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OPERATIONAL 2-LOCAL AUTOMORPHISMS/DERIVATIONS

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Abstract. Let $\phi : \mathcal{A} \to \mathcal{A}$ be a (not necessarily linear, additive or continuous) map of a standard operator algebra. Suppose for any $a, b \in \mathcal{A}$ there is an algebra automorphism $\theta_{a,b}$ of $\mathcal{A}$ such that

$$\phi(a)\phi(b) = \theta_{a,b}(ab).$$

We show that either ϕ or $-\phi$ is a linear Jordan homomorphism. Similar results are obtained when any of the following conditions is satisfied:

$$\phi(a) + \phi(b) = \theta_{a,b}(a + b),$$
$$\phi(a)\phi(b) + \phi(b)\phi(a) = \theta_{a,b}(ab + ba), \quad \text{or}$$
$$\phi(a)\phi(b)\phi(a) = \theta_{a,b}(aba).$$

We also show that a map $\phi : \mathscr{M} \to \mathscr{M}$ of a semi-finite von Neumann algebra $\mathscr{M}$ is a linear derivation if for every $a, b \in \mathscr{M}$ there is a linear derivation $D_{a,b}$ of $\mathscr{M}$ such that

$$\phi(a)b + a\phi(b) = D_{a,b}(ab).$$

1. Introduction

Let $\mathcal{A}$ be a complex (∗-)algebra. We call a map $\theta : \mathcal{A} \to \mathcal{A}$ a (∗-) *automorphism* if θ is bijective, (∗-)linear, and multiplicative. We call θ a *local* (∗-)*automorphism* if for every a in $\mathcal{A}$ there is a (∗-)automorphism θ_a of $\mathcal{A}$, depending on a, such that $\theta(a) = \theta_a(a)$. Although a local automorphism preserves idempotents, square zero elements, central elements, invertibility, and spectrum, it can be nonlinear and/or non-multiplicative (see, e.g., [13]).

The notion of local automorphisms is introduced by Larson and Sourour [11]. They showed that every surjective linear local automorphism of a matrix algebra is either an automorphism or an anti-automorphism, and that of $B(H)$ is an automorphism whenever H is an infinite dimensional Hilbert space (see also Brešar and Šemrl [4]). It is also known that a surjective linear local automorphism of a (resp. purely infinite) C^*-algebra of real rank zero, or a (resp. properly infinite) von Neumann algebra, is a linear Jordan isomorphism (resp. an automorphism), while a linear local automorphism of an abelian C*-algebra is always an algebra homomorphism (see, e.g., [13]).

We call a (not necessarily linear, additive or continuous) map $\theta : \mathcal{A} \to \mathcal{A}$ a 2-*local* (∗-)*automorphism* if for any a, b in $\mathcal{A}$ there is a (∗-)automorphism $\theta_{a,b}$ of $\mathcal{A}$ such

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that $\theta(a) = \theta_{a,b}(a)$ and $\theta(b) = \theta_{a,b}(b)$. It is clear that a 2-local ($*$-)automorphism θ is a local ($*$-)automorphism. It is also known that every 2-local automorphism of a standard operator algebra $\mathcal{A}$ on a Banach space is an algebra homomorphism provided that it is continuous with respect to the weak operator topology or its range contains all finite rank operators, while a surjective 2-local automorphism of a C*-algebra is an algebra automorphism (see [12]). Moreover, every 2-local $*$-automorphism of a von Neumann algebra is an algebra $*$-homomorphism ([5]).

On the other hand, a linear map $D : \mathscr{A} \to \mathscr{A}$ is said to be a *derivation* if

$$D(ab) = D(a)b + aD(b) \quad \text{for all } a, b \in \mathscr{A}.$$

Every derivation on a C*-algebra is (norm) continuous (see [16, Lemma 4.1.3]). Kadison and Sakai showed that every (linear) derivation $D : \mathscr{M} \to \mathscr{M}$ of a von Neumann algebra $\mathscr{M}$ is inner, namely, there is an $s \in \mathscr{M}$ such that

$$D(a) = sa - as \quad \text{for all } a \in \mathscr{M}$$

(see [16, Theorem 4.1.6]). It is also known that every derivation of a simple C^*-algebra with identity is inner ([15]; see also [16, Theorem 4.1.11 and Corollary 4.1.7]), and every derivation from a standard operator algebra into $\mathcal{B}(X)$ is an inner derivation, where X is a Banach space ([6]).

We call a map $\phi : \mathscr{A} \to \mathscr{A}$ a *local derivation* if for any a in $\mathscr{A}$ there is a derivation D_a of $\mathscr{A}$, depending on a, such that $\phi(a) = D_a(a)$. Every linear local derivation of a C^*-algebra is continuous ([9, Theorem 7.5]), and indeed a derivation ([17, Corollary 1]; see also [3, 9]). In particular, every linear local derivation ϕ on a von Neumann algebra is an inner derivation ([10, Theorem A]).

We call a map $\phi : \mathscr{A} \to \mathscr{A}$ a 2-*local derivation* if for any a, b in $\mathscr{A}$ there is a derivation $D_{a,b}$ of $\mathcal{A}$ such that $\phi(a) = D_{a,b}(a)$ and $\phi(b) = D_{a,b}(b)$. It is clear that 2-local derivations are local derivations. Ayupov and Kudaybergenov showed that every 2-local derivation of a von Neumann algebra is a linear local derivation, and thus an inner derivation ([2, Theorem 2.1]).

Recently, Molnár [14] studied some operational forms of 2-local automorphisms/derivations. They are those maps $\phi : \mathscr{A} \to \mathscr{A}$ satisfying one of the following conditions: for any $a, b \in \mathscr{A}$, there is an automorphism $\theta_{a,b}$ or a (linear) derivation $D_{a,b}$ of $\mathcal{A}$ such that

$$\phi(a) + \phi(b) = \theta_{a,b}(a + b), \tag{1.1}$$

$$\phi(a)\phi(b) = \theta_{a,b}(ab), \quad \text{or} \tag{1.2}$$

$$\phi(a)b + a\phi(b) = D_{a,b}(ab). \tag{1.3}$$

It is clear that any 2-local automorphism/derivation is an operational 2-local automorphism/derivation.

Although Proposition 1.1(c) below is stated for standard operator algebras on Hilbert spaces in [14, Theorem 2.7], the statement and the proof there are indeed also good for the case when E is a Banach space.

Proposition 1.1 (Molnár [14]). (a) *Let* $\phi : \mathbf{M}_n(\mathbb{C}) \to \mathbf{M}_n(\mathbb{C})$ *satisfy* (1.1) *and* $n \geq 3$. *Then* ϕ *is either an algebra automorphism or an algebra anti-automorphism.*

(b) *Let* $\phi : \mathbf{M}_n(\mathbb{C}) \to \mathbf{M}_n(\mathbb{C})$ *satisfy* (1.2)*. Then either* ϕ *or* $-\phi$ *is an algebra automorphism.*

(c) *Let* $\mathcal{A}$ *be a standard operator algebra over a complex Banach space* E*. Let* $\phi : \mathcal{A} \to \mathcal{B}(E)$ *satisfy* (1.3)*. Then* ϕ *is a linear derivation.*

In this note, we extend above results of Molnár, among many others, about operational 2-local automorphisms to the setting of standard operator algebras. We also show that any operational 2-local derivation of a semi-finite von Neumann algebra or a unital simple C^*-algebra with a faithful tracial state is a linear local derivation.

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2. Operational 2-local automorphisms

Let E be a (complex) Banach space. Denote by $\mathcal{B}(E)$ the algebra of bounded linear operators on E, and $\mathcal{F}(E)$ its subalgebra of (continuous) finite rank operators. A *standard operator algebra* $\mathcal{A}$ on E is a subalgebra of $\mathcal{B}(E)$ containing $\mathcal{F}(E)$.

A (linear) Jordan isomorphism $J : \mathcal{A}_1 \to \mathcal{A}_2$ between standard operator algebras on Banach spaces E_1 and E_2 is either an algebra isomorphism or an algebra anti-isomorphism. More precisely, J assumes either the form

- $T \mapsto STS^{-1}$ for an invertible bounded linear map $S : E_1 \to E_2$, or
- $T \mapsto ST'S^{-1}$ for an invertible bounded linear map $S : E_1' \to E_2$,

where E' is the Banach dual space of E, and $T' : E_1' \to E_1'$ is the dual map of T. When the second form holds, both E_1 and E_2 are reflexive ([7, Theorem 2.6]).

Proposition 2.1. *Let* $\mathcal{A}$ *be a standard operator algebra* $\mathcal{A}$ *on a complex Banach space* E *of dimension at least three. Let* $\phi : \mathcal{A} \to \mathcal{A}$ *be a map such that the range of* ϕ *contains all rank one idempotents. Suppose for any* $a, b \in \mathcal{A}$ *there is an algebra automorphism* $\theta_{a,b}$ *of* $\mathcal{A}$ *such that*

$$\phi(a) + \phi(b) = \theta_{a,b}(a + b). \tag{2.1}$$

Then the restriction $\phi\mid_{\mathcal{F}(E)}$ *is an algebra automorphism or an anti-automorphism of* $\mathcal{F}(E)$*. If* ϕ *is continuous with respect to the weak operator topology then* ϕ *is an algebra homomorphism or an anti-homomorphism of* $\mathcal{A}$*.*

Proof. With $b = -a$ in (2.1), we see that $\phi(-a) = -\phi(a)$ for any $a \in \mathcal{A}$. Thus, we have $\phi(0) = 0$. With $b = 0$ in (2.1), we see that ϕ is a local automorphism of $\mathcal{A}$. In particular, ϕ sends idempotents to idempotents, and preserves rank. Let p, q be disjoint idempotents in $\mathcal{A}$ of finite rank m and n; that is, $pq = qp = 0$. Then (2.1) implies that $\phi(p) + \phi(q)$ is an idempotent of rank $m + n$. Hence, ϕ sends exactly disjoint idempotents to disjoint idempotents. In particular, ϕ induces a bijection of the set of all rank one idempotents which preserves disjointness in both directions.

Let p be an idempotent of finite rank $n \geq 3$. Then $p' = \phi(p)$ is again an idempotent of rank n. Let $a = pap$ in $\mathcal{F}(E)$. For any finite rank idempotent q disjoint from p, we have

$$\phi(a) + \phi(q) = \theta_{a,q}(a + q)$$

has rank strictly greater than the rank of a. This forces $\phi(a)$ disjoint from $I-\phi(p) = I-p'$. Indeed, suppose that $\phi(a)(I-p')$ has rank m such that $1 \leq m \leq n$. We can write $\phi(a)(I-p') = \sum_{j=1}^{m} y_j \otimes f_j$ with linearly independent vectors $y_1, \ldots, y_m$ in E and linearly independent norm one linear functionals $f_1, \ldots, f_m$ such that $f_j = 0$ on $p'E$ for $j = 1, \ldots, m$. Here, $v \otimes g$ denotes the rank one operator $u \mapsto g(u)v$. Let x_1 be a unit vector in $(I-p')E$ such that $f_1(x_1) = 1$ and $f_2(x_1) = \cdots = f_m(x_1) = 0$. Then the rank one idempotent $q' = x_1 \otimes f_1$ is disjoint from p'. Note that

$$\phi(a) + q' = \phi(a)(I-p') + q' + \phi(a)p' = (x_1 + y_1) \otimes f_1 + \sum_{j=2}^{m} y_j \otimes f_j + \phi(a)p'.$$

Since $(x_1 + y_1) \otimes f_1 + \sum_{j=2}^{m} y_j \otimes f_j$ has rank at most m, which is the rank of $\phi(a)(1-q')$, we see that $\phi(a) + q'$ has rank at most the rank of $\phi(a)$. Let q be a rank one idempotent in $\mathcal{F}(E)$ such that $\phi(q) = q'$. Since $\phi(p)\phi(q) = p'q' = 0$, we have $pq = 0$, and thus $qa = aq = 0$. We then arrive at a contradiction:

$$\operatorname{rank}(\phi(a)) \geq \operatorname{rank}(\phi(a) + \phi(q)) = \operatorname{rank}(a+q) = \operatorname{rank}(a) + 1 = \operatorname{rank}(\phi(a)) + 1.$$

It says $\phi(a)p' = \phi(a)$. Similarly, we see that $p'\phi(a) = \phi(a)$. Therefore,

$$\phi(a) = p'\phi(a)p' \quad \text{for any } a = pap \in \mathcal{F}(E).$$

In other words, ϕ sends $p\mathcal{F}(E)p \cong M_n(\mathbb{C})$ into $p'\mathcal{F}(E)p' \cong M_n(\mathbb{C})$. Applying Proposition 1.1(a), we see that ϕ induces a linear Jordan isomorphism from $p\mathcal{F}(E)p$ onto $p'\mathcal{F}(E)p'$.

For any $a, b \in \mathcal{F}(E)$, let p be an idempotent of big enough finite rank at least three such that $a = pap$ and $b = pap$. Since ϕ induces a linear Jordan isomorphism from $p\mathcal{F}(E)p$ onto $\phi(p)\mathcal{F}(E)\phi(p)$, we have $\phi(\alpha a + \beta b) = \alpha\phi(a) + \beta\phi(b)$ and $\phi(ab+ba) = \phi(a)\phi(b) + \phi(b)\phi(a)$ for all scalars α, β. Since the range of ϕ contains all rank one idempotents and ϕ preserves rank, ϕ restricts to a linear Jordan automorphism from $\mathcal{F}(E)$ onto $\mathcal{F}(E)$.

Finally, since every bounded linear operator on E is a limit of a net of finite rank operators in the weak operator topology, the last assertion is established. □

Definition 2.2. Fix a finite sequence $(i_1, i_2, \ldots, i_m)$ such that $\{i_1, i_2, \ldots, i_m\} = \{1, 2, \ldots, k\}$ and $k \geq 2$. Suppose there is an index i_p different from all other indices i_q. Define a *generalized product* for operators $T_1, \ldots, T_k$ by

$$T_1 * \cdots * T_k \;=\; T_{i_1} \cdots T_{i_m}.$$

Similarly, we define a *generalized Jordan product*

$$T_1 \circ \cdots \circ T_k \;=\; T_{i_1} \cdots T_{i_m} \;+\; T_{i_m} \cdots T_{i_1}.$$

The generalized product covers the usual product $T_1 * \cdots * T_k = T_1 \cdots T_k$ and the Jordan triple product $T_1 * T_2 = T_2 T_1 T_2$. The generalized Jordan product covers the usual Jordan product $T_1 \circ T_2 = T_1 T_2 + T_2 T_1$ and the Jordan ternary product $T_1 \circ T_2 \circ T_3 = T_1 T_2 T_3 + T_3 T_2 T_1$.

Theorem 2.3 ([7, Theorem 3.2] and [8, Theorem 2.2])**.** *Let $\mathcal{A}_1, \mathcal{A}_2$ be standard operator algebras on complex Banach spaces E_1, E_2, respectively. Let $T_1 * \cdots * T_k$ and $T_1 \circ \cdots \circ T_k$ be a generalized product and a generalized Jordan product defined* *as in Definition 2.2, and* $\Phi : \mathcal{A}_1 \to \mathcal{A}_2$ *is a (not necessarily linear, additive or continuous) map. Suppose*

(a) $\Phi(\mathcal{A}_1)$ *contains all finite rank operators on* E_2 *of rank at most two, and the spectrum*

$$\sigma(\Phi(A_1) * \cdots * \Phi(A_k)) = \sigma(A_1 * \cdots * A_k) \tag{2.2}$$

holds whenever any one of $A_1, \ldots, A_k$ *has rank at most one; or*

(b) $\Phi(\mathcal{A}_1)$ *contains all finite rank operators on* E_2 *of rank at most three, and the spectrum*

$$\sigma(\Phi(A_1) \circ \cdots \circ \Phi(A_k)) = \sigma(A_1 \circ \cdots \circ A_k) \tag{2.3}$$

holds whenever any one of $A_1, \ldots, A_k$ *has rank at most one.*

Then there exist a scalar ξ *with* $\xi^m = 1$ *such that* $\Phi = \xi J$ *for a linear Jordan homomorphism* $J : \mathcal{A}_1 \to \mathcal{A}_2$*.*

Moreover, J *will be an algebra homomorphism in case (a) unless the generalized product is symmetric in the sense that* $m = 2p - 1$ *and*

$$(i_1, \ldots, i_{p-1}, i_p, i_{p+1}, \ldots, i_{2p-1}) = (i_{2p-1}, \ldots, i_{p+1}, i_p, i_{p-1}, \ldots, i_1).$$

Theorem 2.4. *Let* $\mathcal{A}$ *be a standard operator algebra on a complex Banach space* E*. Let* $T_1 * \cdots * T_k$ *and* $T_1 \circ \cdots \circ T_k$ *be the generalized product and Jordan products of operators in* $\mathcal{A}$ *defined as in Definition 2.2. Let* $\phi : \mathcal{A} \to \mathcal{A}$ *be a map such that* $\Phi(\mathcal{A})$ *contains all finite rank operators on* E *of rank at most two (resp. three). Suppose for any* $a_1, \ldots, a_k$ *in* $\mathcal{A}$*, in which one of them has rank at most one, there is an algebra (resp. linear Jordan) automorphism* $\theta_{a_1,\ldots,a_k}$ *of* $\mathcal{A}$ *such that*

$$\phi(a_1) * \cdots * \phi(a_k) = \theta_{a_1,\ldots,a_k}(a_1 * \cdots * a_k) \tag{2.4}$$

$$(\textit{resp.} \quad \phi(a_1) \circ \cdots \circ \phi(a_k) = \theta_{a_1,\ldots,a_k}(a_1 \circ \cdots \circ a_k)). \tag{2.5}$$

Then ϕ *assumes either the form*

$$A \mapsto \xi S A S^{-1} \quad \textit{or} \quad A \mapsto \xi S A' S^{-1}$$

for a complex scalar ξ *with* $\xi^m = 1$*, and an invertible bounded linear operator* $S : E \to E$ *or* $S : E' \to E$*.*

Proof. Since both algebra isomorphisms and linear Jordan isomorphisms preserve spectrum, condition (2.4) implies condition (2.2), and condition (2.5) implies condition (2.3). It then follows from Theorem 2.3 the desired assertions. □

Corollary 2.5. *Let* $\mathcal{A}$ *be a standard operator algebra* $\mathcal{A}$ *on a complex Banach space. Let* $\phi : \mathcal{A} \to \mathcal{A}$ *be a map such that the range of* ϕ *contains all finite rank operators on* E *of rank at most two. Consider the following operational* 2*-local automorphism conditions.*

(a) $\phi(a)\phi(b) = \theta_{a,b}(ab)$,
(b) $\phi(a)\phi(b)\phi(a) = \theta_{a,b}(aba)$, *or*
(c) $\phi(a)\phi(b)\phi(a)^2 = \theta_{a,b}(aba^2)$,

where $\theta_{a,b}$ *is an automorphism of* $\mathcal{A}$ *for any* $a, b \in \mathcal{A}$*, in which one of* a, b *has rank at most one.*

If (a) holds, then either ϕ or $-\phi$ is an automorphism. If (b) holds then one of ϕ, $\omega\phi$ or $\omega^2\phi$ is either an automorphism or an anti-automorphism, where ω is a primitive cubic root of unity. If (c) holds then one of $\pm\phi$ and $\pm\sqrt{-1}\phi$ is an automorphism.

Proof. It follows from Theorem 2.4 that $\xi^{-1}\phi$ is an automorphism or an anti-automorphism of $\mathcal{A}$ for some mth root ξ of unity with $m = 2, 3$ or 4. The assertions follow from Theorem 2.3 since the products in (a) and (c) are not symmetric. □

We note that aba and its transpose $(aba)^t = a^t b^t a^t$ are similar if a or b has finite rank, when E is a Hilbert space. Thus the transpose map $a \mapsto a^t$ satisfies condition (b) in Corollary 2.5.

It follows from Theorem 2.4 the same way the following results for operational 2-local and 3-local Jordan automorphisms.

Corollary 2.6. *Let $\mathcal{A}$ be a standard operator algebra $\mathcal{A}$ on a complex Banach space E. Let $\phi : \mathcal{A} \to \mathcal{A}$ be a map such that the range of ϕ contains all finite rank operators on E of rank at most three. Consider the following conditions.*

(a) $\phi(a)\phi(b) + \phi(b)\phi(a) = \theta_{a,b}(ab + ba)$,
(b) $\phi(a)\phi(b)\phi(c) + \phi(c)\phi(b)\phi(a) = \theta_{a,b,c}(abc + cba)$, *or*
(c) $\phi(a)\phi(b)\phi(a)^2 + \phi(a)^2\phi(b)\phi(a) = \theta_{a,b}(aba^2 + a^2ba)$,

where $\theta_{a,b}$ (resp. $\theta_{a,b,c}$) is a linear Jordan automorphism of $\mathcal{A}$ for any $a, b, c \in \mathcal{A}$, in which one of a, b (resp. a, b, c) has rank at most one.

If (a) holds, then ϕ or $-\phi$ is a linear Jordan automorphism. If (b) holds then one of ϕ, $\omega\phi$ or $\omega^2\phi$ is a linear Jordan automorphism, where ω is a primitive cubic root of unity. If (c) holds then one of $\pm\phi$ and $\pm\sqrt{-1}\phi$ is a linear Jordan automorphism.

The assumption that the range of ϕ contains enough operators of small ranks is indispensable for ϕ being surjective. Consider, for example, the map $\phi(a) = RaL$ of $\mathcal{F}(\ell_2)$, where R is the unilateral shift and $L = R^*$ is the backward unilateral shift on the separable Hilbert space ℓ_2. Then ϕ is a linear n-local automorphism of $\mathcal{F}(\ell_2)$ for any $n \geq 1$, and thus satisfies all operational n-local automorphism conditions. However, ϕ is not an automorphism of $\mathcal{F}(\ell_2)$.

Theorem 2.7 ([7, Theorem 4.2] and [8, Theorem 4.1]). *Let $\mathcal{S}(H_1), \mathcal{S}(H_2)$ be the sets of bounded self-adjoint operators on complex Hilbert spaces H_1, H_2. Let $T_1 * \cdots * T_k$ and $T_1 \circ \cdots \circ T_k$ be a generalized product and a generalized Jordan product defined as in Definition 2.2, and $\Phi : \mathcal{A}_1 \to \mathcal{A}_2$ is a (not necessarily linear, additive or continuous) map. Suppose the range of Φ contains all finite rank self-adjoint operators on H_2 of rank at most two (resp. three), and the spectrum*

$$\sigma(\Phi(A_1) * \cdots * \Phi(A_k)) = \sigma(A_1 * \cdots * A_k)$$
$$(\textit{resp.}\quad \sigma(\Phi(A_1) \circ \cdots \circ \Phi(A_k)) = \sigma(A_1 \circ \cdots \circ A_k))$$

holds whenever any one of the self-adjoint operators $A_1, \ldots, A_k$ has rank at most one. Then there exist a scalar $\xi \in \{-1, 1\}$ with $\xi^m = 1$ and a surjective linear isometry $U : H_2 \to H_1$ such that Φ assumes either the form $A \mapsto \xi UAU^*$ or $A \mapsto \xi UA^tU^*$. Here, A^t stands for the transpose of A with respect to a fixed orthonormal basis of H_1.

Note that we do not need to assume the generalized product of self-adjoint operators is again self-adjoint in [7, Theorem 4.2].

One can derive some results about operational 2-local $*$-automorphisms from Theorem 2.7. Below is an example.

Corollary 2.8. *Let H be a complex Hilbert space. Let $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ be a map such that the range of ϕ contains all operators in $\mathcal{B}(H)$ of rank at most two. Suppose that for any $a, b \in \mathcal{B}(H)$ there is a $*$-automorphism $\theta_{a,b}$ of $\mathcal{A}$ such that*

$$\phi(a)^*\phi(b) = \theta_{a,b}(a^*b). \tag{2.6}$$

Then ϕ assumes either the form $a \mapsto UaV$ or $a \mapsto Ua^tV$ for some unitary operators U, V of H.

Proof. Putting $a = b = I$ in (2.6), we see that $\phi(I)^*\phi(I) = I$. Putting $b = a$ in (2.6), we see that $\|\phi(a)\| = \|a\|$ for any $a \in \mathcal{B}(H)$. For any unit vector x in H, let $b \in B(H)$ such that $\phi(b) = x \otimes x$ be the self-adjoint rank one operator $y \mapsto \langle y, x\rangle x$. Together with $a = I$, condition (2.6) implies

$$\|\phi(I)^*x\| = \|(\phi(I)^*x) \otimes x\| = \|\phi(I)^*\phi(b)\| = \|b\| = \|x \otimes x\| = 1.$$

Consequently, the operator $\phi(I)^*$ is an isometry, and thus $\phi(I)\phi(I)^* = I$ as well. In other words, $\phi(I)$ is a unitary operator on H.

Replacing ϕ by the map $\phi(I)^*\phi(\cdot)$, we can assume $\phi(I) = I$. It then follows from (2.6) with $b = I$ that ϕ sends every self-adjoint operator a to a self-adjoint operator $\phi(a) = \theta_{a,I}(a^*)^*$. Therefore, ϕ sends $\mathcal{S}(H)$ into $\mathcal{S}(H)$. Applying Theorem 2.7 with the product $a * b = ab$, we have a unitary operator W of H such that $\phi(b) = \pm WbW^*$ or $\pm Wb^tW^*$ for every self-adjoint operator b in $\mathcal{S}(H)$. Replacing ϕ by the map $\pm W^*\phi(\cdot)W$ or $\pm W^*\phi(\cdot)^tW$, we can further assume that $\phi(b) = b$ for every $b \in \mathcal{S}(H)$. For any $x \in H$, it then follows from (2.6) that the inner product

$$\begin{aligned}\langle \phi(a)^*x, x\rangle &= \operatorname{trace}(\phi(a)^*(x \otimes x)) = \operatorname{trace}(\phi(a)^*\phi(x \otimes x)) \\ &= \operatorname{trace}(\theta_{a,x\otimes x}(a^*x \otimes x)) = \langle a^*x, x\rangle.\end{aligned}$$

The polar identity of inner products ensures that $\phi(a) = a$ for all $a \in \mathcal{B}(H)$.

In the original setting, with $U = \pm\phi(I)W$ and $V = W^*$, we arrive at the desired conclusion. ☐

3. Operational 2-local derivations

Let $\mathscr{M}$ be a semi-finite von Neumann algebra with a normal semi-finite faithful trace τ. In other words, $\tau : \mathscr{M}_+ \to [0, +\infty]$ is a map satisfying that

1. $\tau(x + y) = \tau(x) + \tau(y)$ for all $x, y \in \mathscr{M}_+$;
2. $\tau(tx) = t\tau(x)$ for all $x \in \mathscr{M}_+$ and $t \geq 0$;
3. $\tau(x^*x) = \tau(xx^*)$ for all $x \in \mathscr{M}$;
4. $\tau(x^*x) = 0$ if and only if $x = 0$;
5. $\tau(x) = \sup\{\tau(y) : 0 \leq y \leq x, \tau(y) < +\infty\}$;
6. $\tau(x) = \sup_i \tau(x_i)$ if $x \in \mathscr{M}_+$ is the σ-weak limit of an increasing net x_i in $\mathscr{M}_+$.

Set

$$m_\tau = \{x \in \mathscr{M} : \tau(|x|) < +\infty\},$$

where $|x| = \sqrt{x^*x}$. Then m_τ is a $*$-algebra and is also a two-sided ideal of $\mathscr{M}$. The σ-weak closure of m_τ is $\mathscr{M}$, and for any $x \in \mathscr{M}_+$ there exists an increasing net $\{x_\lambda\}_\lambda$ of positive elements in m_τ with strong (SOT) limit x. Moreover, τ extends to m_τ a complex linear map such that $\tau(ab) = \tau(ba)$ for all $a, b \in m_\tau$. When $\tau(1) = 1$, we say that τ is a normal faithful tracial state; this happens exactly when $\mathscr{M}$ is a finite von Neumann algebra. See, for example, [18].

The following strengthens the results of [1] and [14, Theorem 2.7].

Theorem 3.1. *Let $\mathscr{M}$ be a semi-finite von Neumann algebra with a normal faithful semi-finite trace τ. Let $\phi : \mathscr{M} \to \mathscr{M}$ be a map. Suppose for any $A, B \in \mathscr{M}$, there is a linear derivation $D_{a,b} : \mathscr{M} \to \mathscr{M}$ such that*

$$\phi(a)b + a\phi(b) = D_{a,b}(ab).$$

Then ϕ is a linear derivation.

Proof. Since $\phi(I)I + I\phi(I) = D_{I,I}(I) = 0$, we have $\phi(I) = 0$. Then the condition

$$\phi(a) = \phi(a)I + a\phi(I) = D_{a,I}(a), \quad \forall a \in \mathscr{M},$$

says that ϕ is a local derivation. Since the linear derivation $D_{a,I}$ is inner, we have $\phi(a) \in m_\tau$ whenever $a \in m_\tau$. Consequently, $\phi(m_\tau) \subseteq m_\tau$.

For any $x \in \mathscr{M}, y \in m_\tau$, there is a linear derivation $D_{x,y}$ on $\mathscr{M}$ such that

$$\phi(x)y + x\phi(y) = D_{x,y}(xy).$$

Since $D_{x,y}$ is an inner derivation, there exists $a \in \mathscr{M}$ such that

$$D_{x,y}(xy) = axy - xya \in m_\tau.$$

Consequently,

$$\tau(D_{xy}(xy)) = 0 = \tau(\phi(x)y + x\phi(y)),$$

and thus,

$$\tau(\phi(x)y) = -\tau(x\phi(y)).$$

For any $t_1, t_2 \in \mathbb{C}$, $a_1, a_2 \in \mathscr{M}$, $b \in m_\tau$, we have

$$\begin{aligned}\tau(\phi(t_1a_1 + t_2a_2)b) &= -\tau((t_1a_1 + t_2a_2)\phi(b)) \\ &= -[t_1\tau(a_1\phi(b)) + t_2\tau(a_2\phi(b))] \\ &= \tau([t_1\phi(a_1) + t_2\phi(a_2)]b).\end{aligned}$$

This gives

$$\tau([\phi(t_1a_1 + t_2a_2) - t_1\phi(a_1) - t_2\phi(a_2)]b) = 0.$$

Let

$$w = \phi(t_1a_1 + t_2a_2) - t_1\phi(a_1) - t_2\phi(a_2).$$

We have $\tau(wb) = 0$ for all $b \in m_\tau$. Take an increasing net $\{e_\alpha\}_\alpha$ of projections in m_τ such that $e_\alpha \uparrow_\alpha I$ in $\mathscr{M}$. Since $e_\alpha w^* \in m_\tau$, we have

$$\tau(we_\alpha w^*) = 0 \quad \text{for all } \alpha.$$

Because $we_\alpha w^* \uparrow_\alpha ww^*$ and τ is normal, we have $\tau(ww^*) = 0$. Then $w = 0$ since τ is faithful. Therefore,

$$\phi(t_1a_1 + t_2a_2) - t_1\phi(a_1) - t_2\tau(a_2) = 0$$

for all scalars t_1, t_2 and $a_1, a_2 \in \mathscr{M}$; namely, ϕ is a linear map. Since every linear local derivation of $\mathscr{M}$ is a linear derivation, the proof is complete. □

Theorem 3.2. *Let $\mathscr{A}$ be a unital simple C^*-algebra with a faithful tracial state τ. Let $\phi : \mathscr{A} \to \mathscr{A}$ be a map with the property that for any $a, b \in \mathscr{A}$, there is a linear derivation $D_{a,b} : \mathscr{A} \to \mathscr{A}$ such that*

$$\phi(a)b + a\phi(b) = D_{a,b}(ab).$$

Then ϕ is a linear derivation.

Proof. Recall that every linear derivation of a unital simple C^*-algebra is inner [15]. Argue as in the proof of Theorem 3.1, we see that ϕ is a linear local derivation of $\mathscr{A}$. Since every linear local derivation of a C^*-algebra is a linear derivation ([17, Corollary 1]; see also [3, 9]), the proof is complete. □

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