needed here because the extension of Φ will send the positive part $L^1(\widehat{\mathbb{G}})_+$ of $L^1(\widehat{\mathbb{G}})$ to $L^1(\widehat{\mathbb{H}})_+$.

(c) With the same argument for part (b), but utilizing [3, Theorem 4.5] instead of [3, Theorem 3.16], one obtains the desired assertion. □

Recall that if $\mathbb{G}$ is an ordinary locally compact group, denoted by G, then $L^1(\widehat{\mathbb{G}})$ and $C_0^{\mathrm{u}}(\widehat{\mathbb{G}})^*$ coincide, respectively, with the Fourier algebra $A(G)$ and the Fourier-Stieltjes algebra $B(G)$ of G. The following is a direct consequence of parts (b) and (c) of Proposition 12.

Corollary 13. *Let G_1 and G_2 be two locally compact groups. If there is a metric semi-group isomorphism $\Psi : \mathfrak{S}(A(G_1)^*) \to \mathfrak{S}(A(G_2)^*)$ (or $\Psi : \mathfrak{S}(B(G_1)^*) \to \mathfrak{S}(B(G_2)^*)$), then there exists either a homeomorphic group isomorphism or a homeomorphic group anti-isomorphism $\Theta : G_2 \to G_1$ such that $\Psi(f) = f \circ \Theta$.*

Notice that the corresponding statements of the above for $L^1(G)$ and $M(G)$ also hold. In this case, one obtains a better conclusion that there is a homeomorphic group isomorphism inducing the given metric semi-group isomorphism (i.e. the group anti-isomorphism case is not there; see [16, Theorem 2.4]).

On the other hand, we also have a result for the case of Fourier-Stieltjes algebras of general topological groups. Let G be a topological group and let $B(G)$ be the associated Fourier-Stieltjes algebra. Let $\sigma_u(B(G))$ be the unitary spectrum of $B(G)$ consisting of nonzero multiplicative linear functionals of $B(G)$, which are also the unitary elements in the von Neumann algebra $B(G)^*$. In the weak*-topology,

$\sigma_u(B(G))$ is a topological group (see [13, Proposition 5.4]). Clearly, the set $\Delta(G) = \{\delta_g : g \in G\}$ of point masses is a subgroup of $\sigma_u(B(G))$. One has $\sigma_u(B(G)) = \Delta(G) \cong G$ as topological groups whenever G is locally compact. See [13] for more details.

Corollary 14. *Let G_1 and G_2 be two topological groups. Suppose there is a metric semi-group isomorphism $\Psi : \mathfrak{S}(B(G_1)^*) \to \mathfrak{S}(B(G_2)^*)$. Then there is a Jordan *-isomorphism $\Theta : B(G_2)^* \to B(G_1)^*$ such that its predual map Θ_* extending Ψ. Moreover, Θ restricts to either a homeomorphic group isomorphism or a homeomorphic group anti-isomorphism from $\sigma_u(B(G_2))$ onto $\sigma_u(B(G_1))$.*

Proof. By Proposition 10, Ψ can be extended to a positive isometric isomorphism $\bar{\Psi} : B(G_1) \to B(G_2)$. By [11, Theorem 4.5], there exists a unital Jordan *-isomorphism $\Theta : B(G_2)^* \to B(G_1)^*$ between the von Neumann algebras, whose predual map is precisely $\bar{\Psi}$. The last assertion then follows from [13, Theorem 5.8(d)] and its proof (notice that the unitary $v \in B(G_1)^*$ as in the proof of [13, Theorem 5.8(d)] is the identity element, because $\bar{\Psi}^* = \Theta$ will send the identity of $B(G_2)^*$ to the identity of $B(G_1)^*$). □

Let us end this paper with one more question. Recall that the predual of a von Neumann algebra M can be regarded as the non-commutative $L_1(M)$-space. On top of Question 1 (which is still open in the non-semi-finite case), we also ask the following question concerning non-commutative L_p-spaces.

Question 15. Let M_1 and M_2 be two von Neumann algebras and $p \in (1, \infty)$. Assume there is a metric preserving map $\Phi : L_p(M_1)_+^{\rm sp} \to L_p(M_2)_+^{\rm sp}$ between the positive parts of the unit spheres of the associated non-commutative L_p-spaces. Does there exist a Jordan *-isomorphism $\Theta : M_2 \to M_1$ that induces Φ?

Some progress on related questions can be found in [18–20].

Appendix A. The proof of Proposition 3 using Theorem 2

The aim of this appendix is to give an idea on how to obtain Proposition 3 from Corollary 9 (which itself is a consequence of Thereom 2). For this, let us recall the following results from [17, Lemma 2.4] and [17, Theorem 1.4].

Proposition 16 ([17]). *Let M_1 and M_2 be von Neumann algebras.*

(a) *Suppose that M_1 does not have a type* I_2 *summand. Suppose also that the only metric preserving bijection $\Lambda : \mathfrak{S}(M) \to \mathfrak{S}(M)$ satisfying $\mathbf{s}_{\Lambda(\phi)} = \mathbf{s}_\phi$ ($\phi \in \mathfrak{S}(M)$) is the identity map. Then any metric preserving bijection from $\mathfrak{S}(M_1)$ onto $\mathfrak{S}(M_2)$ is an affine map.*

(b) *If M_1 is of type* I, *then any metric preserving bijection from $\mathfrak{S}(M_1)$ onto $\mathfrak{S}(M_2)$ is an affine map.*

The idea of the proof of Proposition 3 goes as follow. For $k = 1, 2$, let $\mathrm{e}_k \in M_k$ be the central projection such that $\mathrm{e}_k M_k$ is the type I_2 part of M_k. As in the proof of [17, Theorem 1.4], Φ can be decomposed into the direct sum of the following two metric preserving bijections:

$$\Phi' : \mathfrak{S}(\mathrm{e}_1 M_1) \to \mathfrak{S}(\mathrm{e}_2 M_2) \quad \text{and} \quad \Phi'' : \mathfrak{S}((1-\mathrm{e}_1)M_1) \to \mathfrak{S}((1-\mathrm{e}_2)M_2).$$

Proposition 16(b) tells us that Φ' is affine. Moreover, as $(1-\mathrm{e}_1)M_1$ is semi-finite, we know from Corollary 9 and Proposition 16(a) that Φ'' is also affine. By [11, Theorem 4.5], there exist Jordan *-isomorphisms $\Theta' : \mathrm{e}_2 M_2 \to \mathrm{e}_1 M_1$ and $\Theta'' : (1-\mathrm{e}_2)M_2 \to (1-\mathrm{e}_1)M_1$ such that the restrictions of their predual maps on $\mathfrak{S}(\mathrm{e}_1 M_1)$ and $\mathfrak{S}((1-\mathrm{e}_1)M_1)$ are precisely Φ' and Φ'', respectively. Let $\Theta := \Theta' \oplus \Theta''$ and $\Psi := \Theta_*|_{\mathfrak{S}(M_1)}$. As in the proof of [17, Theorem 1.4], one can show that $\Lambda := \Psi^{-1} \circ \Phi$ satisfies the requirement of Corollary 9, and hence we have $\Phi = \Psi$ as claimed.

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